

Shortest Disjoint \mathcal{S} -Paths Via Weighted Linear Matroid Parity

Yutaro Yamaguchi

Osaka University, Osaka 565-0871, Japan
yutaro_yamaguchi@ist.osaka-u.ac.jp

Abstract

Mader's disjoint \mathcal{S} -paths problem unifies two generalizations of bipartite matching: (a) non-bipartite matching and (b) disjoint s - t paths. Lovász (1980, 1981) first proposed an efficient algorithm for this problem via a reduction to matroid matching, which also unifies two generalizations of bipartite matching: (a) non-bipartite matching and (c) matroid intersection. While the weighted versions of the problems (a)–(c) in which we aim to minimize the total weight of a designated-size feasible solution are known to be solvable in polynomial time, the tractability of such a weighted version of Mader's problem has been open for a long while.

In this paper, we present the first solution to this problem with the aid of a linear representation for Lovász' reduction (which leads to a reduction to linear matroid parity) due to Schrijver (2003) and polynomial-time algorithms for a weighted version of linear matroid parity announced by Iwata (2013) and by Pap (2013). Specifically, we give a reduction of the weighted version of Mader's problem to weighted linear matroid parity, which leads to an $O(n^5)$ -time algorithm for the former problem, where n denotes the number of vertices in the input graph. Our reduction technique is also applicable to a further generalized framework, packing non-zero A -paths in group-labeled graphs, introduced by Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour (2006). The extension leads to the tractability of a broader class of weighted problems not restricted to Mader's setting.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, G.2.1 Approximation, G.2.2 Graph Theory

Keywords and phrases Mader's \mathcal{S} -paths, packing non-zero A -paths in group-labeled graphs, linear matroid parity, weighted problems, tractability

Digital Object Identifier 10.4230/LIPIcs.ISAAC.2016.63

1 Introduction

1.1 Mader's \mathcal{S} -paths

Let $G = (V, E)$ be an undirected graph. For a prescribed vertex set $A \subseteq V$ with its partition \mathcal{S} (i.e., \mathcal{S} is a family of disjoint nonempty subsets of A whose union is A), an A -path is a path between distinct vertices in A that does not intersect A in between, and an \mathcal{S} -path is an A -path whose end vertices belong to distinct elements of \mathcal{S} . Each vertex in A is called a *terminal*.

Mader's disjoint \mathcal{S} -paths problem is to find a maximum number of vertex-disjoint \mathcal{S} -paths in a given undirected graph with a terminal set partitioned as \mathcal{S} . This problem can formulate (a) the *non-bipartite matching problem* (finding a maximum matching in a given undirected graph) and (b) the *disjoint s - t paths problem* (finding a maximum number of openly disjoint paths between two specified vertices s and t in a given undirected graph), which are both



© Yutaro Yamaguchi;

licensed under Creative Commons License CC-BY

27th International Symposium on Algorithms and Computation (ISAAC 2016).

Editor: Seok-Hee Hong; Article No. 63; pp. 63:1–63:13

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

fundamental problems generalizing the *bipartite matching problem* (finding a maximum matching in a given bipartite graph).

Mader's problem was first mentioned by Gallai [8], and Mader [19] gave a good characterization by a min-max duality theorem. Lovász [16, 17] proposed the first polynomial-time algorithm via a reduction to the matroid matching problem, and later Schrijver [23, Section 73.1a] pointed out that Lovász' reduction enjoys a linear representation, which leads to a reduction of Mader's problem to the linear matroid parity problem (see Section 1.2 for the definitions). Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour [5] recently introduced a further generalized framework, called packing non-zero A -paths in group-labeled graphs (see Section 3.1 for the definition). They provided a min-max duality theorem extending Mader's theorem, and Chudnovsky, Cunningham, and Geelen [4] developed a direct combinatorial polynomial-time algorithm for that generalized problem.

In this paper, we focus on the following weighted version of Mader's problem, whose tractability has been open for a long while: for a given nonnegative length of each edge, to minimize the total length of a designated number of disjoint \mathcal{S} -paths. For a family \mathcal{P} of disjoint paths, we denote by $E(\mathcal{P})$ the set of edges traversed by some path in \mathcal{P} .

Shortest Disjoint \mathcal{S} -paths Problem

Input: An undirected graph $G = (V, E)$, a terminal set $A \subseteq V$ with its partition \mathcal{S} , a nonnegative edge length $\ell \in \mathbb{R}_{\geq 0}^E$, and a positive integer $k \in \mathbb{Z}_{>0}$.

Goal: Find a family \mathcal{P} of k vertex-disjoint \mathcal{S} -paths in G with $\ell(\mathcal{P}) := \sum_{e \in E(\mathcal{P})} \ell_e$ minimum.

Related work

Karzanov [13] presented a polynomial-time algorithm for a similar weighted problem in the edge-disjoint A -paths setting (which is a special case of Mader's setting), whose full proof was left to an unpublished paper [12]. Karzanov's problem can be solved by finding shortest k vertex-disjoint \mathcal{S} -paths for all possible k , where the number of iterations is at most $|A|/2$ and can be reduced to $O(\log |A|)$ by binary search. Hirai and Pap [10] discussed the tractability of a generalization of Karzanov's setting, in which each pair of two terminals has weight. Pap [21] dealt with another weighted version of Mader's problem, in which weight is defined only on terminal pairs (no edge length or cost).

Björklund and Husfeldt [1] recently devised a randomized algorithm for the *shortest 2-disjoint paths problem* (minimizing the total length of two vertex-disjoint paths between two pairs (s_1, t_1) and (s_2, t_2) of specified vertices in a given undirected graph), which is the first polynomial-time one. Inspired by their work, Kobayashi and Toyooka [14] developed one for finding a shortest non-zero s - t path in a group-labeled graph, and so did Hirai and Namba [9] for finding shortest disjoint $(A + B)$ -paths in an undirected graph that cover all the vertices in $A \cup B$ when $|A| + |B|$ is fixed, where an $(A + B)$ -path is an A -path or a B -path for disjoint vertex sets A and B . The weighted version of packing non-zero A -paths includes finding a shortest non-zero s - t path (see Section 3.5).

1.2 Matroid Matching

The matroid matching problem introduced by Lawler [15] also unifies two fundamental generalizations of bipartite matching: (a) the non-bipartite matching problem and (c) the *matroid intersection problem* (finding a maximum-cardinality common independent set in given two matroids on the same ground set). This problem cannot be solved in polynomial time in general, but is known to be tractable as well as to admit a good characterization

when the matroid in question is linearly represented (or in a more general situation) due to Lovász [16, 17] and Dress and Lovász [6].

We here describe the problem formulation. A *2-polymatroid* is a pair (S, f) of a finite set S (called the *ground set*) and an integer-valued set function $f: 2^S \rightarrow \mathbb{Z}$ such that

- $0 \leq f(X) \leq 2|X|$ for each $X \subseteq S$,
- $f(X) \leq f(Y)$ for each $X \subseteq Y \subseteq S$, and
- $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for each $X, Y \subseteq S$.

A subset $X \subseteq S$ is called a *matching* in a 2-polymatroid (S, f) if $f(X) = 2|X|$, and a *parity base* if $f(X) = 2|X| = f(S)$.

The *matroid matching problem* is to find a maximum matching in a given 2-polymatroid. In this paper, we utilize the following weighted version of this problem: for a given weight on the ground set, to minimize the total weight of a parity base.

Weighted Matroid Matching Problem

Input: A 2-polymatroid (S, f) and a weight $w \in \mathbb{R}^S$.

Goal: Find a parity base $B \subseteq S$ in (S, f) with $w(B) := \sum_{e \in B} w_e$ minimum.

A 2-polymatroid (S, f) is said to be *linearly represented* over a field \mathbb{F} if we are given a matrix $Z = (Z_e)_{e \in S} \in \mathbb{F}^{r \times 2S}$ obtained by concatenating $r \times 2$ matrices $Z_e \in \mathbb{F}^{r \times 2}$ ($e \in S$) such that $f(X) = \text{rank } Z(X)$ for every $X \subseteq S$, where r is a positive integer and $Z(X) = (Z_e)_{e \in X}$ denotes the submatrix of Z obtained by extracting the corresponding columns. A subset $X \subseteq S$ is called a *matching* or *parity base* for Z if it is a matching or parity base, respectively, in the corresponding 2-polymatroid.

When the input 2-polymatroid is linearly represented, the matroid matching problem is called the *linear matroid parity problem*, for which various efficient algorithms have been developed originated by Lovász [17], e.g., by Gabow and Stallmann [7], Orlin [20], and Cheung, Lau, and Leung [3]. The same weighted version is formulated as follows.

Weighted Linear Matroid Parity Problem

Input: A finite set S , a matrix $Z \in \mathbb{F}^{r \times 2S}$ over a field \mathbb{F} , and a weight $w \in \mathbb{R}^S$.

Goal: Find a parity base $B \subseteq S$ for Z with $w(B)$ minimum.

Camerini, Galbiati, and Maffioli [2] first showed that the weighted linear matroid parity problem can be solved in pseudopolynomial time, and later Cheung et al. [3] devised a faster pseudopolynomial-time algorithm. As announced by Iwata [11], this problem is solved in $O(r^3|S|)$ time, which is strongly polynomial. Independently, Pap [22] also announced a strongly polynomial-time algorithm for an equivalent weighted problem, for which no estimate of the running time was given.

1.3 Results

We present a reduction of the shortest disjoint \mathcal{S} -paths problem to the weighted linear matroid parity problem.

► **Theorem 1.** *The shortest disjoint \mathcal{S} -paths problem reduces to the weighted linear matroid parity problem.*

Our reduction leads to the first polynomial-time algorithm for the shortest disjoint \mathcal{S} -paths problem with the aid of weighted linear matroid parity algorithms due to Iwata [11] and Pap [22]. This is also the first successful application of weighted linear matroid parity.

As seen in Section 2, when $|V| = n$, our reduction results in an $O(n) \times O(n^2)$ matrix and requires $O(n^3)$ time in addition to solving weighted linear matroid parity, and hence we obtain the following running time bound by using Iwata's algorithm.

► **Corollary 2.** *One can solve the shortest disjoint \mathcal{S} -paths problem in $O(|V|^5)$ time.*

Moreover, our reduction technique can be extended to the same weighted version of packing non-zero A -paths. With the aid of reductions of packing non-zero A -paths to matroid matching and to linear matroid parity due to Tanigawa and the author [24] and the author [25], respectively, we obtain reductions between the weighted versions. While we can see the tractability via a reasonable reduction to weighted linear matroid parity only when the group in question satisfies some representability condition (see Theorem 11), it holds for a variety of groups including one for formulating Mader's setting. We refer the readers to Section 3 for the details.

The rest of this paper is organized as follows. Section 2 is devoted to presenting our reduction of the shortest disjoint \mathcal{S} -paths problem to the weighted linear matroid parity problem. In Section 3, we show that our reduction technique can be utilized to reduce the same weighted version of packing non-zero A -paths to weighted matroid matching and to weighted linear matroid parity. Finally, in Section 4, we conclude this paper with some open problems related to our work.

2 Reduction

In this section, we give a reduction of the shortest disjoint \mathcal{S} -paths problem to the weighted linear matroid parity problem. We first review Schrijver's linear representation for Lovász' reduction in Section 2.1, and then present our reduction. For the sake of convenience, we restate the two problems.

Shortest Disjoint \mathcal{S} -paths Problem

Input: An undirected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, a terminal set $A \subseteq V$ with its partition \mathcal{S} , a nonnegative edge length $\ell \in \mathbb{R}_{\geq 0}^E$, and a positive integer $k \in \mathbb{Z}_{>0}$.

Goal: Find a family \mathcal{P} of k vertex-disjoint \mathcal{S} -paths in G with $\ell(\mathcal{P})$ minimum.

Weighted Linear Matroid Parity Problem

Input: A finite set S , a matrix $Z \in \mathbb{F}^{r \times 2S}$ over a field \mathbb{F} , and a weight $w \in \mathbb{R}^S$.

Goal: Find a parity base $B \subseteq S$ for Z with $w(B)$ minimum.

Our reduction procedure is summarized as follows. We first construct an auxiliary graph G' from a given undirected graph G (see Section 2.2), which is the most important contribution of this paper. This step requires $O(n^2)$ time. Next, following Schrijver's linear representation, we make a matrix Z over the field \mathbb{Q} of rationals associated with the auxiliary graph G' , and define a weight w from the edge length ℓ in a natural way (see Section 2.3). This step takes $O(n^3)$ time. Finally, we show that the following two facts (see Claim 5), which complete the reduction within $O(n^3)$ time in total:

- for any family \mathcal{P} of k vertex-disjoint \mathcal{S} -paths in G , there exists a parity base B for Z with $w(B) = \ell(\mathcal{P})$;
- for any parity base B for Z , there exists a family \mathcal{P} of k vertex-disjoint \mathcal{S} -paths in G with $\ell(\mathcal{P}) \leq w(B)$, which can be found easily, in $O(n)$ time.

2.1 Associated Matrix for Mader's \mathcal{S} -paths

In this section, we review Schrijver's linear representation [23, Section 73.1a] for Lovász' reduction of Mader's problem to matroid matching. For a given undirected graph $G = (V, E)$ with a terminal set $A \subseteq V$ partitioned as $\mathcal{S} = \{A_1, A_2, \dots, A_t\}$, we construct an associated matrix $Z \in \mathbb{Q}^{2V \times 2E}$, where a 2×2 submatrix corresponds to each vertex $v \in V$ and each edge $e \in E$. We assume that every connected component of G has at least one \mathcal{S} -path.

Associate each edge $e = uv \in E$ with a 2-dimensional linear subspace of $(\mathbb{Q}^2)^V$,

$$L_e := \{x \in (\mathbb{Q}^2)^V \mid x(u) + x(w) = \mathbf{0}, x(v) = \mathbf{0} \ (v \in V \setminus \{u, w\})\}. \tag{1}$$

For each terminal $a \in A_i$ ($i = 1, 2, \dots, t$), define a 1-dimensional linear subspace

$$Q_a := \{x \in (\mathbb{Q}^2)^V \mid x(a) \in \langle \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle, x(v) = \mathbf{0} \ (v \in V \setminus \{a\})\}, \tag{2}$$

where $\langle y \rangle := \{py \mid p \in \mathbb{F}\}$ for a vector $y \in \mathbb{F}^r$ over a field \mathbb{F} .

Let $Q := \sum_{a \in A} Q_a$ and $\mathcal{E} := \{L_e/Q \mid e \in E\}$. Let us construct a matrix $Z \in \mathbb{Q}^{2V \times 2E}$ associated with \mathcal{E} so that $\text{rank } Z(F) = \dim(L_F/Q)$ for every $F \subseteq E$, where $L_F := \sum_{e \in F} L_e$. This can be done by arranging an appropriate basis of $L_e/Q \in \mathcal{E}$ (which is regarded as taken from the original space $(\mathbb{Q}^2)^V$) for each edge $e \in E$ (see also Remark at the end of this section). Then, the matchings for Z are characterized as follows, from which we can derive a useful characterization of the parity bases for Z (Lemma 4).

► **Lemma 3** (Schrijver [23, (73.18)]). *An edge set $F \subseteq E$ is a matching for Z if and only if F forms a forest such that every connected component has at most two terminals in A , which belong to distinct elements of \mathcal{S} if there are two.*

► **Lemma 4.** *An edge set $F \subseteq E$ is a parity base for Z if and only if F forms a spanning forest of G such that every connected component has exactly two terminals in A , which belong to distinct elements of \mathcal{S} .*

Proof. We first see that $\text{rank } Z = 2|V| - |A|$. By the above construction, it suffices to show the equality for each connected component, and assume that G itself is connected. Fix an \mathcal{S} -path in G (recall the existence assumption), and let $a, b \in A$ be its end vertices. Then, by Lemma 3, $\dim(L_E/(Q_a + Q_b)) \geq 2(|V| - 1)$ (consider a spanning tree of G with a restricted terminal set $\{a, b\}$ instead of A). Since $L_E/(Q_a + Q_b)$ is a linear subspace of the $(2|V| - 2)$ -dimensional quotient space $(\mathbb{Q}^2)^V/(Q_a + Q_b)$, these two spaces are identified. Hence, $\text{rank } Z = \dim(L_E/Q) = \dim((\mathbb{Q}^2)^V/Q) = \dim(\mathbb{Q}^2)^V - \dim Q = 2|V| - |A|$.

["If" part] Suppose that $F \subseteq E$ forms such a spanning forest of G . Then it immediately follows from Lemma 3 that F is a matching for Z . Since every connected component has exactly two terminals in A , the spanning forest formed by F consists of exactly $|A|/2$ connected components, which implies $|F| = |V| - \frac{1}{2}|A| = \frac{1}{2} \text{rank } Z$.

["Only if" part] Suppose that $F \subseteq E$ is a parity base for Z . We then have $|F| = \frac{1}{2} \text{rank } Z = |V| - \frac{1}{2}|A|$. Since F forms a forest by Lemma 3, the subgraph (V, F) is also a forest, which consists of $|A|/2$ connected components including isolated vertices if exist. Then, by Lemma 3 and the pigeonhole principle, every connected component of (V, F) has exactly two terminals in A (which belong to distinct elements of \mathcal{S}), and hence F must be spanning. ◀

► **Remark.** The above construction does not define a unique associated matrix $Z \in \mathbb{Q}^{2V \times 2E}$, and one is obtained as follows. We first compute the Kronecker product $B_G \otimes I_2 \in \mathbb{Q}^{2V \times 2E}$ of the incidence matrix $B_G \in \{-1, 0, 1\}^{V \times E} \subseteq \mathbb{Q}^{V \times E}$ of G (where each edge in G is assumed

to be arbitrarily oriented) and the 2×2 identity matrix $I_2 \in \mathbb{Q}^{2 \times 2}$. Note that $B_G \otimes I_2$ is a matrix obtained by arranging a basis of L_e for each edge $e \in E$. We then obtain Z by adding to each column of $B_G \otimes I_2$ a multiple of a vector $x \in (\mathbb{Q}^2)^V$ with $\langle x \rangle = Q_a$ for each terminal $a \in A$ (e.g., x is defined by $x(a) := \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $x(v) := \mathbf{0}$ ($v \in V \setminus \{a\}$) when $a \in A_i$) so that the first row of the corresponding submatrix $Z_a \in \mathbb{Q}^{2 \times 2E}$ has only zero. This procedure takes $O(|V| \cdot |E|)$ time in total.

2.2 Construction of Auxiliary Graph

As the first step of our reduction, we construct an auxiliary undirected graph $G' = (V', E')$ with a terminal set $A' \subseteq V'$ partitioned as \mathcal{S}' from a given undirected graph $G = (V, E)$ with a terminal set $A \subseteq V$ partitioned as \mathcal{S} . We assume that there exists a feasible solution, i.e., G has k vertex-disjoint \mathcal{S} -paths, and then we have $|A| \geq 2k$.

The construction is summarized as follows (see also Fig. 1). Add $(|A| - 2k)$ extra terminals so that each extra terminal is adjacent to all the original terminals in A , and let A_0 be the set of those extra terminals. Besides, add two other extra terminals b_1, b_2 so that b_1 and b_2 are adjacent and b_1 is adjacent to all the non-terminal vertices in $V \setminus A$. Finally, define $\mathcal{S}' := \mathcal{S} \cup \{A_0, \{b_1\}, \{b_2\}\}$.

Formally, for the vertex set, let a'_i ($i = 1, 2, \dots, |A| - 2k$) and b_j ($j = 1, 2$) be distinct new vertices not in V , and define

$$\begin{aligned} A_0 &:= \{a'_i \mid i = 1, 2, \dots, |A| - 2k\}, \\ V' &:= V \cup A_0 \cup \{b_1, b_2\}, \\ A' &:= A \cup A_0 \cup \{b_1, b_2\}, \\ \mathcal{S}' &:= \mathcal{S} \cup \{A_0, \{b_1\}, \{b_2\}\}. \end{aligned}$$

For the edge set, define

$$\begin{aligned} E_1 &:= \{e_{ia} = a'_i a \mid a'_i \in A_0, a \in A\}, \\ E_2 &:= \{e_v = b_1 v \mid v \in V \setminus A\}, \\ E' &:= E \cup E_1 \cup E_2 \cup \{e' = b_1 b_2\}. \end{aligned}$$

Note that, since $|A_0| \leq |A| = O(n)$ and we may assume that G has no parallel edges, we have $|V'| = O(n)$ and $|E'| = O(m + n^2) = O(n^2)$.

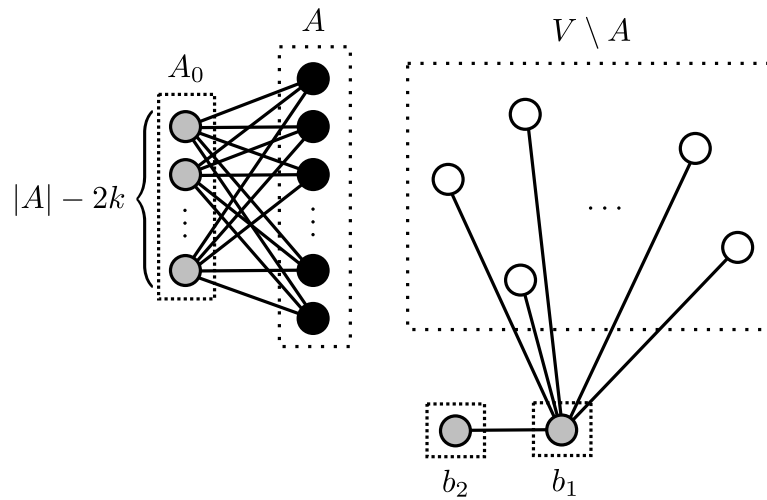
2.3 Completion of Reduction

For the auxiliary graph $G' = (V', E')$ with a terminal set $A' \subseteq V'$ partitioned as \mathcal{S}' obtained in Section 2.2, we construct an associated matrix $Z \in \mathbb{Q}^{2V' \times 2E'}$ defined in Section 2.1. Note that the construction requires $O(n^3)$ time, because $|V'| = O(n)$ and $|E'| = O(n^2)$. Define a weight $w \in \mathbb{R}^{E'}$ as follows: for each $e \in E'$,

$$w_e := \begin{cases} \ell_e & (e \in E), \\ 0 & (e \in E' \setminus E). \end{cases} \quad (3)$$

Note that $w(F) = w(F \cap E) = \sum_{e \in F \cap E} \ell_e$ for every $F \subseteq E'$, and $w(F_1) \leq w(F_2)$ for every $F_1 \subseteq F_2 \subseteq E'$ (recall that ℓ is nonnegative).

Our reduction is completed by the following claim, which implies that one can efficiently transform any minimum-weight parity base for Z into an optimal solution to the shortest disjoint \mathcal{S} -paths problem.



■ **Figure 1** How to construct the auxiliary graph (the original edges are omitted).

► **Claim 5.** *The following relations hold between feasible solutions of the two problems.*

- (i) *For any family \mathcal{P} of k vertex-disjoint \mathcal{S} -paths in G , there exists a parity base $B_{\mathcal{P}} \subseteq E'$ for Z with $B_{\mathcal{P}} \cap E = E(\mathcal{P})$ (hence $w(B_{\mathcal{P}}) = \ell(\mathcal{P})$).*
- (ii) *For any parity base $B \subseteq E'$ for Z , there exists a family \mathcal{P}_B of k vertex-disjoint \mathcal{S} -paths in G with $E(\mathcal{P}_B) \subseteq B \cap E$ (hence $\ell(\mathcal{P}_B) \leq w(B)$), which can be found in $O(n)$ time.*

Proof. (i) Let \mathcal{P} be a family of k vertex-disjoint \mathcal{S} -paths in G , which are also \mathcal{S}' -paths in G' . Then, exactly $(|A| - 2k)$ terminals in A are exposed by \mathcal{P} . By the definition of $A_0 \subseteq A' \setminus A$ and $E_1 \subseteq E' \setminus E$, there exist a perfect matching in G' between those exposed terminals and A_0 , which form $(|A| - 2k)$ vertex-disjoint \mathcal{S}' -paths in G' . Besides, G' has one more \mathcal{S}' -path consisting of the single edge $e' = b_1 b_2 \in E' \setminus E$. We can obtain a family \mathcal{P}' of $(|A| - k + 1)$ vertex-disjoint \mathcal{S}' -paths in G' by adding these paths to \mathcal{P} .

Let $U \subseteq V \setminus A$ be the set of non-terminal vertices that are not intersected by any path in \mathcal{P}' . By the definition of $E_2 \subseteq E' \setminus E$, there exists an edge $e_v = b_1 v \in E_2$ for every $v \in U$. Let $B_{\mathcal{P}} := E(\mathcal{P}') \cup \{e_v \mid v \in U\}$. Then, obviously $B_{\mathcal{P}} \cap E = E(\mathcal{P})$, and $B_{\mathcal{P}}$ is indeed a parity base for Z by Lemma 4.

(ii) Let $B \subseteq E'$ be a parity base for Z . Then, by Lemma 4, B forms a spanning forest of G such that every connected component has exactly two terminals in A' , which belong to distinct elements of \mathcal{S}' .

By the construction of G' , the terminal b_2 can be connected only to the terminal b_1 , and hence $e' \in B$. In addition, every terminal in A_0 can be connected only to the original terminals in A , and hence every connected component containing a terminal in A_0 has an original terminal in A . This implies that $2k$ original terminals in A are distributed to k connected components in the spanning forest so that every connected component has two terminals belonging to distinct elements of \mathcal{S} . Since every additional edge in $E' \setminus E$ incident to at least one extra terminal in $A' \setminus A$, the restriction $B \cap E$ forms a forest in G that contains k vertex-disjoint \mathcal{S} -paths. Those \mathcal{S} -paths can be extracted in $O(n)$ time by the breadth first search in the forest from each terminal in A . ◀

3 Extension to Non-zero A -paths

In this section, we discuss a possible extension of our reduction technique to a further generalized framework, packing non-zero A -paths in group-labeled graphs. We first describe necessary definitions and the background in Section 3.1, and next show how to adjust our reduction to this framework in Section 3.2. In Sections 3.3 and 3.4, we discuss reductions to weighted matroid matching in general and to weighted linear matroid parity under some representability condition for the group in question, respectively. Finally, in Section 3.5, we briefly describe several special cases solvable via weighted linear matroid parity.

3.1 Packing Non-zero A -paths

Let Γ be a group. A Γ -labeled graph is a directed graph $G = (V, E)$ with each edge labeled by an element of Γ , i.e., with a mapping $\psi_G: E \rightarrow \Gamma$ called a *label function*. The *label* of a walk $W = (v_0, e_1, v_1, e_2, v_2, \dots, e_l, v_l)$ in the underlying graph of G (i.e., $v_i \in V$ for each $i = 0, 1, \dots, l$ and either $e_i = v_{i-1}v_i \in E$ or $e_i = v_iv_{i-1} \in E$ for each $i = 1, 2, \dots, l$) is defined as $\psi_G(W) := \psi_G(e_1) \cdots \psi_G(e_l)$ if $e_i = v_{i-1}v_i$ for every $i = 1, 2, \dots, l$, and otherwise by replacing the corresponding label $\psi_G(e_i)$ with its inverse $\psi_G(e_i)^{-1}$ for each i with $e_i = v_iv_{i-1}$. A walk is called *balanced* (or a *zero walk*) if its label is the identity element 1_Γ of Γ , and *unbalanced* (or a *non-zero walk*) otherwise.

The problem of *packing non-zero A -paths*, introduced by Chudnovsky et al. [5], is to find a maximum number of vertex-disjoint non-zero A -paths in a given Γ -labeled graph. It generalizes Mader's disjoint \mathcal{S} -paths problem as well as several interesting problems such as ones in topological graph theory (see [5, Section 2] for more details). For this problem, a min-max duality theorem and a polynomial-time algorithm were given by Chudnovsky et al. [5] and Chudnovsky et al. [4], respectively. Extending Lovász' reduction [16] of Mader's problem to tractable matroid matching, Tanigawa and the author [24] gave a reduction of this problem to tractable matroid matching. Furthermore, the author [25] clarified when Schrijver's reduction [23, Section 73.1a] (cf. Section 2.1) of Mader's problem to linear matroid parity can be extended to this problem.

The weighted version of packing non-zero A -paths is similarly formulated as follows.

Shortest Disjoint Non-zero A -paths Problem

Input: A Γ -labeled graph $G = (V, E)$, a terminal set $A \subseteq V$, a nonnegative edge length $\ell \in \mathbb{R}_{\geq 0}^E$, and a positive integer $k \in \mathbb{Z}_{>0}$.

Goal: Find a family \mathcal{P} of k vertex-disjoint non-zero A -paths in G with $\ell(\mathcal{P})$ minimum.

3.2 Adjustment for Non-zero A -paths

Similarly to Section 2.2, we construct an auxiliary Γ -labeled graph $G' = (V', E')$ with a terminal set $A' \subseteq V'$ from a given Γ -labeled graph $G = (V, E)$ with a terminal set $A \subseteq V$. The underlying graph and terminal set are defined in the same way, and only the difference is that we have to define a label for each additional edge in $E' \setminus E$ instead of defining a partition \mathcal{S}' of the terminal set A' .

It in fact works well to define the labels of additional edges in $E' \setminus E$ so that each \mathcal{S}' -path consisting of a single additional edge in the auxiliary graph constructed in Section 2.2 is replaced by a non-zero A' -path. Formally, we extend $\psi_G: E \rightarrow \Gamma$ to $\psi_{G'}: E' \rightarrow \Gamma$ as follows:

for each edge $e \in E'$,

$$\psi_{G'}(e) := \begin{cases} \psi_G(e) & (e \in E), \\ \alpha & (e \in E' \setminus E), \end{cases}$$

where $\alpha \in \Gamma \setminus \{1_\Gamma\}$ is an arbitrary nonidentity element.

3.3 Reduction to Weighted Matroid Matching

Tanigawa and the author [24] gave a reduction of packing non-zero A -paths to tractable matroid matching by constructing a 2-polymatroid (E, f) associated with a given Γ -labeled graph $G = (V, E)$ with a terminal set $A \subseteq V$, which extends Lovász' idea [16] for Mader's problem. The associated 2-polymatroid (E, f) is defined as follows (cf. [24, Lemma 3.1]): for each $F \subseteq E$,

$$f(F) := \sum_{F' \in \text{comp}(F)} (2|V(F')| - 2 + \rho(F') - |V(F') \cap A|),$$

where $\text{comp}(F)$ denotes the partition of F according to the connected components, and $\rho: 2^E \rightarrow \mathbb{Z}_{\geq 0}$ is defined as

$$\rho(F') := \begin{cases} 2 & (|V(F') \cap A| \geq 1 \text{ and } F' \text{ has a non-zero } A\text{-path or a non-zero cycle}), \\ 1 & \left(\begin{array}{l} |V(F') \cap A| \geq 1 \text{ and } F' \text{ has no non-zero } A\text{-path and no non-zero cycle, or} \\ |V(F') \cap A| = 0 \text{ and } F' \text{ has a non-zero cycle,} \end{array} \right), \\ 0 & (\text{otherwise}). \end{cases}$$

The matchings and parity bases in this associated 2-polymatroid (E, f) are characterized analogously to Lemmas 3 and 4, respectively, as follows.

► **Lemma 6** (Tanigawa–Yamaguchi [24, Lemma 3.2]). *An edge set $F \subseteq E$ is a matching in (E, f) if and only if F forms a forest such that every connected component has at most one A -path in G , which is non-zero if exists.*

► **Lemma 7.** *An edge set $F \subseteq E$ is a parity base in (E, f) if and only if F forms a spanning forest of G such that every connected component has exactly one A -path in G , which is non-zero.*

Thus, by the same argument as Section 2.3, we can complete an analogous reduction of the shortest disjoint non-zero A -paths problem to weighted matroid matching, in which we construct the associated 2-polymatroid for the auxiliary Γ -labeled graph $G' = (V', E')$ with the terminal set $A' \subseteq V'$ obtained in Section 3.2, and define a weight $w \in \mathbb{R}^{E'}$ as (3).

► **Theorem 8.** *The shortest disjoint non-zero A -paths problem reduces to the weighted matroid matching problem.*

3.4 Reduction to Weighted Linear Matroid Parity

The author [25] showed that, under some representability condition for the group Γ , Schrijver's linear representation can be extended to the non-zero A -paths setting. We define $\text{PGL}_2(\mathbb{F}) := \text{GL}_2(\mathbb{F}) / \{pI_2 \mid p \in \mathbb{F} \setminus \{0\}\}$, where $\text{GL}_2(\mathbb{F})$ denotes the general linear group of degree 2 over a field \mathbb{F} (i.e., the set of all nonsingular 2×2 matrices over \mathbb{F} with the ordinary multiplication) and $I_2 \in \text{GL}_2(\mathbb{F})$ the 2×2 identity matrix. Recall that we denote by $\langle y \rangle$ the 1-dimensional linear subspace spanned by a vector y .

► **Theorem 9** (Yamaguchi [25, Theorem 1]). *Let Γ be a group and \mathbb{F} a field. If there exists a homomorphism $\rho: \Gamma \rightarrow \text{PGL}_2(\mathbb{F})$ such that $\rho(\alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ for every $\alpha \in \Gamma \setminus \{1_\Gamma\}$, then packing non-zero A -paths in Γ -labeled graphs reduces to linear matroid parity over \mathbb{F} .*

For a given Γ -labeled graph $G = (V, E)$ with a terminal set $A \subseteq V$, an associated matrix $Z \in \mathbb{F}^{2V \times 2E}$ is constructed in a similar way to Section 2.1. In the construction, we just modify the definitions (1) and (2) of the linear subspaces L_e and Q_a , respectively, as follows (cf. [25, Section 2.2]):

$$L_e := \{x \in (\mathbb{F}^2)^V \mid \rho(\psi_G(e))x(u) + x(w) = \mathbf{0}, x(v) = \mathbf{0} \ (v \in V \setminus \{u, w\})\} \quad (e = uw \in E),$$

$$Q_a := \{x \in (\mathbb{F}^2)^V \mid x(a) \in \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle, x(v) = \mathbf{0} \ (v \in V \setminus \{a\})\} \quad (a \in A).$$

While Lemmas 3 and 4 are not extended straightforward, the parity bases for this associated matrix Z are characterized as follows by a similar argument based on a characterization [25, Lemmas 9 and 10] of the matchings for Z .

► **Lemma 10.** *An edge set $F \subseteq E$ is a parity base for Z if F forms a spanning forest of G such that every connected component has exactly one A -path in G , which is non-zero; only if F forms a spanning subgraph of G such that every connected component has either no cycle and exactly one A -path in G , which is non-zero, or no terminal in A and exactly one cycle.*

We here show the correctness of an analogous reduction of the shortest disjoint non-zero A -paths problem to the weighted linear matroid parity problem under the representability condition of Γ in Theorem 9. We construct the associated matrix $Z \in \mathbb{F}^{2V' \times 2E'}$ for the auxiliary Γ -labeled graph $G' = (V', E')$ with the terminal set $A' \subseteq V'$ obtained in Section 3.2, and define a weight $w \in \mathbb{R}^{E'}$ as (3).

The key observation is that any cycle in G' intersecting no terminal in A' is contained in G . Hence, for any parity base $B \subseteq E'$ for Z that has some cycles (at most one in each connected component by the “only if” part of Lemma 10), there exists a parity base $B' \subseteq E'$ for Z with $w(B') \leq w(B)$ that has no cycle (obtained by replacing edges in the cycles with edges in $E_2 \subseteq E' \setminus E$). Thus we can show an analogous statement to Claim 5 (in particular the part (ii)) in Section 2.3, which suffices to complete the reduction. Note that, even if a minimum-weight parity base for Z with some cycles is obtained by solving weighted linear matroid parity, we can ignore the cycles, whose length must be totally zero.

► **Theorem 11.** *Let Γ be a group and \mathbb{F} a field. If there exists a homomorphism $\rho: \Gamma \rightarrow \text{PGL}_2(\mathbb{F})$ such that $\rho(\alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ for every $\alpha \in \Gamma \setminus \{1_\Gamma\}$, then the shortest disjoint non-zero A -paths problem on Γ -labeled graphs reduces to the weighted linear matroid parity problem over \mathbb{F} , which can be solved in $O(|V|^5)$ time.*

3.5 Applications

As shown in [25, Section 3], a variety of groups enjoy such projective representations ρ in Theorems 9 and 11, called *coherent representations*. We here describe several examples with its applications (see [25, Section 3] for more details).

Mader’s \mathcal{S} -paths. Mader’s \mathcal{S} -paths are formulated as non-zero A -paths in \mathbb{Z} -labeled graphs as follows, where \mathbb{Z} denotes the additive group on the integers. For an undirected graph $G = (V, E)$ with a terminal set $A \subseteq V$ partitioned as $\mathcal{S} := \{A_1, A_2, \dots, A_t\}$, let $A_0 := V \setminus A$.

Orient each edge in E arbitrarily, and define a label function ψ on the set of oriented edges as follows: for each edge uw with $u \in A_i$ and $w \in A_j$, let $\psi(uw) := i - j \in \mathbb{Z}$. Then, a non-zero A -path in the resulting \mathbb{Z} -labeled graph is indeed an \mathcal{S} -path in G , and vice versa.

The group \mathbb{Z} admits a coherent representation over \mathbb{Q} in general, and hence our reduction to weighted linear matroid parity works well for any \mathbb{Z} -labeled graph.

Odd A -paths. A parity constraint is formulated by using the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, which enjoys a coherent representation over an arbitrary field. Let $G = (V, E)$ be a \mathbb{Z}_2 -labeled graph with a terminal set $A \subseteq V$, and define $E_1 := \{e \in E \mid \psi_G(e) = 1\}$, where the direction of each edge has no meaning since $-0 = 0$ and $-1 = 1$ in \mathbb{Z}_2 . Then, each non-zero A -path in G is an A -path traversing an odd number of edges in E_1 , and vice versa.

Extending the well-known fact that one can find a shortest odd s - t path via the minimum-weight perfect matching problem, one can see that shortest disjoint odd A -paths in an undirected graph can be obtained by finding a minimum-weight perfect matching in an appropriate auxiliary graph. Our reduction exaggerates the problem in a sense, but provides a unified view through the matroid matching framework.

Shortest non-zero s - t path. In general, not only \mathbb{Z} and \mathbb{Z}_2 , all cyclic groups admit coherent representations. Hence, by the fundamental theorem of finitely generated abelian groups, when $k = 1$ and Γ is a finitely generated abelian group given with its decomposition into q cyclic groups, one can find a shortest non-zero A -path in a Γ -labeled graph by solving the weighted linear matroid parity problem repeatedly q times. When $A = \{s, t\}$ for distinct vertices $s, t \in V$ in particular, one can find a shortest non-zero s - t path in $O(q|V|^5)$ time. In contrast with an algebraic, randomized algorithm due to Kobayashi and Toyooka [14], which is restricted to the case when Γ is a “finite” abelian group, our result leads to a combinatorial, deterministic solution to a more general case.

Dihedral groups. Even when Γ is non-abelian, there exists a solvable case admitting a coherent representation. A simple example is the dihedral group D_n of degree $n \geq 3$, i.e., $D_n = \langle r, R \mid r^n = R^2 = \text{id}, rR = Rr^{n-1} \rangle$.

4 Concluding Remarks

In this paper, we have presented a reduction of a weighted version of Mader’s disjoint \mathcal{S} -paths problem to weighted linear matroid parity, which leads to the first polynomial-time algorithm for the former problem with the aid of weighted linear matroid parity algorithms due to Iwata [11] and Pap [22]. It should be emphasized that this is not only a solution to a longstanding open problem but also essentially the first successful application of weighted linear matroid parity. We have also discussed a possible extension of our reduction to a generalized framework, packing non-zero A -paths in group-labeled graphs. The shortest disjoint non-zero A -paths problem always reduces to weighted matroid matching, and to weighted linear matroid parity under some representability condition for the group in question.

There remain two open problems related to this work. One is whether the shortest disjoint non-zero A -paths problem is tractable in general or not. While Chudnovsky et al. [4] developed a direct combinatorial algorithm for packing non-zero A -paths, it is quite nontrivial whether their algorithm can be extended to the weighted version. The other is on weighted matroid matching. In Lovász’ matroid matching algorithm [17], a linear representation of the input 2-polymatroid is not necessarily required, and it is sufficient to be able to compute an

appropriate projection whenever we encounter a nontrivial double circuit (see also [16, 18]). This fact enables us to solve packing non-zero A -paths efficiently via matroid matching in general. While strongly polynomial-time weighted linear matroid parity algorithms were developed by Iwata [11] and Pap [22], these algorithms rely on the linearity. If one can extend Lovász' matroid matching algorithm to the weighted case, then it is expected that a broader class of the weighted matroid matching problem turns out to be tractable, which also leads to a positive answer to the first open problem.

Acknowledgments. I am deeply grateful to Satoru Iwata for giving a large number of helpful comments and suggestions. I also appreciate insightful and instructive comments of the anonymous referees.

References

- 1 A. Björklund and T. Husfeldt. Shortest two disjoint paths in polynomial time. In *Proceedings of ICALP 2014*, pages 211–222, 2014.
- 2 P. M. Camerini, G. Galbiati, and F. Maffioli. Random pseudo-polynomial algorithms for exact matroid problems. *Journal of Algorithms*, 13(2):258–273, 1992.
- 3 H. Y. Cheung, L. C. Lau, and K. M. Leung. Algebraic algorithms for linear matroid parity problems. *ACM Transactions on Algorithms*, 10(3):No. 10, 2014.
- 4 M. Chudnovsky, W. H. Cunningham, and J. Geelen. An algorithm for packing non-zero A -paths in group-labelled graphs. *Combinatorica*, 28(2):145–161, 2008.
- 5 M. Chudnovsky, J. Geelen, B. Gerards, L. Goddyn, M. Lohman, and P. Seymour. Packing non-zero A -paths in group-labelled graphs. *Combinatorica*, 26(5):521–532, 2006.
- 6 A. Dress and L. Lovász. On some combinatorial properties of algebraic matroids. *Combinatorica*, 7(1):39–48, 1987.
- 7 H. N. Gabow and M. Stallmann. An augmenting path algorithm for linear matroid parity. *Combinatorica*, 6(2):123–150, 1986.
- 8 T. Gallai. Maximum-Minimum Sätze und verallgemeinerte Faktoren von Graphen. *Acta Mathematica Academiae Scientiarum Hungaricae*, 12(1):131–173, 1961.
- 9 H. Hirai and H. Namba. Shortest $(A + B)$ -path packing via hafnian. *arXiv preprints*, arXiv:1603.08073, 2016.
- 10 H. Hirai and G. Pap. Tree metrics and edge-disjoint \mathcal{S} -paths. *Mathematical Programming, Ser. A*, 147(1):81–123, 2014.
- 11 S. Iwata. A weighted linear matroid parity algorithm. In *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pages 251–259, 2013.
- 12 A. V. Karzanov. Edge-disjoint T -paths of minimum total cost. Technical Report STAN-CS-92-1465, Department of Computer Science, Stanford University, 1993.
- 13 A. V. Karzanov. Multiflows and disjoint paths of minimum total cost. *Mathematical Programming*, 78(2):219–242, 1997.
- 14 Y. Kobayashi and S. Toyooka. Finding a shortest non-zero path in group-labeled graphs via permanent computation. *Algorithmica*, to appear.
- 15 E. L. Lawler. *Combinatorial Optimization: Networks and Matroids*. Holt, Rinehart and Winston, 1976.
- 16 L. Lovász. Matroid matching and some applications. *Journal of Combinatorial Theory, Ser. B*, 28(2):208–236, 1980.
- 17 L. Lovász. The matroid matching problem. *Algebraic Methods in Graph Theory: Colloquia Mathematica Societatis János Bolyai*, 25(2):495–517, 1981.
- 18 L. Lovász and M. D. Plummer. *Matching Theory*. Akadémiai Kiadó, 1986.

- 19 W. Mader. Über die Maximalzahl kreuzungsfreier H -Wege. *Archiv der Mathematik*, 31(1):387–402, 1978.
- 20 J.B. Orlin. A fast, simpler algorithm for the matroid parity problem. In *Proceedings of IPCO 2008*, pages 240–258, 2008.
- 21 G. Pap. A polynomial time algorithm for weighted node-disjoint S -paths. In *Proceedings of the 7th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications*, pages 322–331, 2011.
- 22 G. Pap. Weighted linear matroid matching. In *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pages 411–413, 2013.
- 23 A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer-Verlag, 2003.
- 24 S. Tanigawa and Y. Yamaguchi. Packing non-zero A -paths via matroid matching. *Discrete Applied Mathematics*, 214(11):169–178, 2016.
- 25 Y. Yamaguchi. Packing A -paths in group-labelled graphs via linear matroid parity. *SIAM Journal on Discrete Mathematics*, 30(1):474–492, 2016.