

Bisimulation for Weakly Expressive Coalgebraic Modal Logics

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Abstract

Research on the expressiveness of coalgebraic modal logics with respect to semantic equivalence notions has so far focused mainly on finding logics that are able to distinguish states that are not behaviourally equivalent (such logics are said to be expressive). In other words, the notion of behavioural equivalence is taken as the starting point, and the expressiveness of the logic is evaluated against it. However, for some applications, modal logics that are not expressive are of independent interest. Such an example is given by contingency logic. We can now turn the question of expressiveness around and ask, given a modal logic, what is a suitable notion of semantic equivalence? In this paper, we propose a notion of Λ -bisimulation which is parametric in a collection Λ of predicate liftings. We study the basic properties of Λ -bisimilarity, and prove as our main result a Hennessy-Milner style theorem, which shows that (for finitary functors) Λ -bisimilarity exactly matches the expressiveness of the coalgebraic modal logic arising from Λ .

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1 Introduction

Coalgebraic modal logic, as in [21, 13], is a framework in which modal logics for specifying coalgebras can be developed parametric in the signature of the modal language and the coalgebra type functor T . Given a base logic (usually classical propositional logic), modalities are interpreted via so-called predicate liftings for the functor T . These are natural transformations that turn a predicate over the state space X into a predicate over TX . Given that T -coalgebras come with general notions of T -bisimilarity [23] and behavioral equivalence [14], coalgebraic modal logics are designed to respect those. In particular, if two states are behaviourally equivalent then they satisfy the same formulas. If the converse holds, then the logic is said to be expressive. and we have a generalisation of the classic Hennessy-Milner theorem [9] which states that over the class of image-finite Kripke models, two states are Kripke bisimilar if and only if they satisfy the same formulas in Hennessy-Milner logic.

General conditions for when an expressive coalgebraic modal logic for T -coalgebras exists have been identified in [22, 3, 24]. A condition that ensures that a coalgebraic logic is

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expressive is when the set of predicate liftings chosen to interpret the modalities is *separating* [22]. Informally, a collection of predicate liftings is separating if they are able to distinguish non-identical elements from TX . This line of research in coalgebraic modal logic has thus taken as starting point the semantic equivalence notion of behavioral equivalence (or T -bisimilarity), and provided results for how to obtain an expressive logic. However, for some applications, modal logics that are not expressive are of independent interest. Such an example is given by *contingency logic* (see e.g. [6, 19]). We can now turn the question of expressiveness around and ask, given a modal language, what is a suitable notion of semantic equivalence?

In this paper, we propose a notion of Λ -bisimulation which is parametric in a collection Λ of predicate liftings, and therefore tailored to the expressiveness of a given coalgebraic modal logic. The definition relies on the notion of Z -coherent pairs, where Z is a relation between the state spaces of the relevant coalgebras. Coherent pairs were introduced in [8] when studying coalgebraic semantic equivalence notions in neighbourhood frames. In particular, we see that if T is the neighbourhood functor and Λ consists of the usual neighbourhood modality, then Λ -bisimulation amounts to the notion of precocongruence for neighbourhood frames from [8]. We observe that coherent pairs have an abstract characterisation in terms of pullbacks and pushouts which makes it possible to prove most of our results using general category theoretical arguments. This suggests to us that Λ -bisimulations are a natural concept, which may be useful when considering coalgebraic modal logics over other categories than **Sets**. Moreover, we show that Λ -bisimulations, like T -bisimulations, form a complete lattice, and we show how they relate to T -bisimulations, behavioural equivalences and precocongruences. We also discuss their relationship to similar notions proposed by Gorin & Schröder [7] and Enqvist [4]. Our main result is a finitary Hennessy-Milner theorem (which does not assume Λ is separating): If T is finitary, then two states are Λ -bisimilar if and only if they satisfy the same modal Λ -formulas.

Overview. In Section 2 we fix notation and introduce the notion of coherent sets. In Section 3 we define our notion of Λ -bisimulation, study its properties, and relate it to other existing equivalence notions. Our Hennessy-Milner theorem is proved in Section 4. The paper concludes with a discussion of future and related work in Section 5.

2 Preliminaries

We will work in the category **Sets** of sets and functions. The contravariant powerset functor $Q: \mathbf{Sets} \rightarrow \mathbf{Sets}^{\text{op}}$ sends a set X to the powerset of X and a function $f: X \rightarrow Y$ to the inverse image map $Qf = f^{-1}: QY \rightarrow QX$. We assume familiarity with basic coalgebraic concepts and only provide the basic definitions. For an introduction, we refer to [23]. Given a functor $T: \mathbf{Sets} \rightarrow \mathbf{Sets}$, a T -coalgebra is a pair $(X, \gamma: X \rightarrow TX)$. A T -coalgebra morphism from (X, γ) to (Y, δ) is a function $f: X \rightarrow Y$ such that $Tf \circ \gamma = \delta \circ f$.

2.1 Coalgebraic modal logic

Coalgebraic modal logic [21] is a uniform framework in which modal logics for coalgebras can be developed parametric in the type functor T and a choice of predicate lifting.

Syntax. Given a *similarity type* Λ , which is a set of modal operators with finite arities, we define the syntax of coalgebraic modal logic as follows.

► **Definition 1.** The set \mathcal{L}_Λ of Λ -formulas is generated by the following grammar:

$$\mathcal{L}_\Lambda \ni \varphi ::= \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \underbrace{\heartsuit(\varphi, \dots, \varphi)}_{n \text{ times}} \quad (\heartsuit \in \Lambda, n\text{-ary})$$

We use the standard definitions of the Boolean operators \perp, \vee and \rightarrow .

A T -coalgebraic semantics of \mathcal{L}_Λ -formulas is given by providing a Λ -structure $(T, (\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda})$ where T is a functor on **Sets**, and for each n -ary $\heartsuit \in \Lambda$, $\llbracket \heartsuit \rrbracket$ is an n -ary predicate lifting, i.e., $\llbracket \heartsuit \rrbracket : Q^n \Rightarrow QT$ is a natural transformation. Different choices of predicate liftings yield different Λ -structures and consequently different logics.

Semantics. Given a Λ -structure $(T, (\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda})$, the truth of \mathcal{L}_Λ -formulas in a T -coalgebra $\mathbb{X} = (X, \gamma : X \rightarrow TX)$ is defined as follows:

$$\begin{aligned} \mathbb{X}, x &\models \top && \text{always} \\ \mathbb{X}, x &\models \neg\varphi && \text{iff } \mathbb{X}, x \not\models \varphi \\ \mathbb{X}, x &\models \varphi \wedge \psi && \text{iff } \mathbb{X}, x \models \varphi \text{ and } \mathbb{X}, x \models \psi \\ \mathbb{X}, x &\models \heartsuit(\varphi_1, \dots, \varphi_n) && \text{iff } \gamma(x) \in \llbracket \heartsuit \rrbracket_X(\llbracket \varphi_1 \rrbracket_{\mathbb{X}}, \dots, \llbracket \varphi_n \rrbracket_{\mathbb{X}}). \end{aligned}$$

where $\llbracket \varphi \rrbracket_{\mathbb{X}} = \{x \in X \mid \mathbb{X}, x \models \varphi\}$ for all $\varphi \in \mathcal{L}_\Lambda$. Two states x in \mathbb{X} and y in \mathbb{Y} are modally equivalent (notation: $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$), if they satisfy the same \mathcal{L}_Λ -formulas, i.e., $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$ if for all $\varphi \in \mathcal{L}_\Lambda$, $\mathbb{X}, x \models \varphi$ iff $\mathbb{Y}, y \models \varphi$.

Pattinson in [22] introduced the notion of a *separating* set of predicate liftings when studying expressive logics.

► **Definition 2.** A set $(\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda}$ of predicate liftings for a functor T is *separating* (for T) if every $t \in TX$ is uniquely determined by the set $\{((A_1, \dots, A_n), \heartsuit) \in (\mathcal{P}X)^n \times \Lambda \mid t \in \llbracket \heartsuit \rrbracket_X(A_1, \dots, A_n)\}$. That is, if $t_1, t_2 \in TX$ and $t_1 \neq t_2$ then there is an n -ary $\heartsuit \in \Lambda$ and $A_1, \dots, A_n \in \mathcal{P}X$ such that $t_1 \in \llbracket \heartsuit \rrbracket(A_1, \dots, A_n)$ and $t_2 \notin \llbracket \heartsuit \rrbracket(A_1, \dots, A_n)$, or vice versa.

We provide some examples of modal languages and their coalgebraic semantics.

► **Example 3.** Coalgebras for the covariant powerset functor \mathcal{P} are Kripke frames. The similarity type $\Lambda = \{\Box\}$ for the basic modal language (without proposition letters) is given the usual Kripke semantics by interpreting \Box via the predicate lifting $\llbracket \Box \rrbracket_X(A) = \{B \in \mathcal{P}X \mid B \subseteq A\}$, which is separating for \mathcal{P} , cf. [22].

Proposition letters can be included in the language by interpreting them as nullary predicate liftings. More precisely, given a set AtProp of proposition letters, the basic modal language over AtProp is obtained from the similarity type $\Lambda = \{\Box\} \cup \text{AtProp}$. This language is given its usual semantics in Kripke models which are coalgebras for the functor $TX = \mathcal{P}(X) \times \mathcal{P}(\text{AtProp})$ by taking the Λ -structure $(T, (\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda})$ where $\llbracket \Box \rrbracket_X(A) = \{(B, P) \in \mathcal{P}(X) \times \mathcal{P}(\text{AtProp}) \mid B \subseteq A\}$ and $\llbracket p \rrbracket_X(A) = \{(B, P) \in \mathcal{P}(X) \times \mathcal{P}(\text{AtProp}) \mid p \in P\}$.

► **Example 4.** The language of *contingency logic* [6] corresponds to the modal similarity type $\Lambda = \{\Delta\}$ and it is interpreted over Kripke frames (i.e. \mathcal{P} -coalgebras) via the predicate lifting $\llbracket \Delta \rrbracket_X(A) = \{B \in \mathcal{P}X \mid B \subseteq A \text{ or } B \subseteq A^c\}$. It is straightforward to check that $\llbracket \Delta \rrbracket$ is not separating for \mathcal{P} .

► **Example 5.** Neighbourhood frames are coalgebras for the functor $\mathcal{N} = Q^{\text{op}}Q$. We obtain the neighbourhood semantics of the basic modal language, where $\Lambda = \{\Box\}$, by taking $\llbracket \Box \rrbracket_X(A) = \{B \in \mathcal{N}X \mid A \in B\}$, which is separating for \mathcal{N} .

► **Example 6.** Neighbourhood semantics of contingency logic [5] is obtained by taking $T = \mathcal{N}$, $\Lambda = \{\Delta\}$, and $\llbracket \Delta \rrbracket_X(X) = \{B \in \mathcal{N}X \mid A \in B \text{ or } A^c \in B\}$. As in the Kripke case, $\llbracket \Delta \rrbracket$ is not separating for \mathcal{N} .

► **Example 7.** The language of *instantial neighbourhood logic (INL)* [25] arises from the similarity type $\Lambda = \{\square_n \mid n \in \mathbb{N}\}$ where \square_n is $n + 1$ -ary for all $n \in \mathbb{N}$. The semantics of instantial neighbourhood logic is obtained by taking $T = \mathcal{PP}$ and $\llbracket \square_n \rrbracket_X(A_1, \dots, A_n, B) = \{N \in \mathcal{PP}X \mid \exists U \in N : U \subseteq B \text{ and for all } i = 1, \dots, n : U \cap A_i \neq \emptyset\}$. The collection $\{\llbracket \square_n \rrbracket \mid n \in \mathbb{N}\}$ is separating for \mathcal{PP}_ω , where $\mathcal{P}_\omega(X)$ are all finite subsets of X : Suppose $N, N' \in \mathcal{P}(\mathcal{P}_\omega(X))$ and $B \in N \setminus N'$ with $B = \{x_1, \dots, x_n\}$. Then $\llbracket \square_n \rrbracket(\{x_1\}, \dots, \{x_n\}, B)$ contains N , but not N' . It is not hard to see that any finite subset of $\{\llbracket \square_n \rrbracket \mid n \in \mathbb{N}\}$ is not separating for \mathcal{PP}_ω .

2.2 Relations and Coherence

Let $R \subseteq X \times Y$ be a relation. The converse of R is written $R^{-1} \subseteq Y \times X$. The R -image of $U \subseteq X$ is the set $R[U] = \{y \in Y \mid \exists x \in U : (x, y) \in R\}$. If $R \subseteq X \times X$ is an equivalence relation, then we write $[x]_R$ (or simply $[x]$) for the equivalence class of x . The composition of $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is $R; S \subseteq X \times Z$.

We will use pullbacks and pushouts in what follows. We recall the concrete constructions in **Sets**, and refer to [17] for the general definitions. In **Sets**, a pullback $(B, g_l: B \rightarrow X, g_r: B \rightarrow Y)$ of two functions $f_l: X \rightarrow Z$ and $f_r: Y \rightarrow Z$ can be concretely constructed by taking $B = pb(f_l, f_r) = \{(x, y) \in X \times Y \mid f_l(x) = f_r(y)\}$ and $g_l = \pi'_l: B \rightarrow X$ and $g_r = \pi'_r: B \rightarrow Y$ to be the projections. Pushouts are the dual notion of pullbacks. Given a relation $R \subseteq X \times Y$ the pushout of the projections $\pi_l: R \rightarrow X$ and $\pi_r: R \rightarrow Y$, is obtained concretely as follows.

The relation R can be seen as a relation R_{X+Y} on the coproduct $X + Y$ by composing the projections with the coproduct injections $\text{in}_l: X \rightarrow X + Y$ and $\text{in}_r: Y \rightarrow X + Y$. More precisely, $R_{X+Y} = (\text{in}_l \times \text{in}_r)(R) = \{(\text{in}_l(x), \text{in}_r(y)) \mid (x, y) \in R\}$. Let \bar{R} be the smallest equivalence relation on $X + Y$ that contains R_{X+Y} . Then we take $P = (X + Y)/\bar{R}$ to be the set of \bar{R} -equivalence classes with associated quotient map $q: X + Y \rightarrow P$, and we take $p_l = q \circ \text{in}_l: X \rightarrow P$, $p_r = q \circ \text{in}_r: Y \rightarrow P$. Then (P, p_l, p_r) is a pushout of π_l and π_r . The situation is illustrated with the diagram below.

$$\begin{array}{ccccc}
 & & R & & \\
 & \swarrow \pi_l & & \searrow \pi_r & \\
 X & \xrightarrow{\text{in}_l} & X + Y & \xleftarrow{\text{in}_r} & Y \\
 & \searrow p_l & \downarrow q & \swarrow p_r & \\
 & & P & &
 \end{array}$$

Our definition of Λ -bisimulation relies on the notion of coherent pairs, which was introduced in [8]. We recall the definition and some basic facts.

► **Definition 8** (*R-coherent pairs*). Let $R \subseteq X \times Y$ be a relation with projections $\pi_l: R \rightarrow X$ and $\pi_r: R \rightarrow Y$, and let $U \subseteq X$ and $V \subseteq Y$. The pair (U, V) is *R-coherent* if $R[U] \subseteq V$ and $R^{-1}[V] \subseteq U$. In case $R \subseteq X \times X$ and $U \subseteq X$, then we say that U is *R-coherent* if (U, U) is *R-coherent*.

Note that if R is an equivalence relation on a set X and $U \subseteq X$, then U is *R-coherent* iff U is *R-closed*, i.e., U is a union of *R*-equivalence classes. We make some easy, but useful observations. Further properties of coherent sets may be found in Lemma 2.2 and 2.3 of [8].

► **Lemma 9.**

1. Let $R \subseteq X \times Y$ be a relation with projections $\pi_l: R \rightarrow X$ and $\pi_r: R \rightarrow Y$, and let $U \subseteq X$ and $V \subseteq Y$. The following are equivalent:
 - (a) (U, V) is R -coherent.
 - (b) $\pi_l^{-1}[U] = \pi_r^{-1}[V]$.
 - (c) (U, V) is in the pullback of $Q\pi_l$ and $Q\pi_r$.
 - (d) for all $(x, y) \in R$, $x \in U$ iff $y \in V$.
 - (e) $U + V$ is R_{X+Y} -coherent.
2. If $R \subseteq X \times X$ is reflexive and (U, V) is R -coherent, then $U = V$.

Due to Lemma 9(1.c), we will refer to the concrete pullback $(pb(Q\pi_l, Q\pi_r), \pi'_l, \pi'_r)$ as the *pullback of R -coherent pairs*.

The following lemma shows that there is a fundamental connection between coherent pairs and pushouts of relations. It is also key in proving Propositions 20 and 21 later.

► **Lemma 10.** *Let $R \subseteq X \times Y$ be a relation, and let (P, p_l, p_r) be the pushout of R . The triple (QP, Qp_l, Qp_r) is also pullback of $(QR, Q\pi_l, Q\pi_r)$, and hence it is isomorphic to $(pb(Q\pi_l, Q\pi_r), \pi'_l, \pi'_r)$, the pullback of R -coherent pairs.*

Proof. This lemma holds for the general reason that the contravariant powerset functor $Q: \mathbf{Sets} \rightarrow \mathbf{Sets}^{\text{op}}$ is a left adjoint of itself, more precisely of $Q^{\text{op}}: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets}$, and that left adjoints preserve colimits. Hence Q turns the pushout into a pullback. Since pullbacks are unique up to isomorphism, the result follows. The isomorphism is given concretely by the map $h: QP \rightarrow pb(Q\pi_l, Q\pi_r)$ defined for all $A \in QP$ by $h(A) = (Qp_l(A), Qp_r(A))$. We verify that $(Qp_l(A), Qp_r(A))$ is R -coherent. So let $(x, y) \in R$. It follows that $p_l(x) = p_r(y)$, and hence $x \in Qp_l(A)$ iff $p_l(x) \in A$ iff $p_r(y) \in A$ iff $y \in Qp_r(A)$. To see that h is injective, suppose $A, A' \subseteq P$ and $a \in A \setminus A'$. The maps p_l and p_r are jointly surjective. If $a \in p_l[X]$, then there is a $x \in Qp_l(A)$ such that $p_l(x) = a$. If also $x \in Qp_l(A')$, then $p_l(x) = a \in A'$, a contradiction. Similarly, if $a \in p_r[Y]$, then there is a $y \in Qp_r(A)$ such that $p_r(y) = a$, and it must be the case that $y \notin Qp_r(A')$. Hence $h(A) \neq h(A')$. To see why h is surjective, it can be verified that if (U, V) is R -coherent, and we take $A \subseteq P$ to be $A = p_l[U] \cup p_r[V]$, then $h(A) = (U, V)$. For example, to see why $Qp_l(p_l[U]) = U$, first note that the inclusion \supseteq always holds. Equality follows from the fact that (U, V) is R -coherent. Finally, we remark that (QP, Qp_l, Qp_r) is a competitor to the pullback of R -coherent pairs precisely because $(Qp_l(A), Qp_r(A))$ is R -coherent for all $A \subseteq P$. ◀

3 Λ -bisimulation

In this section, we introduce the notion of Λ -bisimulation between T -coalgebras, and investigate its properties. This notion is parametric in the choice of a signature Λ and a Λ -structure $(T, (\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda})$. In the remaining part of the paper, we therefore assume that we have fixed a functor $T: \mathbf{Sets} \rightarrow \mathbf{Sets}$, and for each $\heartsuit \in \Lambda$, a predicate lifting $\llbracket \heartsuit \rrbracket$ of appropriate arity. From now on, by abuse of language, we will also refer to Λ as the set of these predicate liftings. Moreover, we let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ denote arbitrary T -coalgebras.

3.1 Definition and Basic Properties

Our definition of Λ -bisimulations is as follows.

► **Definition 11** (Λ -bisimulation). Let $Z \subseteq X \times Y$ be a relation and let $(pb(Q\pi_l, Q\pi_r), \bar{\pi}_l, \bar{\pi}_r)$ be the associated pullback of Z -coherent pairs. The relation Z is a Λ -bisimulation between \mathbb{X} and \mathbb{Y} , if for all $\heartsuit \in \Lambda$, with \heartsuit n -ary:

$$Q\pi_l \circ Q\gamma \circ [\heartsuit]_{\mathbb{X}} \circ \bar{\pi}_l^n = Q\pi_r \circ Q\delta \circ [\heartsuit]_{\mathbb{Y}} \circ \bar{\pi}_r^n \quad (1)$$

where $\bar{\pi}_l^n: pb(Q\pi_l, Q\pi_r)^n \rightarrow (QX)^n$ and $\bar{\pi}_r^n: pb(Q\pi_l, Q\pi_r)^n \rightarrow (QY)^n$ are the pointwise projections, for example, $\bar{\pi}_l((U_1, V_1), \dots, (U_n, V_n)) = (U_1, \dots, U_n)$. In other words, the relation Z is a Λ -bisimulation if whenever $(x, y) \in Z$, then for all $\heartsuit \in \Lambda$, n -ary, and all Z -coherent pairs $(U_1, V_1), \dots, (U_n, V_n)$, we have that

$$\gamma(x) \in [\heartsuit]_{\mathbb{X}}(U_1, \dots, U_n) \quad \text{iff} \quad \delta(y) \in [\heartsuit]_{\mathbb{Y}}(V_1, \dots, V_n). \quad (\text{Coherence})$$

We write $\mathbb{X}, x \sim_{\Lambda} \mathbb{Y}, y$, if there is a Λ -bisimulation between \mathbb{X} and \mathbb{Y} that contains (x, y) . A Λ -bisimulation on a T -coalgebra \mathbb{X} is a Λ -bisimulation between \mathbb{X} and \mathbb{X} .

The next lemma provides an easy observation about dual modal operators that we will use further in the examples.

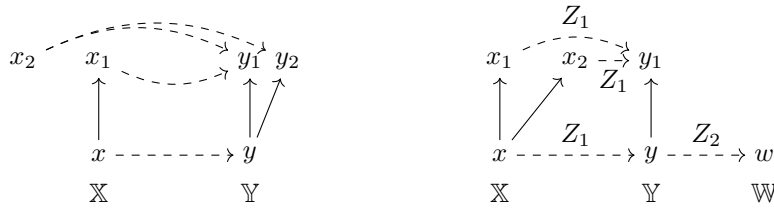
► **Lemma 12.** Let $\heartsuit, \heartsuit' \in \Lambda$ be two n -ary dual modalities, that is $\heartsuit = \neg\heartsuit'\neg$. A relation Z is a \heartsuit -bisimulation between \mathbb{X} and \mathbb{Y} iff Z is a \heartsuit' -bisimulation between \mathbb{X} and \mathbb{Y} .

Proof. First, $\heartsuit = \neg\heartsuit'\neg$ means that for all sets W , and all $A_1, \dots, A_n \subseteq W$, we have that $[\heartsuit]_W(A_1, \dots, A_n) = ([\heartsuit']_W(A_1^c, \dots, A_n^c))^c$. We note that if $Z \subseteq X \times Y$, $U \subseteq X$ and $V \subseteq Y$, then the pair (U, V) is Z -coherent iff (U^c, V^c) is Z -coherent. Hence, $\gamma(x) \in [\heartsuit]_{\mathbb{X}}(U_1, \dots, U_n)$ iff $\gamma(x) \notin [\heartsuit']_{\mathbb{X}}(U_1^c, \dots, U_n^c)$ iff $\delta(y) \notin [\heartsuit']_{\mathbb{Y}}(V_1^c, \dots, V_n^c)$ iff $\delta(y) \in [\heartsuit]_{\mathbb{Y}}(V_1, \dots, V_n)$. ◀

We provide some examples of our notion of Λ -bisimulation.

► **Example 13.** Taking $T = \mathcal{P}$ (i.e. T -coalgebras are Kripke frames) and $\Lambda = \{\Box\}$ (or $\Lambda = \{\Diamond\}$), then a relation Z between Kripke frames $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ is a \Box -bisimulation if for all $(x, y) \in Z$ and all Z -coherent pairs (U, V) : $\gamma(x) \subseteq U$ iff $\delta(y) \subseteq V$. An easy proof shows that if Z is a Kripke bisimulation then Z is a \Box -bisimulation. However, a Λ -bisimulation may not be a Kripke bisimulation. Consider the following Kripke frames: $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$, where $X = \{x, x_1, x_2\}$, $\gamma(x) = \{x_1\}$, $Y = \{y, y_1, y_2\}$ and $\delta(y) = \{y_1, y_2\}$. It can be easily checked that the relation $Z = \{(x, y), (x_1, y_1), (x_2, y_1), (x_2, y_2)\}$ is a \Box -bisimulation, but it is not a Kripke bisimulation, since the successor y_2 of y is not related to a successor of x . The situation is depicted below on the left, where Z is indicated by dashed lines. Still, when considering the associated bisimilarity notions, we find that Λ -bisimilarity coincides with Kripke bisimilarity. This follows from our Proposition 22, using that \Box (and \Diamond) is separating and \mathcal{P} preserves weak pullbacks.

This choice of T and Λ demonstrates that, in general, Λ -bisimulations are not closed under relational composition. To see this, let $\mathbb{X} = (X, \gamma)$, $\mathbb{Y} = (Y, \delta)$ and $\mathbb{W} = (W, \alpha)$ be the three Kripke frames depicted below on the right together with the two relations $Z_1 \subseteq X \times Y$ and $Z_2 \subseteq Y \times W$ (indicated by dashed lines): It is straightforward to check that Z_1 and Z_2 are Λ -bisimulations, but the composition $Z_1; Z_2 = \{(x, w)\}$ is not, because $(\{x, x_1\}, \{w\})$ is $Z_1; Z_2$ -coherent and $\gamma(x) \not\subseteq \{x, x_1\}$ and $\alpha(w) \subseteq \{w\}$.



► **Example 14.** Taking $T = \mathcal{N}$ (i.e. neighbourhood frames) and $\Lambda = \{\Box\}$, where \Box is the neighbourhood modality from Example 5, we find that a relation Z is a \Box -bisimulation between neighbourhood frames $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ if for all $(x, y) \in Z$ and all Z -coherent (U, V) : $U \in \gamma(x)$ iff $V \in \delta(y)$. This shows that Λ -bisimulations are the same as *precocongruences* which were introduced in [8], due to [8, Proposition 3.16]. We will discuss the relation between precocongruences and Λ -bisimulations further in subsection 3.2.

► **Example 15.** Taking $T = \mathcal{P}$ and $\Lambda = \{\Delta\}$, where Δ is the contingency modality from Example 4, then a Z is Δ -bisimulation between Kripke frames $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ if for all $(x, y) \in Z$ and all Z -coherent (U, V) : $\gamma(x) \subseteq U$ or $\gamma(x) \subseteq U^c$ iff $\delta(y) \subseteq V$ or $\delta(y) \subseteq V^c$. This is exactly the definition of a *rel- Δ -bisimulation* which was introduced in [2]. Prop. 3.4 in [6] tells us that Δ -bisimilarity does not imply \Box -bisimilarity. Note that in [2] the relation \sim_Λ is denoted $\sim_\Delta^{\text{betw}}$.

► **Example 16.** Taking $T = \mathcal{N}$ and $\Lambda = \{\Delta\}$, where Δ is the neighbourhood contingency modality from Example 6, then by instantiating (**Coherence**) for Δ , we have that Z is a Δ -bisimulation between neighbourhood frames $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ if for all $(x, y) \in Z$ and all Z -coherent (U, V) : $U \in \gamma(x)$ or $U^c \in \gamma(x)$ iff $V \in \delta(y)$ or $V^c \in \delta(y)$. This is exactly the definition of a *nbh- Δ -bisimulation* which was introduced in [2].

The following proposition shows that Λ -bisimulations enjoy many of the properties known to hold for Kripke bisimulations. In particular, even though Λ -bisimulations do not need to be closed under composition (cf. Example 13), we can still show that on a single T -coalgebra, \sim_Λ is an equivalence relation.

► **Proposition 17.** Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be T -coalgebras.

1. The identity relation $\text{Id} \subseteq X \times X$ is a Λ -bisimulation on \mathbb{X} .
2. If $Z \subseteq X \times Y$ is a Λ -bisimulation between \mathbb{X} and \mathbb{Y} then $Z^{-1} \subseteq Y \times X$ is a Λ -bisimulation between \mathbb{Y} and \mathbb{X} .
3. Λ -bisimulations are closed under arbitrary unions: If $Z_i \subseteq X \times Y$, $i \in I$, are Λ -bisimulations, then so is $\bigcup_{i \in I} Z_i$.
4. The relation \sim_Λ is the largest Λ -bisimulation between \mathbb{X} and \mathbb{Y} .
5. The relation \sim_Λ on \mathbb{X} is an equivalence relation.

Proof.

Item 1-2: are straightforward to check. We omit the details.

Item 3: Let $Z_i \subseteq X \times Y$, $i \in I$, be Λ -bisimulations, and let $Z = \bigcup_{i \in I} Z_i$. To show that Z is a Λ -bisimulation, assume that $(x, y) \in Z$, $\heartsuit \in \Lambda$, and (U, V) is a Z -coherent pair. From $(x, y) \in Z$ it follows that $(x, y) \in Z_i$ for some $i \in I$, and since $Z_i \subseteq Z$ we also have that (U, V) is Z_i -coherent. Hence $\gamma(x) \in \llbracket \heartsuit \rrbracket_X(U) \iff \delta(y) \in \llbracket \heartsuit \rrbracket_X(V)$. Which implies that Z is a Λ -bisimulation.

Item 4: Follows immediately from item 3.

Item 5: We show that if Z is a Λ -bisimulation on \mathbb{X} , then the equivalence closure of Z is again a Λ -bisimulation on \mathbb{X} , which suffices due to item 4. So let Z be a Λ -bisimulation on \mathbb{X} . By items 1 and 2, we may assume that Z is reflexive and symmetric. The result follows by showing that the transitive closure $Z^+ = \bigcup_{n \geq 1} Z^n$ is a Λ -bisimulation. Due to item 3 it suffices to show that for all $n \geq 1$, Z^n is a Δ -bisimulation. The proof is by induction on n . The base case ($n = 1$) holds by assumption on Z . Assume it holds for n . Induction step ($n + 1$): First note that if (U, U'') is Z^{n+1} -coherent, then since Z^{n+1} is reflexive, it follows that $U = U''$. Now suppose $(x, x') \in Z^n$, $(x', x'') \in Z$ and (U, U) is Z^{n+1} -coherent. Since Z and Z^n are reflexive and $Z^{n+1} = Z^n; Z$, it follows that

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$Z \subseteq Z^{n+1}$ and $Z^n \subseteq Z^{n+1}$, and hence (U, U) is Z -coherent as well as Z^n -coherent. We then have

$$\begin{aligned} \gamma(x) \in \llbracket \heartsuit \rrbracket_X(U) &\iff \gamma(x') \in \llbracket \heartsuit \rrbracket_X(U) \quad (\text{by induction hypothesis}) \\ &\iff \gamma(x'') \in \llbracket \heartsuit \rrbracket_X(U) \quad (\text{since } Z \text{ is a } \Lambda\text{-bisimulation}). \end{aligned}$$

Hence Z^{n+1} is a Λ -bisimulation which concludes the proof. \blacktriangleleft

Λ -bisimulations were designed to match the expressiveness of the modal language. In the next proposition we show that indeed, Λ -bisimilar states satisfy the same \mathcal{L}_Λ -formulas.

► **Proposition 18.** *If $\mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$ then $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$.*

Proof. Let $\mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$, so there exists a Λ -bisimulation $Z \subseteq X \times Y$ such that $(x, y) \in Z$. The proof is by induction on φ . The only interesting part is the modal case of the inductive step. Assume that φ is of the form $\heartsuit\psi$. By induction hypothesis, $(\llbracket \psi \rrbracket_{\mathbb{X}}, \llbracket \psi \rrbracket_{\mathbb{Y}})$ is Z -coherent. Since Z is a Λ -bisimulation, we have $\gamma(x) \in \llbracket \heartsuit \rrbracket_X(\llbracket \psi \rrbracket_{\mathbb{X}})$ iff $\delta(y) \in \llbracket \heartsuit \rrbracket_Y(\llbracket \psi \rrbracket_{\mathbb{Y}})$, which means that $\mathbb{X}, x \models \heartsuit\psi$ iff $\mathbb{Y}, y \models \heartsuit\psi$. \blacktriangleleft

3.2 Comparison with other notions

In this part, we compare our notion of Λ -bisimulation to the established notions of T -bisimulations and behavioural equivalence. It turns out that Λ -bisimulations are closest to the notion called *precongruences* in [8]. Finally, we also compare our notion to other similar proposals by Gorín and Schröder [7].

First we recall the definitions of behavioural equivalence, T -bisimulations and precongruences.

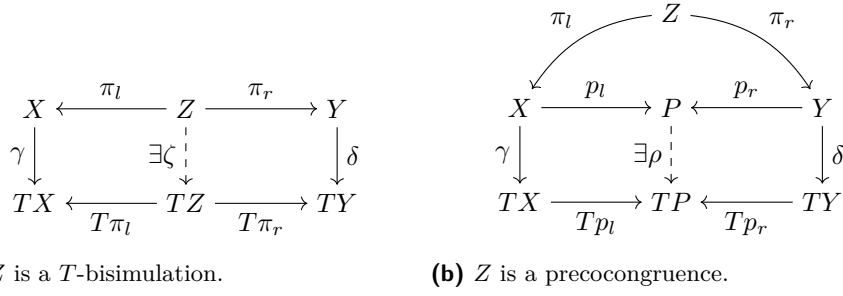
► **Definition 19.** Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be T -coalgebras.

- *Behavioural equivalence.* Two states $x \in X$ and $y \in Y$ are *behaviourally equivalent* (notation: $\mathbb{X}, x \sim_{bh} \mathbb{Y}, y$), if there is a T -coalgebra $\mathbb{E} = (E, \epsilon)$ and a pair of T -coalgebra morphisms $f : \mathbb{X} \rightarrow \mathbb{E}$ and $g : \mathbb{Y} \rightarrow \mathbb{E}$ such that $f(x) = g(y)$.
- *T -bisimulation.* A relation $Z \subseteq X \times Y$ is a *T -bisimulation* between X and Y , if there exists a function $\zeta : Z \rightarrow TZ$ such that the projections $\pi_l : Z \rightarrow X$ and $\pi_r : Z \rightarrow Y$ are T -coalgebra morphisms, i.e., the diagram in Figure 1a commutes. Two states $x \in X$ and $y \in Y$ are *T -bisimilar* (notation: $\mathbb{X}, x \sim_T \mathbb{Y}, y$) if there is a T -bisimulation between \mathbb{X} and \mathbb{Y} linking x and y .
- *Precongruence.* Let $Z \subseteq X \times Y$ be a relation with pushout (P, p_l, p_r) . Z is a *precongruence* between \mathbb{X} and \mathbb{Y} if there exists a function $\rho : P \rightarrow TP$ such that the pushout morphisms $p_l : X \rightarrow P$ and $p_r : Y \rightarrow P$ are T -coalgebra morphisms, i.e., if the diagram in Figure 1b commutes. If two states $x \in X$ and $y \in Y$ are related by some precongruence, we write $\mathbb{X}, x \sim_p \mathbb{Y}, y$.

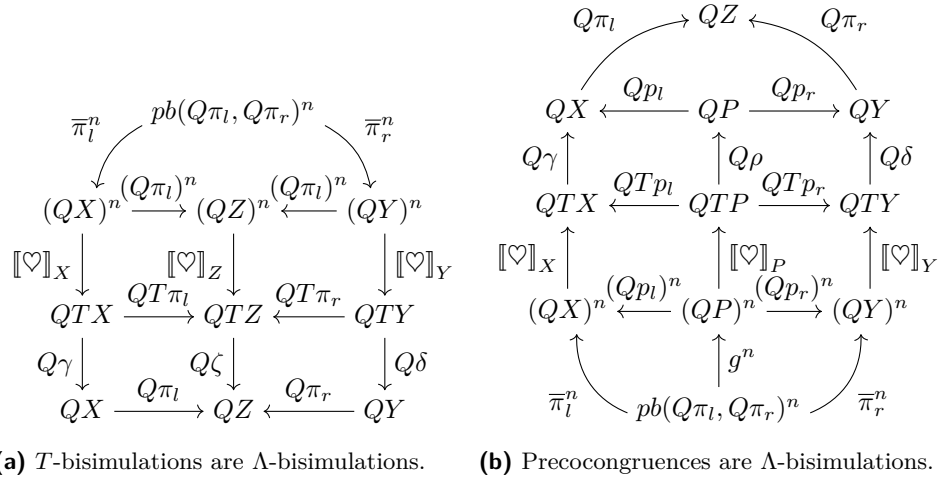
In the following proposition we give the first comparison between precongruences, T -bisimulations and Λ -bisimulations.

► **Proposition 20.** *Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be T -coalgebras, and Z be a relation between X and Y .*

1. *If Z is a T -bisimulation then Z is a Λ -bisimulation.*
2. *If Z is a precongruence then Z is a Λ -bisimulation.*



■ **Figure 1**



■ **Figure 2**

Proof.

Item 1: Apply Q to the diagram of T -bisimulation (Figure 1a), and take the pullback of $Q\pi_l$ and $Q\pi_r$. Then, by naturality of $[\![\heartsuit]\!]_Z$, and the fact that π_l and π_r are coalgebra morphisms, the diagram in Figure 2a commutes and hence, Z is a Λ -bisimulation.

Item 2: Let $Z \subseteq X \times Y$ be a pre-cocongruence relation with pushout (P, p_l, p_r) , and let (U, V) be Z -coherent. By Lemma 10, there is a map $g: pb(Q\pi_l, Q\pi_r) \rightarrow QP$ such that $Qp_l \circ g = \bar{\pi}_l$ and $Qp_r \circ g = \bar{\pi}_r$. Then, by naturality of $[\![\heartsuit]\!]_Z$ and the fact that p_l and p_r are T -coalgebra morphisms, it follows that the outer part of the diagram in Figure 2b commutes. Hence, Z is a Λ -bisimulation. ◀

The next proposition shows that, if Λ is separating, then we have the converse of Proposition 20(2).

► **Proposition 21.** *If Λ is separating and $Z \subseteq X \times Y$ is a Λ -bisimulation between \mathbb{X} and \mathbb{Y} , then Z is a pre-cocongruence between \mathbb{X} and \mathbb{Y} .*

Proof. Let $Z \subseteq X \times Y$ be a Λ -bisimulation with projections $\pi_l: Z \rightarrow X$ and $\pi_r: Z \rightarrow Y$, and pushout (P, p_l, p_r) . We need to define $\rho: P \rightarrow TP$ such that $\rho \circ p_l = Tp_l \circ \gamma$ and $\rho \circ p_r = Tp_r \circ \delta$. We obtain such a ρ from the universal property of the pushout, if we can show that for all $(x, y) \in Z$: $Tp_l(\gamma(x)) = Tp_r(\delta(y))$. To prove this, since Λ is separating, it suffices to show that for arbitrary $\heartsuit \in \Lambda$, n -ary, and $A_1, \dots, A_n \subseteq P$, $Tp_l(\gamma(x)) \in [\![\heartsuit]\!]_P(A_1, \dots, A_n)$ iff $Tp_r(\delta(y)) \in [\![\heartsuit]\!]_P(A_1, \dots, A_n)$, which is equivalent to, $Q\pi_l \circ Q\gamma \circ QTp_l \circ [\![\heartsuit]\!]_P = Q\pi_r \circ Q\delta \circ QTp_r \circ [\![\heartsuit]\!]_P$. This holds because of the commutativity of the diagram in Figure 21, where the map h is obtained from Lemma 10. ◀

$$\begin{array}{ccccc}
QX & \xrightarrow{Q\pi_l} & QZ & \xleftarrow{Q\pi_r} & QY \\
Q\gamma \uparrow & & & & \uparrow Q\delta \\
QTX & \xleftarrow{QTp_l} & QTP & \xrightarrow{QTp_r} & QTY \\
\llbracket \heartsuit \rrbracket_X \uparrow & & \uparrow \llbracket \heartsuit \rrbracket_P & & \uparrow \llbracket \heartsuit \rrbracket_Y \\
(QX)^n & \xleftarrow{(Qp_l)^n} & (QP)^n & \xrightarrow{(Qp_r)^n} & (QY)^n \\
\uparrow \bar{\pi}_l^n & \searrow & \downarrow h^n & \swarrow & \uparrow \bar{\pi}_r^n \\
& & pb(Q\pi_l, Q\pi_r)^n & &
\end{array}$$

■ **Figure 3** Proof of Proposition 21.

It was shown in [8, Proposition 3.10] that, in general, T -bisimilarity implies precocongruence equivalence which in turn implies behavioural equivalence. This fact together with Proposition 21 tells us that Λ -bisimilarity implies behavioural equivalence, whenever Λ is separating. Moreover, it is well known [23] that if T preserves weak pullbacks, then T -bisimilarity coincides with behavioural equivalence. Hence in this case, by Proposition 21, it follows that Λ -bisimilarity coincides with T -bisimilarity and behavioural equivalence. The following proposition summarises our discussion so far.

► **Proposition 22.** *Let Λ be a set of predicate liftings for T .*

1. $\mathbb{X}, x \sim_T \mathbb{Y}, y \implies \mathbb{X}, x \sim_p \mathbb{Y}, y \implies \mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$.

2. *If Λ is separating, then*

$$\mathbb{X}, x \sim_p \mathbb{Y}, y \iff \mathbb{X}, x \sim_\Lambda \mathbb{Y}, y \implies \mathbb{X}, x \sim_{bh} \mathbb{Y}, y.$$

3. *If Λ is separating and T preserves weak pullbacks, then all four notions coincide:*

$$\mathbb{X}, x \sim_T \mathbb{Y}, y \iff \mathbb{X}, x \sim_p \mathbb{Y}, y \iff \mathbb{X}, x \sim_\Lambda \mathbb{Y}, y \iff \mathbb{X}, x \sim_{bh} \mathbb{Y}, y.$$

The next lemma states that similar to the fact that T -coalgebra morphisms preserve and reflect behavioural equivalence, one can show that they preserve and reflect Λ -bisimilarity as well. We will use this fact to prove the Hennessy-Milner theorem in Section 4.

► **Proposition 23.** *If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a T -coalgebra morphism, then for all $x, x' \in X$:*

$$\mathbb{X}, x \sim_\Lambda \mathbb{X}, x' \iff \mathbb{Y}, f(x) \sim_\Lambda \mathbb{Y}, f(x').$$

Proof. For the direction from left to right, assume $\mathbb{X}, x \sim_\Lambda \mathbb{X}, x'$. Then, there exists a Λ -bisimulation Z on \mathbb{X} such that $(x, x') \in Z$. We show that $(f \times f)(Z) = \{(f(x), f(x')) \in Y \times Y \mid (x, x') \in Z\}$ is a Λ -bisimulation. Let $(f(x), f(x')) \in (f \times f)(Z)$ and $\heartsuit \in \Lambda$. Note that if (U, V) is $(f \times f)(Z)$ -coherent, then the pair $(f^{-1}[U], f^{-1}[V])$ is Z -coherent. By naturality and the fact that f is a coalgebra morphism, we have $\delta(f(x)) \in \llbracket \heartsuit \rrbracket_Y(U)$ iff $\gamma(x) \in \llbracket \heartsuit \rrbracket_X(f^{-1}[U])$, and $\delta(f(x')) \in \llbracket \heartsuit \rrbracket_Y(V)$ iff $\gamma(x') \in \llbracket \heartsuit \rrbracket_X(f^{-1}[V])$. Since Z is a Λ -bisimulation and $(f^{-1}[U], f^{-1}[V])$ is Z -coherent, we obtain $\delta(f(x)) \in \llbracket \heartsuit \rrbracket_Y(U)$ iff $\delta(f(x')) \in \llbracket \heartsuit \rrbracket_Y(V)$. A similar argument shows that if Z is a Λ -bisimulation on \mathbb{Y} then $(f^{-1} \times f^{-1})(Z) = \{(x, x') \in X \times X \mid (f(x), f(x')) \in Z\}$ is a Λ -bisimulation on X . ◀

3.2.1 Λ -bisimulations: a different approach

Gorín and Schröder introduced in [7] a similar notion of Λ -bisimulation. To distinguish their notion from the one presented here, we refer to it as GS - Λ -bisimulation. One difference with

our work is that Gorín and Schröder assume that Λ is a set of *monotone* predicate liftings. For convenience, we recall their definition here, which can be stated without the assumption of monotonicity. A relation $Z \subseteq X \times Y$ is a *GS- Λ -bisimulation* if whenever $(x, y) \in Z$ then for all $\heartsuit \in \Lambda$ and for all $A \subseteq X$ and $B \subseteq Y$

$$\gamma(x) \in \llbracket \heartsuit \rrbracket_X(A) \Rightarrow \delta(y) \in \llbracket \heartsuit \rrbracket_Y(Z[A]) \quad \text{and} \quad \delta(y) \in \llbracket \heartsuit \rrbracket_Y(B) \Rightarrow \gamma(x) \in \llbracket \heartsuit \rrbracket_X(Z^{-1}[B]).$$

Under the assumption that all $\heartsuit \in \Lambda$ are monotone, it is straightforward to show that a *GS- Λ -bisimulation* is also Λ -bisimulation. Example 13 demonstrates that there exists a choice of T and monotone Λ such that the two notions differs at the level of relations. Namely, the relation Z given there is a Λ -bisimulation, but not a *GS- Λ -bisimulation*. To see this, take $A = \{x_1, x_2\}$. We have that $\gamma(x) = \{x_1\} \subseteq A$, but $\delta(y) = \{y_1, y_2\} \not\subseteq Z[A] = \{y_1\}$. However, one can show that under the assumption that Λ is monotone, difunctional (also called zig-zag closed) Λ -bisimulations are *GS- Λ -bisimulations*, and that the relation \sim_Λ between any two T -coalgebras is difunctional. Hence the two bisimilarity notions coincide. In [7, Theorem 26] it was shown that when Λ is separating and monotone, then *GS- Λ -bisimilarity* coincides with behavioural equivalence, and hence under these assumptions, Λ -bisimilarity coincides both with *GS- Λ -bisimilarity* and with behavioural equivalence.

► **Proposition 24.** *If Λ is separating and monotone, then*

$$\mathbb{X}, x \sim_{GS-\Lambda} \mathbb{Y}, y \iff \mathbb{X}, x \sim_\Lambda \mathbb{Y}, y \iff \mathbb{X}, x \sim_{bh} \mathbb{Y}, y.$$

We point out that our results on Λ -bisimulation do not require Λ to be monotone. Furthermore, our aims and results differ from those of [7] where the starting point was to investigate simulations between T -coalgebras. In this context, *GS- Λ -bisimulations* arose naturally as two-way simulations. The results in [7] focus on identifying conditions that ensure that *GS- Λ -bisimilarity* coincides with behavioural equivalence and/or T -bisimilarity. Our approach is to accept that the language is not expressive, and show that Λ -bisimilarity allows us to generalise several results that are known to hold for expressive languages.

► **Example 25.** Consider INL from Example 7 (i.e. $T = \mathcal{PP}$). Since $\llbracket \square_n \rrbracket$ is monotone [25], it follows that \square_n -bisimilarity coincides with *GS- \square_n -bisimilarity*. Note that $\llbracket \square_0 \rrbracket_X(A) = \{N \in \mathcal{PP}(X) \mid \exists U \in N : U \subseteq A\}$ is like the monotone neighbourhood modality (which is usually interpreted in \mathcal{N} -coalgebras). It is straightforward to prove that *GS- \square_0 -bisimulations* coincide with monotonic bisimulations (see e.g. [25]). For $n \geq 1$, one can show that Z is a *GS- \square_n -bisimulation* iff for all $(x, y) \in Z$: (Here $A \subseteq_n B$ means that $A \subseteq B$ and $|A| \leq n$.)
(forth) $_n \forall U \neq \emptyset : U \in \gamma(x) \implies \forall U' \subseteq_n U. \exists V \neq \emptyset : V \in \delta(y), V \subseteq Z[U]$ and $U' \subseteq Z^{-1}[V]$.
(back) $_n \forall V \neq \emptyset : V \in \delta(y) \implies \forall V' \subseteq_n V. \exists U \neq \emptyset : U \in \gamma(x), U \subseteq Z^{-1}[V]$ and $V' \subseteq Z[U]$.
The proof uses the fact that if $\{x_1, \dots, x_n\} \subseteq A \in N$ then $N \in \square_n(\{x_1, \dots, \{x_n\}, A)$.

3.3 Λ -morphisms

Given the fact that the graph of a T -coalgebra morphism is a T -bisimulation (cf. [23, Theorem 2.5.]), it is natural to define a *Λ -morphism from \mathbb{X} to \mathbb{Y}* to be a function $f: X \rightarrow Y$ for which the graph $Gr(f) = \{(x, f(x)) \mid x \in X\}$ is a Λ -bisimulation. It then follows from Proposition 20(1) that T -coalgebra morphisms are also Λ -morphisms. Moreover, one can show that Λ -homomorphisms are closed under composition (unlike Λ -bisimulations). Therefore, T -coalgebras together with Λ -morphisms form a category.

In Enqvist [4], a weak notion of morphism for T -coalgebras was proposed which, like ours, is parametric in a set Λ of predicate liftings. To distinguish this notion from ours, we refer

to it as E - Λ -morphism. We briefly recall the definition (which we state only for unary \heartsuit , as is the case in [4]). A function $f: X \rightarrow Y$ is an E - Λ -morphism from \mathbb{X} to \mathbb{Y} if for all $B \subseteq Y$, $x \in X$, and $\heartsuit \in \Lambda$: $\delta(f(x)) \in \llbracket \heartsuit \rrbracket_Y(B)$ implies $\gamma(x) \in \llbracket \heartsuit \rrbracket_X(f^{-1}[B])$. Taking $Z = Gr(f)$, it can easily be seen that a pair (U, V) is Z -coherent iff $U = f^{-1}[V] = Q(V)$. It then follows that Λ -morphisms are E - Λ -morphisms. Since only one direction of the **(Coherence)** condition needs to hold for E - Λ -morphisms, it is straightforward to construct an example of a E - Λ -morphism which is not a Λ -morphism.

We do not investigate our notion of Λ -morphisms further in the present paper. Several interesting questions could be asked, though. We discuss those in Section 5.

4 Hennessy-Milner Theorem

This section is devoted to proving the main technical result of the paper: a coalgebraic Hennessy-Milner theorem for our notion of Λ -bisimilarity.

As we saw in Proposition 18, \mathcal{L}_Λ -formulas are invariant under Λ -bisimulation. Given that our modal language has only finite conjunctions, we will need to assume our coalgebra functor is finitary. This is the analogue of restricting to image-finite Kripke frames, as is done in the classic Hennessy-Milner theorem. However, there is another issue. As shown in [2, Example 1(4)], even between finite \mathcal{P} -coalgebras, it is possible for two states to fail to be Λ -bisimilar while still satisfying the same modal \mathcal{L}_Λ -formulas. We recall this example here for convenience. We are in the setting of contingency logic over Kripke frames from Example 4, i.e. $\Lambda = \{\Delta\}$. Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be two \mathcal{P} -coalgebras, where $X = \{x, x_1, x_2\}$, $\gamma(x) = \{x_1, x_2\}$, $\gamma(x_i) = \emptyset$ for $i = 1, 2$, $Y = \{y\}$ and $\delta(y) = \emptyset$. The relation $Z = \{(x, y), (x_1, x_2), (x_2, x_2)\}$ is a Λ -bisimulation on the coproduct of \mathbb{X} and \mathbb{Y} (we omit injection maps for readability). Since the coproduct injections are T -coalgebra morphisms, they are also Λ -morphisms, and hence $\mathbb{X}, x \equiv_\Lambda \mathbb{Y}, y$. However, it is not hard to show that there is no Λ -bisimulation between \mathbb{X} and \mathbb{Y} linking x and y . The solution in [2] was to define a notion of bisimilarity via the coproduct of Kripke/neighbourhood frames. We take a similar approach here.

► **Definition 26.** Two states x in \mathbb{X} and y in \mathbb{Y} are Λ_+ -bisimilar (notation: $\mathbb{X}, x \sim_{\Lambda_+} \mathbb{Y}, y$) if $\mathbb{X} + \mathbb{Y}, \text{in}_l(x) \sim_\Lambda \mathbb{X} + \mathbb{Y}, \text{in}_r(y)$.

On a single T -coalgebra, the relations \sim_Λ and \sim_{Λ_+} coincide, but in general they differ.

► **Proposition 27.** For all $x, x' \in X$ and $y \in Y$,

1. $\mathbb{X}, x \sim_\Lambda \mathbb{Y}, y$ implies $\mathbb{X}, x \sim_{\Lambda_+} \mathbb{Y}, y$. The implication is strict.
2. $\mathbb{X}, x \sim_\Lambda \mathbb{X}, x'$ iff $\mathbb{X}, x \sim_{\Lambda_+} \mathbb{X}, x'$.

Proof.

Item 1. Let $Z \subseteq X \times Y$ be a Λ -bisimulation between \mathbb{X} and \mathbb{Y} . We show that the relation

$X + Y \xleftarrow{\text{in}_l \circ \pi_l} Z \xrightarrow{\text{in}_r \circ \pi_r} X + Y$ is a Λ -bisimulation on $\mathbb{X} + \mathbb{Y} = (X + Y, \zeta)$. The proof follows from the commutativity of the diagram below in which $\heartsuit \in \Lambda$ is arbitrary. The commutativity follows from observing that $pb(Q(\text{in}_l \circ \pi_l), Q(\text{in}_r \circ \pi_r))$ with $\hat{\pi}_l \circ Q\text{in}_l$ and $\hat{\pi}_r \circ Q\text{in}_r$ is a competitor to the pullback $(pb(Q\pi_l, Q\pi_r), \bar{\pi}_l, \bar{\pi}_r)$. This yields a mediating map (dashed arrow) such that the upper part of the diagram commutes. The lower, outer parts commute due to naturality of $\llbracket \heartsuit \rrbracket$ and the inclusions being T -coalgebra morphisms.

$$\begin{array}{c}
\begin{array}{ccc}
& pb(Q(\text{in}_l \circ \pi_l), Q(\text{in}_r \circ \pi_r))^n & \\
\hat{\pi}_l^n \swarrow & \downarrow & \searrow \hat{\pi}_r^n \\
& pb(Q\pi_l, Q\pi_r)^n & \\
\bar{\pi}_l^n \swarrow & & \searrow \bar{\pi}_r^n
\end{array} \\
\begin{array}{ccccccc}
Q(X+Y)^n & \xrightarrow{(Q\text{in}_l)^n} & (QX)^n & \xrightarrow{(Q\pi_l)^n} & (QZ)^n & \xleftarrow{(Q\pi_r)^n} & (QY)^n \xleftarrow{(Q\text{in}_r)^n} Q(X+Y)^n \\
\downarrow \llbracket \heartsuit \rrbracket_{X+Y} & & \downarrow \llbracket \heartsuit \rrbracket_X & & \downarrow \llbracket \heartsuit \rrbracket_Y & & \downarrow \llbracket \heartsuit \rrbracket_{X+Y} \\
QT(X+Y) & \xrightarrow{QT\text{in}_l} & QTX & & QTY & \xleftarrow{QT\text{in}_r} & QT(X+Y) \\
\downarrow Q\zeta & & \downarrow Q\gamma & & \downarrow Q\delta & & \downarrow Q\zeta \\
Q(X+Y) & \xrightarrow{Q\text{in}_l} & QX & \xrightarrow{Q\pi_l} & QZ & \xleftarrow{Q\pi_r} & QY \xleftarrow{Q\text{in}_r} Q(X+Y)
\end{array}
\end{array}$$

Item 2. (\Rightarrow) follows from item 1. For (\Leftarrow), assume that Z is a Λ -bisimulation on $\mathbb{X} + \mathbb{X} = (X + X, \zeta)$. We show that $Z' = \{(w, w') \in X \times X \mid \exists i, j \in \{r, l\} : (\text{in}_i(w), \text{in}_j(w')) \in Z\}$ is a Λ -bisimulation on \mathbb{X} . First, note that is (U, V) if Z' -coherent, then $(U + U, V + V)$ is Z -coherent. Let $(x, x') \in Z'$, then $(\text{in}_i(x), \text{in}_j(x')) \in Z$, for some $i, j \in \{l, r\}$. Since Z is a Λ -bisimulation, it follows that:

$$\zeta(\text{in}_i(x)) \in \llbracket \heartsuit \rrbracket_{X+X}(U + U) \iff \zeta(\text{in}_j(x')) \in \llbracket \heartsuit \rrbracket_{X+X}(V + V) \quad (2)$$

To complete the proof, it remains to show that for every $U \subseteq X$

$$\gamma(x) \in \llbracket \heartsuit \rrbracket_X[U] \iff \zeta(\text{in}_i(x)) \in \llbracket \heartsuit \rrbracket_{X+X}(U + U) \quad (i = l, r) \quad (3)$$

But this follows from naturality and the fact that inclusion maps are T -coalgebra morphism. Item 2 then follows from (2) and (3). \blacktriangleleft

Due to Proposition 27(1), we define Hennessy-Milner classes of T -coalgebras with respect to $\sim_{\Lambda+}$.

► Definition 28. A class \mathbf{C} of T -coalgebras is a *Hennessy-Milner class*, if for every \mathbb{X} and \mathbb{Y} in \mathbf{C} , we have $\mathbb{X}, x \equiv_{\Lambda} \mathbb{Y}, y$ iff $\mathbb{X}, x \sim_{\Lambda+} \mathbb{Y}, y$.

As a first step towards our main result, we show that the class of finite T -coalgebras is a Hennessy-Milner class. We will use the following terminology. Given a T -coalgebra $\mathbb{X} = (X, \gamma)$, a subset $U \subseteq X$ is *modally coherent* if U is \equiv_{Λ} -closed. (Recall that \equiv_{Λ} denotes the modal equivalence relation.) The next lemma provides us with a characterisation of modally coherent sets.

► Lemma 29. Let \mathbb{X} be a finite T -coalgebra. For all $U \subseteq X$, U is modally coherent iff U is definable by a modal \mathcal{L}_{Λ} -formula.

Proof. It can be proved using the same line of argumentation as in the proof of [8, Lemma 4.5]. If $U = \llbracket \varphi \rrbracket_{\mathbb{X}}$ for some $\varphi \in \mathcal{L}_{\Lambda}$, then clearly U is modally coherent. For the converse implication, assume U is modally coherent, i.e., U is a union of modal equivalence classes: $U = \bigcup_{i \in I} [x_i]_{\equiv_{\Lambda}}$. Since X is finite, we may assume that I is finite. For $i, j \in I$ and $i \neq j$, there is a modal \mathcal{L}_{Λ} -formula $\delta_{i,j}$ such that $x_i \models \delta_{i,j}$ and $x_j \models \neg \delta_{i,j}$, so by taking $D_i = \{\delta_{i,j} \mid i, j \in I, i \neq j\}$, we have $[x_i]_{\equiv_{\Lambda}} = \bigcap_{i \in I} \llbracket D_i \rrbracket \subseteq X$. Since I is finite, D_i is finite. Defining $\delta_i = \bigwedge D_i$ for each $i \in I$, we then have $U = \bigcup_{i \in I} \llbracket \delta_i \rrbracket_{\mathbb{X}}$. Therefore, U is definable by the formula $\delta = \bigvee \delta_i$. \blacktriangleleft

Now, we have the finite version of Hennessy-Milner theorem for Λ -bisimulation.

► **Theorem 30.** *Let $\mathbb{X} = (X, \gamma)$ and $\mathbb{Y} = (Y, \delta)$ be finite T -coalgebras, and let Λ be a set of predicate liftings for T .*

1. *For all states $x, x' \in X$: $\mathbb{X}, x \equiv_{\Lambda} \mathbb{X}, x'$ iff $\mathbb{X}, x \sim_{\Lambda} \mathbb{X}, x'$.*
2. *For all states $x \in X$ and $y \in Y$: $\mathbb{X}, x \equiv_{\Lambda} \mathbb{Y}, y$ iff $\mathbb{X}, x \sim_{\Lambda+} \mathbb{Y}, y$.*

Proof.

Item 1: The direction from right to left has been shown in Proposition 18. For the other direction, we show that \equiv_{Λ} is a Λ -bisimulation. Let $x, x' \in X$ be such that $\mathbb{X}, x \equiv_{\Lambda} \mathbb{X}, x'$, and let $\heartsuit \in \Lambda$. For simplicity, we just give the argument for unary $\llbracket \heartsuit \rrbracket$. The n -ary generalisation is straightforward. Let $U \subseteq X$ be modally coherent. By Lemma 29, U is definable by a \mathcal{L}_{Λ} -formula ψ . We therefore have $x \in \llbracket \heartsuit \psi \rrbracket_{\mathbb{X}}$ iff $x' \in \llbracket \heartsuit \psi \rrbracket_{\mathbb{X}}$ because x and x' are modally equivalent. It follows that $\gamma(x) \in \llbracket \heartsuit \rrbracket_X(U)$ iff $\gamma(x') \in \llbracket \heartsuit \rrbracket_X(U)$. Hence, \equiv_{Λ} is a Λ -bisimulation on \mathbb{X} .

Item 2: Follows from item 1 and the fact that the inclusion maps preserve truth of modal formulas: $\mathbb{X}, x \sim_{\Lambda+} \mathbb{Y}, y$ iff $\mathbb{X} + \mathbb{Y}, \text{in}_l(x) \sim_{\Lambda} \mathbb{X} + \mathbb{Y}, \text{in}_r(y)$ iff $\mathbb{X} + \mathbb{Y}, \text{in}_l(x) \equiv_{\Lambda}$ iff $\mathbb{X}, x \equiv_{\Lambda} \mathbb{Y}, y$. ◀

We leverage the result for finite T -coalgebras to coalgebras for finitary functors.

► **Theorem 31 (Finitary Hennessy-Milner theorem).** *Suppose T is a finitary functor, and $\mathbb{X} = (X, \gamma)$, $\mathbb{Y} = (Y, \delta)$ are T -coalgebras.*

1. *For all states $x, x' \in X$: $\mathbb{X}, x \equiv_{\Lambda} \mathbb{X}, x'$ iff $\mathbb{X}, x \sim_{\Lambda} \mathbb{X}, x'$.*
2. *For every $x \in X$ and $y \in Y$: $\mathbb{X}, x \equiv_{\Lambda} \mathbb{Y}, y$ iff $\mathbb{X}, x \sim_{\Lambda+} \mathbb{Y}, y$.*

Proof.

Item 1: Let $x, x' \in X$ be such that $\mathbb{X}, x \equiv_{\Lambda} \mathbb{X}, x'$. By [1, Theorem 4.1] there exists a finite subcoalgebra $\mathbb{X}_0 = (X_0, \gamma_0)$ of \mathbb{X} with $x, x' \in X_0$. Since, the inclusion $\text{in}_{X_0} : X_0 \rightarrow X$ is a T -coalgebra morphism and hence preserves truth of formulas, it follows that $\mathbb{X}_0, x \equiv_{\Lambda} \mathbb{X}_0, x'$. By Theorem 30(1) we obtain $\mathbb{X}_0, x \sim_{\Lambda} \mathbb{X}_0, x'$, and from Proposition 23, using again that in_{X_0} is a T -coalgebra morphism that $\mathbb{X}, x \sim_{\Lambda} \mathbb{X}, x'$.

Item 2: can be proved using item 1 in a similar way as item 2 of Theorem 30. ◀

5 Discussion and Future Work

We have shown that our notion of Λ -bisimulation gives rise to a Hennessy-Milner theorem, and thus it fits exactly the expressiveness of the modal language. The coherence condition in the definition of Λ -bisimulation is, however, a non-local property as one would need to compute all coherent pairs over the state space in order to verify that two states are Λ -bisimilar. For concrete instances of Λ -bisimulations, it would be desirable to have a local back-and-forth style characterisation, similar to, e.g., the usual ones for Kripke frames, and the zig-zag conditions for Δ -bisimulations over Kripke frames in [6]. Such a local condition would obtain if Λ -bisimilarity could be characterised in terms of relation liftings. In the case that Λ is separating, respectively monotone, Λ -bisimilarity coincides with pre-congruences, respectively GS- Λ -bisimilarity, both of which have a relation lifting characterisation, cf. [8, 7]. We would like to investigate whether approaches such as those of [15, 18] can be used to obtain a relation lifting characterisation of Λ -bisimilarity under weaker conditions.

In [2], a Van Benthem characterisation theorem was proved for contingency logic over neighbourhood frames, that is, over neighbourhood frames, contingency logic is the fragment of first order logic which is invariant under Λ -bisimilarity, where $\Lambda = \{\Delta\}$. We would like to generalise this result and show a coalgebraic version for Λ -bisimilarity, using as

correspondence language *coalgebraic predicate logic (CPL)*, which was introduced in [16] as a first order correspondence language of coalgebraic modal logic.

We hardly explored the notion of Λ -morphisms in the present paper. It would be interesting to know which constructions are possible in the category of T -coalgebras and Λ -morphisms. For example, in [2] it was shown that for $T = \mathcal{N}$ and $\Lambda = \{\Delta\}$, one can construct Λ -quotients, i.e., quotients of T -coalgebras with respect to Λ -bisimilarity. We would like to know whether this is possible, in general. That would mean that we can minimise T -coalgebras with respect to Λ -bisimilarity. Finally, we would also like to know if a final object can be constructed from satisfied theories using techniques along the lines of [12, 20], and whether the Hennessy-Milner theorem for Λ -bisimilarity fits into the more abstract picture where a coalgebraic modal logic is obtained via a dual adjunctions, as in e.g. [11, 10].

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