Comparison of Max-Plus Automata and Joint Spectral Radius of Tropical Matrices

Laure Daviaud*[1], Pierre Guillon[2], and Glenn Merlet[3]

1 **MIMUW, University of Warsaw, Warsaw, Poland**
ldaviaud@mimuw.edu.pl
2 **Université d'Aix-Marseille CNRS, Centrale Marseille, I2M, UMR 7373, Marseille, France**
pierre.guillon@math.cnrs.fr
3 **Université d'Aix-Marseille CNRS, Centrale Marseille, I2M, UMR 7373, Marseille, France**
glenn.merlet@univ-amu.fr

Abstract

Weighted automata over the tropical semiring $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$ are closely related to finitely generated semigroups of matrices over $\mathbb{Z}_{\max}$. In this paper, we use results in automata theory to study two quantities associated with sets of matrices: the joint spectral radius and the ultimate rank. We prove that these two quantities are not computable over the tropical semiring, *i.e.* there is no algorithm that takes as input a finite set of matrices Γ and provides as output the joint spectral radius (resp. the ultimate rank) of Γ. On the other hand, we prove that the joint spectral radius is nevertheless approximable and we exhibit restricted cases in which the joint spectral radius and the ultimate rank are computable. To reach this aim, we study the problem of comparing functions computed by weighted automata over the tropical semiring. This problem is known to be undecidable, and we prove that it remains undecidable in some specific subclasses of automata.

1998 ACM Subject Classification F.4.3 Formal Languages

Keywords and phrases max-plus automata, max-plus matrices, weighted automata, tropical semiring, joint spectral radius, ultimate rank

Digital Object Identifier 10.4230/LIPIcs.MFCS.2017.19

1 Introduction

Weighted automata were introduced by Schützenberger in [25] as a quantitative extension of nondeterministic finite automata. They compute functions from the set of words over a finite alphabet to the set of values of a semiring, allowing one to model quantities such as costs, gains or probabilities. In this paper, we particularly focus on max-plus automata: automata weighted within the tropical semiring $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$. A max-plus automaton is thus a nondeterministic finite automaton whose transitions are weighted by integers. The value associated to a word w depends on the runs labelled by w: the weight of a given run is the sum of the weights of the transitions in the run, and the weight of w is the maximum of the weights of the accepting runs labelled by w. This kind of automata is

* The first author was partly supported by ANR Project ELICA ANR-14-CE25-0005, by ANR Project RECRE ANR-11-BS02-0010 and by project LIPA that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 683080).

particularly suitable to model gain maximisation, to study worst-case complexity [8] and to describe discrete event systems [12, 14]. The so-called linear presentation gives a matrix representation of such an automaton. More precisely, there is a canonical way to associate a max-plus automaton with a finitely generated semigroup of matrices over $\mathbb{Z}_{\max}$. Usually, the matrix representation is used to provide algebraic proofs of automata results. In this paper, we use results in automata theory to study two quantities related to sets of matrices: the joint spectral radius and the ultimate rank. The joint spectral radius generalises the notion of spectral radius for sets of matrices. The ultimate rank unifies some usual other notions of ranks. We link some comparison problems on max-plus automata with the computation of these two quantities. This leads to (1) proving results about comparison problems in some restricted classes of max-plus automata that we believe to be interesting for themselves and (2) applying these results to the study of the computability of the joint spectral radius and the ultimate rank.

Decidability questions about the description of functions computed by max-plus automata have been intensively studied. In his celebrated paper [20], Krob proves the undecidability of the equivalence problem for max-plus automata: there is no algorithm to decide if two max-plus automata compute the same function. In fact, his proof gives a stronger result: it is undecidable to determine whether a max-plus automaton computes a positive function. More recent approaches are based on a reduction from the halting problem of two-counter machines [1, 6]. By various reductions, this leads to the undecidability of several properties of automata with weights in different versions of the tropical semiring: $(\mathbb{N} \cup \{-\infty\}, \max, +)$, $(\mathbb{N} \cup \{+\infty\}, \min, +)$... The reader is referred to [21] for a survey on these questions.

By encoding the alphabet, and splitting each transition into several transitions with weight 1 and -1, it can be derived that the undecidability remains even if the automata are restricted to have weights within $\{-1, 1\}$ (Theorem 2). In [10], Gaubert and Katz notice that the undecidability of the comparison also remains true even if the number of states of the automata is bounded by a certain integer d. This extension is based on Krob's original proof and on the use of a universal diophantine equation. However, they ask for a more direct proof that would allow one to control the bound d. As an attempt to answer this question, we extend the proofs through two-counter machines. This allows a much sharper bound on the number of states for which comparison is undecidable (Theorem 2) than the one that would have followed from a universal diophantine equation.

The class of functions computed by max-plus automata that have all their states both initial and final is strictly included in the class of functions computed by max-plus automata. It is closely related to the study of finitely generated semigroup of tropical matrices. In this paper, we prove that comparison remains undecidable in this restricted class (Theorem 4). (In [1], it is proved that the undecidability remains for min-plus automata with all states final; we can deduce the same result for max-plus automata.)

The tropical (sub-)joint spectral radius is a natural counterpart of the usual joint spectral radius over the semiring $(\mathbb{R}, +, \times)$. Although the latter is a well-studied notion when considering the semiring $(\mathbb{R}, +, \times)$ (see [18] and the references therein) only few results are known when considering the tropical semiring. As far as we know, the best known result concerning its computability is given in [4], where it is shown that the joint spectral radius is NP-hard to compute and to approximate for tropical matrices. The tropical spectral radius as we defined it latter in this paper can be used to approximate the usual one in the spirit of [2] or [9], and from an applied point of view, for a max-plus linear system, it corresponds to the cycle time for an optimal scheduling of tasks, already studied in [12, 11, 14].

We drastically improve the NP-hardness result, by proving that the joint spectral radius is not computable in the tropical semiring, *i.e.* there is no algorithm that takes as input a finite set Γ of matrices and provides as output the joint spectral radius of Γ (Theorem 8). As a corollary of this result, we also get the uncomputability of the ultimate rank, a notion introduced – and a question raised – in [15] (Theorem 9).

On the other hand, we also give positive results. By making a link with a result in [7] about approximate comparison of max-plus automata, we prove that the joint spectral radius is approximable in EXPSPACE (Theorem 10). We also show that, when restricted to matrices with only finite rational entries, computing the joint spectral radius or the ultimate rank is PSPACE-complete (Theorem 12).

2 Definitions and first properties

We introduce definitions and notation of tropical matrices and max-plus automata.

2.1 Tropical matrices

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative binary operation '$\cdot$'. If, furthermore, the product has a neutral element 1, $(S, \cdot, 1)$ is called a *monoid*. The monoid is said to be *commutative* if $\cdot$ is commutative. A *semiring* $(S, \oplus, \otimes, 0_S, 1_S)$ is a set S equipped with two binary operations $\oplus$ and $\otimes$ such that $(S, \oplus, 0_S)$ is a commutative monoid, $(S - \{0_S\}, \otimes, 1_S)$ is a monoid, 0_S is absorbing for $\otimes$, and $\otimes$ distributes over $\oplus$. We shall use the *tropical semiring*:

$$\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

Note that $0_{\mathbb{Z}_{\max}} = -\infty$ and $1_{\mathbb{Z}_{\max}} = 0$. We may also use the restriction of $\mathbb{Z}_{\max}$ to the nonnegative integers, $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ denoted by $\mathbb{N}_{\max}$.

Semigroups of matrices. Let S be a semiring. The set of matrices with d rows and d' columns over S is denoted $\mathcal{M}^{d \times d'}(S)$, or simply $\mathcal{M}^d(S)$ if $d = d'$. The set of all matrices over S is $\mathcal{M}(S)$. As usual, the product AB for two matrices A, B (provided that the width of A and the height of B, denoted d, coincide) is defined as:

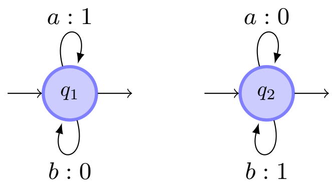
$$(AB)_{i,j} = \bigoplus_{1 \leqslant k \leqslant d} (A_{i,k} \otimes B_{k,j}) \quad \left(= \max_{1 \leqslant k \leqslant d} (A_{i,k} + B_{k,j}) \quad \text{for } S = \mathbb{Z}_{\max}\right)$$

The diagonal matrix with 1_S (*i.e.* 0 for $\mathbb{Z}_{\max}$) on the diagonal, and 0_S (*i.e.* $-\infty$ for $\mathbb{Z}_{\max}$) elsewhere is denoted I_d. It is a standard result that $(\mathcal{M}^d(S), \cdot, I_d)$ is a monoid.

For a positive integer k, we denote by M^k the product of M by itself k times. Moreover, $\|M\|_\infty$ denotes the maximal entry of a matrix M (it is not a norm). For $k \in \mathbb{Z}_{\max}$ and $A \in \mathcal{M}(\mathbb{Z}_{\max})$, $k \odot A$ is defined by $(k \odot A)_{ij} = k + A_{ij}$. For a set of matrices Γ, this notation is extended by $k \odot \Gamma = \{k \odot A | \, A \in \Gamma\}$. Finally, if $\Gamma \subset \mathcal{M}^d(S)$, we denote by $\langle \Gamma \rangle$ the submonoid generated by Γ.

Graph of a matrix. Any square matrix $A \in \mathcal{M}^d(\mathbb{Z}_{\max})$, for a positive integer d, can be represented by a graph $\mathcal{G}(A)$: the vertices are the indices $1, \ldots, d$, and there is an edge from i to j, labelled $A_{i,j}$, if and only if the latter is finite. The *spectral radius* $\rho(A)$ of a square matrix $A \in \mathcal{M}^d(\mathbb{Z}_{\max})$, for some positive integer d, known to be the limit $\lim_{n \to +\infty} \frac{1}{n} \|A^n\|_\infty$, can be seen as the maximal average weight of the cycles in $\mathcal{G}(A)$:

$$\rho(A) = \sup_{\substack{\ell \in \mathbb{N} \setminus \{0\} \\ 1 \leqslant i_1, \ldots, i_\ell \leqslant d}} \left(\frac{1}{\ell} \left(A_{i_1,i_2} + A_{i_2,i_3} + \ldots + A_{i_{\ell-1},i_\ell} + A_{i_\ell,i_1} \right) \right)$$



$$\mu(a) = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & -\infty \\ -\infty & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Figure 1 Graph and matrix representations of a max-plus automaton.

The *critical graph* $\mathcal{G}_c(A)$ is the union of cycles $(i_1, \ldots, i_\ell)$ that achieve this maximum. Its *strongly connected components* are the maximal subgraphs $C \subseteq \mathcal{G}_c(A)$ such that for any vertices i, j of C there is a path from i to j in $\mathcal{G}_c(A)$. The *cyclicity of a strongly connected component* is the greatest common divisor of the length of its cycles. The *cyclicity of* $\mathcal{G}_c(A)$ is the least common multiple of the cyclicities of its strongly connected components. The reader is referred to [3, 16, 5] for more detailed explanations.

2.2 Max-plus automata

We give the definition of max-plus automata that can be viewed as graphs or as sets of matrices. A *max-plus automaton* $\mathcal{A}$ over the alphabet Σ with d states is a map μ from Σ to $\mathcal{M}^d(\mathbb{Z}_{\max})$ together with an initial vector $I \in \mathcal{M}^{1\times d}(\{0, -\infty\})$ and a final vector $F \in \mathcal{M}^{d\times 1}(\{0, -\infty\})$[1]. The map μ is uniquely extended into a morphism, also denoted μ, from the semigroup Σ^+ of nonempty finite words over alphabet Σ into $\mathcal{M}^d(\mathbb{Z}_{\max})$. The *function computed by the automaton*, $[\![\mathcal{A}]\!]$, maps each word $w \in \Sigma^+$ to $I\mu(w)F \in \mathbb{Z}_{\max}$. Sometimes, 0 will denote the function constantly equal to 0, and $\geqslant$ the induced partial order over functions $\Sigma^+ \to \mathbb{Z}_{\max}$ (so that we can write things like $[\![\mathcal{A}]\!] \geqslant 0$).

Another way to represent a max-plus automaton is in terms of graphs. Given a map μ from Σ^+ to $\mathcal{M}^d(\mathbb{Z}_{\max})$, the corresponding automaton has d states $q_1, \ldots q_d$, that correspond to the lines, or to the columns of the matrices. There is a transition from q_i to q_j labelled by a letter $a \in \Sigma$, with weight $\mu(a)_{i,j}$, if and only if the latter is finite. The initial (resp. final) states are the states q_i such that $I_i = 0$ (resp. $F_i = 0$). A run over the word w is a path (a sequence of compatible transitions) in the graph, labelled by w. Its weight is the sum of the weights of the transitions. The weight of a given word w is the maximum of the weights of the accepting runs (runs going from an initial state to a final state) labelled by w. The weight of w, given by the graph representation, is exactly the value $I\mu(w)F$, given by the matrix presentation.

Given a positive integer d and a max-plus automaton $\mathcal{A}$ defined by some map $\mu : \Sigma \to \mathcal{M}^d(\mathbb{Z}_{\max})$, we denote $\Gamma_{\mathcal{A}} = \{\mu(a) \,|\, a \in \Sigma\}$. Then the set of weights on the transitions of $\mathcal{A}$ corresponds to the finite entries appearing in matrices of $\Gamma_{\mathcal{A}}$.

▶ **Example 1.** Figure 1 gives the matrix and graph presentations of a max-plus automaton with 2 states, both initial (ingoing arrow) and final (outgoing arrow), over the alphabet $\{a, b\}$. The function that it computes associates a word w to the value $\max(|w|_a, |w|_b)$ where $|w|_x$ denotes the number of occurrences of the letter $x \in \{a, b\}$ in w.

This work aims to link results in automata theory with the study of semigroups of matrices. Concepts defined over semigroups of matrices correspond to concepts over the

[1] Note that, unlike variants in the literature, our weighted automata have no input or output weight (that is, I and F have entries in $\{0, -\infty\}$), but this does not restrict the set of computed functions.

subclass of automata in which all states are both initial and final, because if $M \in \mathcal{M}^d(\mathbb{Z}_{\max})$, and I and F have only 0 entries, then $IMF = \|M\|_\infty$, so that for this class of automata, $[\![\mathcal{A}]\!](w) = \|\mu(w)\|_\infty$ for every word w.

3 Undecidability of the comparison of max-plus automata

We are interested in the comparison problem, *i.e.* deciding, given two max-plus automata $\mathcal{A}$ and $\mathcal{B}$, whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$. There exist (at least) two different approaches to prove the undecidability of this problem. The original one by Krob [20] is a reduction from the tenth problem of Hilbert about diophantine equations. The proof is nicely written in [21], where it encodes a homogeneous polynomial P of degree 4 on n variables with integer coefficients into a max-plus automaton $\mathcal{A}$ computing a function with values in $\mathbb{N}$, such that $P-1$ has a root in $\mathbb{N}^n$ if and only if there is a word w such that $[\![\mathcal{A}]\!](w) = 0$, *i.e.* if $[\![\mathcal{A}]\!] \geq 1$. A more recent approach [1, 6], consists of a reduction from the halting problem of a two-counter machine. This computational model was introduced by Minsky [23, 24], and is as powerful as a Turing machine. It can be viewed as a finite-state machine with two counters. The equality of the counters with 0 can be tested and the counters can be incremented and decremented if not 0. The idea of the proof is to embed it into a max-plus automaton.

3.1 Restriction on the parameters

Different parameters can be taken into account when dealing with the size of a max-plus automaton: we will focus on the number of states, the maximal and minimal weights appearing on the transitions and the size of the alphabet. When considering the matrix representation of an automaton $\mathcal{A}$, these parameters correspond respectively to the dimension, the maximal and minimal finite entries and the number of matrices in $\Gamma_{\mathcal{A}}$.

Regarding the size of the alphabet, by a classical encoding from an arbitrary alphabet to a two-letter alphabet, one can see that the comparison problem remains undecidable when restricting to the class of automata on the binary alphabet.

Regarding the two other parameters, if they are both bounded, the problem becomes decidable since we are now only considering a finite number of max-plus automata. What is more interesting to study is when one of the parameter is bounded and not the other. We will see that this problem remains undecidable in these cases, and our purpose is to give bounds on these parameters that allow to keep the undecidability.

On the one hand, Gaubert and Katz notice in [10] that the original proof of Krob, applied to some specific diophantine equations gives that the problem remains undecidable when bounding the number of states. They also raised the question of finding an alternative proof that could allow to control this number of states. We roughly counted how many states we would obtain by using a so-called universal diophantine equation given in [17], of degree 4 with 58 unknowns. At the very least, we would be able to bound the number of states by 8700 on a 6-letter alphabet. On the other hand, the proof via two-counter machines allows to drastically improve this number, as we are going to see.

Define $\texttt{Pos}^k(S)$ (resp. $\texttt{Pos}^k_d(S)$) as the following problem: *Given a max-plus automaton $\mathcal{A}$ on a k-letter alphabet, with weights in $S \subseteq \mathbb{Z}_{\max}$ (resp. and d states), determine whether $[\![\mathcal{A}]\!] \geqslant 0$.*

▶ **Theorem 2.** *Problems $\texttt{Pos}^2(\{-1,1\})$ and $\texttt{Pos}^6_{553}(\mathbb{Z}_{\max})$ are undecidable.*

The first statement is easily derived from the general undecidability: given a max-plus automaton $\mathcal{A}$, one can first decompose all the transitions from $\mathcal{A}$ into a sequence of transitions with weights 1 or -1 and 0 thanks to a suitable encoding of the alphabet, such that $\mathcal{A}$ computes a non negative function if and only if the newly created automaton computes a non negative function. Using then a usual encoding of any alphabet in a two-letter alphabet, we can obtain an automaton over two letters and with weights still in $\{-1, 0, 1\}$. Then, we transform again the automaton by multiplying all the weights by 2, so that a transition weighted by 1 (resp. -1) is now weighted by 2 (resp. -2). Finally, we decompose a transition labelled by a letter a with weight 2 (resp. -2) into two transitions each labelled by a with weight 1 (resp. -1), and a transition labelled by a with weight 0 into two transitions each labelled by a, one with weight 1 and the other with weight -1. For example, the word ab^2a is now encoded by $a^2b^4a^2$. All the words that are not composed with square of the letters (which form a rational language) are given weight $\geqslant 0$ (by using only weights in $\{1, -1\}$). It is easy to see that the obtained automaton computes a nonnegative function if and only if $[\![\mathcal{A}]\!] \geqslant 0$. Moreover the obtained automaton is on a two letter alphabet with weights within $\{-1, 1\}$.

The undecidability of $\texttt{Pos}^6_{553}(\mathbb{Z}_{\max})$ is a contribution of the present paper. The halting problem is undecidable for a universal two-counter machine on any input (the initial value of the first counter). We construct a max-plus automaton simulating the runs of a two-counter machine where the input $n \in \mathbb{N}$ is now encoded by an additional widget involving two edges with weights n and $-n$. Thus we construct a max-plus automaton with a fixed number of states (depending only on the number of states of the universal two-counter machine) but arbitrary weights (induced by the value of the input of the universal two-counter machine).

3.2 Restriction on initial and final states

It is easy to see that max-plus automata having all their states initial and final compute only subadditive functions, that is to say functions f such that for any two words u and v, $f(uv) \leqslant f(u) + f(v)$. In particular, the support of such a function is closed by taking factors (if uvw is in the support then v is also in the support). Thus, as for unweighted automata [19], this subclass of automata defines a strict subclass of functions of the functions computed by max-plus automata. However, the following lemma shows that the comparison problem remains undecidable within this subclass.

▶ **Lemma 3.** *Let Σ be a finite alphabet and $\star \notin \Sigma$ be a special symbol. Given a max-plus automaton $\mathcal{A}$ on Σ, with d states and weights within a set S, one can build a max-plus automaton $\mathcal{A}'$ on $\Sigma' = \Sigma \cup \{\star\}$, with $d+1$ states, all of which are initial and final, and weights within $S \cup \{0\}$, such that:*

$$\min\left(\inf_{u \in \Sigma^+} \frac{[\![\mathcal{A}]\!](u)}{|u|}, 0\right) \leq \inf_{w \in \Sigma'^+} \frac{[\![\mathcal{A}']\!](w)}{|w|} \quad \textit{and} \quad \inf_{u \in \Sigma'^+} \frac{[\![\mathcal{A}']\!](u)}{|u|} \leq \inf_{w \in \Sigma^+} \frac{[\![\mathcal{A}]\!](w)}{|w|+1}$$

In particular, $[\![\mathcal{A}]\!] \geqslant 0$ if and only if $[\![\mathcal{A}']\!] \geqslant 0$.

Proof. Consider a max-plus automaton $\mathcal{A}$ defined by a map $\mu : \Sigma \to \mathcal{M}^d(\mathbb{Z}_{\max})$, an initial vector I and a final vector F, and a new symbol $\star$. Let $\Sigma' = \Sigma \cup \{\star\}$. The idea is to construct a new automaton $\mathcal{A}'$ by adding a new state q and transitions from every final state of $\mathcal{A}$ to every initial state of $\mathcal{A}$ as well as transitions from every final state of $\mathcal{A}$ to q, loops around q and transitions from q to every initial state of $\mathcal{A}$, all labelled by $\star$ with weight 0. All the states of the new automaton $\mathcal{A}'$ are initial and final. Let us note μ', I' and F' defining this new automaton.

Any word $w \in \Sigma'^{+} \backslash \{\star\}^*$ can be written: $w = \star^{n_0} w_1 \star^{n_1} w_2 \star^{n_2} \dots w_k \star^{n_k}$ where for all $1 \leqslant i \leqslant k$, $w_i \in \Sigma^+$ and for all $0 < i < k$, $n_i > 0$, $n_0 \geqslant 0$ and $n_k \geqslant 0$. We get:

$$\llbracket \mathcal{A}' \rrbracket (w) = \|\mu'(w)\|_\infty \geq \sum_{i=1}^{k} I\mu(w_i)F$$

since the weight of $\star$ is 0. This is at least $\sum_{i=1}^{k} |w_i| \inf_u \frac{\llbracket \mathcal{A} \rrbracket (u)}{|u|}$. If $\llbracket \mathcal{A} \rrbracket \geq 0$, then we get $\llbracket \mathcal{A}' \rrbracket (w) \geq 0$. Otherwise, $\inf_u \frac{\llbracket \mathcal{A} \rrbracket (u)}{|u|} < 0$, and since $\sum_i |w_i| \leq |w|$, we get $\frac{\llbracket \mathcal{A}' \rrbracket (w)}{|w|} \geq \inf_u \frac{\llbracket \mathcal{A} \rrbracket (u)}{|u|}$. Moreover, since the weights of the words in $\{\star\}^*$ is 0 in $\mathcal{A}'$, then the first inequality holds.

The other inequality is obtained by observing how arcs labelled $\star$ are positioned in $\mathcal{A}'$. Indeed, if a transition labelled by $\star$ is taken, then it has to start from a final state or q, and has to end in an initial state or q. Moreover, no other letter labels a transition starting or ending in q. So, when reading a word $w \in \Sigma^+$ between two $\star$, this word is read on a run that was already an existing accepting run in $\mathcal{A}$.

Thus, we see that for all words $w \in \Sigma^+$ and all $k \in \mathbb{N}$: $\llbracket \mathcal{A}' \rrbracket ((\star w)^k \star) = k \llbracket \mathcal{A} \rrbracket (w)$, so that:

$$\inf_{u \in \Sigma'^{+}} \frac{\llbracket \mathcal{A}' \rrbracket (u)}{|u|} \leq \inf_{\substack{w \in \Sigma^+ \\ k \in \mathbb{N}}} \frac{\llbracket \mathcal{A}' \rrbracket ((\star w)^k \star)}{k(|w|+1)+1} \leq \inf_{\substack{w \in \Sigma^+ \\ k \in \mathbb{N}}} \frac{\llbracket \mathcal{A} \rrbracket (w)}{|w|+1+1/k} . \qquad ◀$$

As a corollary of this lemma and of the previous results on the undecidability of comparison, we get the following theorem.

▶ **Theorem 4.** *The restrictions of Problems* $\mathtt{Pos}^3(\{-1,0,1\})$ *and* $\mathtt{Pos}^7_{554}(\mathbb{Z}_{\max})$ *to automata whose states are all initial and final are still undecidable.*

Simultaneously to the latter construction, one can also encode the alphabet into the binary alphabet in a smart way, yielding to the following result:

▶ **Theorem 5.** *The restrictions of Problems* $\mathtt{Pos}^2(\{-1,0,1\})$ *and* $\mathtt{Pos}^2_{3319}(\mathbb{Z}_{\max})$ *to automata whose states are all initial and final are still undecidable.*

4 Joint spectral radius and ultimate rank of tropical matrices

4.1 Joint spectral radius

The definition of spectral radius extends to the *joint spectral radius* of a set $\Gamma \subseteq \mathcal{M}^d(\mathbb{Z}_{\max})$ of matrices, as follows:

$$\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \|M_1 \cdots M_\ell\|_\infty \;\middle|\; M_1, \dots, M_\ell \in \Gamma \right\}$$

The following lemma, which gives other equivalent definitions[2], is a known application of Fekete's subadditive lemma (see for example [11, Theorem 3.4]).

[2] Note that here we use the inf definition for the joint spectral radius instead of the sup definition used in the literature (see [18] and references therein). The latter is easy to compute in $\mathbb{Z}_{\max}$ since it is the spectral radius of the generators' tropical sum, unlike the notion considered here (sometimes called lower spectral radius or joint spectral subradius).

▶ **Lemma 6.** *For any set* Γ *of matrices in* $\mathcal{M}^d(\mathbb{Z}_{\max})$*, we have:*

$$\rho(\Gamma) = \lim_{\ell\to\infty} \min \left\{ \frac{1}{\ell} \left\| M_1 \cdots M_\ell \right\|_\infty \middle| M_1, \ldots, M_\ell \in \Gamma \right\} \tag{1}$$

$$= \inf_{\ell>0} \left\{ \frac{1}{\ell} \rho(M_1 \cdots M_\ell) \middle| M_1, \ldots, M_\ell \in \Gamma \right\} \tag{2}$$

Proof. Let $u_\ell = \inf \{ \|M_1 \cdots M_\ell\|_\infty \mid M_1, \ldots, M_\ell \in \Gamma\}$. The sequence $(u_\ell)_\ell$ is subadditive *i.e.* for all ℓ, ℓ', $u_{\ell+\ell'} \leqslant u_\ell + u_{\ell'}$. Indeed for all $M_1, \ldots, M_{\ell+\ell'} \in \Gamma$, $\|M_1 \cdots M_{\ell+\ell'}\|_\infty \leqslant \|M_1 \cdots M_\ell\|_\infty + \|M_{\ell+1} \cdots M_{\ell+\ell'}\|_\infty$. Thus by Fekete's lemma, $\lim_{\ell\to\infty} \frac{u_\ell}{\ell}$ is well defined and $\inf_{\ell>0} \frac{u_\ell}{\ell} = \lim_{\ell\to\infty} \frac{u_\ell}{\ell}$, which implies (1).

For the second equality, let us set: $\rho'(\Gamma) = \inf_{\ell>0} \left\{ \frac{1}{\ell}\rho(M_1 \cdots M_\ell) \middle| M_1, \ldots, M_\ell \in \Gamma \right\}$. Since for all matrices M, $\rho(M) \leqslant \|M\|_\infty$, we have $\rho'(\Gamma) \leqslant \rho(\Gamma)$. Let us show the reverse inequality. For all $\varepsilon > 0$, there is $\ell > 0$ and $M_1, \ldots, M_\ell \in \Gamma$ such that $\frac{1}{\ell}\rho(M_1 \cdots M_\ell) \leqslant \rho'(\Gamma) + \varepsilon$. By definition, it means that $\frac{1}{\ell} \lim_n \frac{1}{n} \|(M_1 \cdots M_\ell)^n\|_\infty \leqslant \rho'(\Gamma) + \varepsilon$, or equivalently, $\lim_n \frac{1}{n\ell} \|(M_1 \cdots M_\ell)^n\|_\infty \leqslant \rho'(\Gamma) + \varepsilon$. By definition, $\rho(\Gamma) \leqslant \lim_n \frac{1}{n\ell} \|(M_1 \cdots M_\ell)^n\|_\infty$, thus, for all $\varepsilon > 0$, $\rho(\Gamma) \leqslant \rho'(\Gamma) + \varepsilon$, that concludes the proof. ◀

It can be easily seen that $\rho(k \odot \Gamma) = \rho(\Gamma) + k$.

4.2 Ultimate rank

In the classical setting of a field, the notion of rank enjoys many equivalent definitions. These notions do not coincide in the case of $\mathbb{Z}_{\max}$. However, it was noticed in [15] that they coincide on the limit points of the powers of the matrix, when properly normalized (or considered projectively). This is formalized in [15, Theorem 5.2], and equivalent to the following definition: the *ultimate rank* $\mathrm{urk}(M)$ of a matrix $M \in \mathcal{M}^d(\mathbb{Z}_{\max})$ is the sum of the cyclicities of the strongly connected components of its critical graph. Clearly, $\mathrm{urk}(M) = 0$ (M has empty critical graph) if and only if $\rho(M) = -\infty$, and this corresponds to the nilpotency of M. As for the joint spectral radius, this notion can be generalized to sets of matrices. The *ultimate rank* of a set $\Gamma \subseteq \mathcal{M}^d(\mathbb{Z}_{\max})$ of matrices is: $\mathrm{urk}(\Gamma) = \min \{ \mathrm{urk}(M) | \, M \in \langle\Gamma\rangle\}$.

Clearly, $\mathrm{urk}(\Gamma) = 0$ if and only if $\rho(\Gamma) = -\infty$, and this corresponds to the mortality of the semigroup generated by Γ. It can be seen (or read in [15, Theorem 5.2]) that the ultimate rank is a projective notion: $\mathrm{urk}(k \odot \Gamma) = \mathrm{urk}(\Gamma)$ for any $k \in \mathbb{Z}$.

In some interesting cases, $\mathrm{urk}(\Gamma)$ is indeed the reached minimum of the ranks in the semigroup, so that it is the dimension of the limit set of the action of Γ on $\mathbb{R}^d$. Those cases include sets with irreducible fixed structure (all matrices have the same infinite entries), and sets of matrices with no line of $-\infty$ that contain one matrix with only finite entries. This is implicitely used in [22] and allows to extend some nice properties of products of random matrices from matrices with the so-called memory-loss property (case $\mathrm{urk}(\Gamma) = 1$) to more general ones (see [22, Corollary 1.2]).

4.3 Uncomputability and link with automata

Finitely generated semigroups of matrices exactly correspond to max-plus automata that have all their states initial and final. In particular, the following lemma links the computation of the joint spectral radius of the former to the comparison of the latter.

▶ **Lemma 7.** *Let* $\mathcal{A}$ *be a max-plus automaton over an alphabet* Σ *whose all states are both initial and final. The following statements are equivalent.*

1. $[\![\mathcal{A}]\!] \geqslant 0$.
2. *For all matrices* M *in* $\langle\Gamma_{\mathcal{A}}\rangle$, $\|M\|_\infty \geqslant 0$.

3. *For all matrices M in $\langle\Gamma_{\mathcal{A}}\rangle$, $\rho(M) \geqslant 0$.*
4. *$\rho(\Gamma_{\mathcal{A}}) \geqslant 0$.*

According to the terminology in [1], this also corresponds to the case when $\mathcal{A}$ is called *universal with threshold* 0.

Proof. Items 1. and 2. are equivalent since all the states of $\mathcal{A}$ are both initial and final. Thus, for all words w, $[\![\mathcal{A}]\!](w) = \|\mu(w)\|_\infty$. Moreover, $\langle\Gamma_{\mathcal{A}}\rangle$ is exactly the set $\{\mu(w) | w \in \Sigma^+\}$. Items 2. and 3. are equivalent by definition of the joint spectral radius. Finally, Items 3. and 4. are equivalent by Lemma 6. ◀

The uncomputability of the joint spectral radius is deduced from the equivalence in Lemma 7 and from Theorem 4 and 5. More precisely, define $\mathtt{JSR}^k(S)$ (resp. $\mathtt{JSR}^k_d(S)$) as the following problem: *Given a finite set of k matrices with coefficients in $S \subseteq \mathbb{Z}_{\max}$ (resp. and dimension d), determine whether their joint spectral radius is greater than or equal to* 0.

▶ **Theorem 8.** *Problems $\mathtt{JSR}^2(\{-\infty,-1,0,1\})$ and $\mathtt{JSR}^7_{554}(\mathbb{Z}_{\max})$ are undecidable.*

Proof. The undecidability comes from a reduction from the problem stated in Theorem 4 and 5. Consider a max-plus automaton $\mathcal{A}$ whose states are all initial and final. By Lemma 7, $[\![\mathcal{A}]\!] \geqslant 0$ if and only if $\rho(\Gamma_{\mathcal{A}}) \geqslant 0$. Thus $\mathtt{JSR}^2(\{-\infty,-1,0,1\})$ and $\mathtt{JSR}^7_{554}(\mathbb{Z}_{\max})$ are undecidable. ◀

By reduction from Theorem 8, we prove that the ultimate rank is also uncomputable. Define $\mathtt{UR}^k(S)$ (resp. $\mathtt{UR}^k_d(S)$) as the following problem: *Given a finite set of k matrices with coefficients in S (resp. and dimension d), determine whether the ultimate rank of the semigroup that they generate is equal to* 1.

▶ **Theorem 9.** *Problems $\mathtt{UR}^2(\{-\infty,-1,0,1\})$ and $\mathtt{UR}^7_{1109}(\mathbb{Z}_{\max})$ are undecidable.*

Proof. From any matrix M, one can build: $\widehat{M} = \begin{bmatrix} M & -\infty & -\infty \\ -\infty & M & -\infty \\ -\infty & -\infty & 0 \end{bmatrix}$. It is then clear that, for any finite family of matrices Γ, the semigroup generated by $\widehat{\Gamma} = \left\{ \widehat{M} \middle| M \in \Gamma \right\}$ is $\left\langle \widehat{\Gamma} \right\rangle = \left\{ \widehat{M} \middle| M \in \langle\Gamma\rangle \right\}$.

If M has size d and entries in S, then $\widehat{M}$ has size $2d+1$ and entries in $S \cup \{-\infty, 0\}$. Moreover, if $\rho(M) < 0$, then the critical graph of $\widehat{M}$ is simply the loop over the last vertex (last line of the matrix $\widehat{M}$), so that $\mathrm{urk}(\widehat{M}) = 1$. Otherwise, the critical graph of $\widehat{M}$ contains at least two copies of that of M (which is nonempty), so that $\mathrm{urk}(\widehat{M}) \geq 2$. Thus, $\rho(M) \geqslant 0$ if and only if $\mathrm{urk}(\widehat{M}) \geq 2$. By reduction from the undecidable problems of Theorem 8, we can deduce that $\mathtt{UR}^2(\{-\infty,-1,0,1\})$ and $\mathtt{UR}^7_{1109}(\mathbb{Z}_{\max})$ are undecidable. ◀

▶ Remark. As noted above, the joint spectral radius and ultimate rank are not altered through translation by a constant; thus uncomputability is preserved with other restrictions over the entries. Regarding the joint spectral radius, the comparison to 0 may no longer be undecidable, but the comparison to some other constants will be. For instance, if $\Gamma \subset \mathcal{M}(\mathbb{N}_{\max})$, positivity is always true, but the question whether $\rho(\Gamma) \geqslant 1$ is undecidable.

4.4 Approximation of the joint spectral radius

Still by using results in automata theory, we prove that even though the joint spectral radius is not computable in general, it is approximable and computable in restricted cases in the following sense.

▶ **Theorem 10.** *There is an algorithm that, given a finite set* Γ *of matrices and* $n \in \mathbb{N} \setminus \{0\}$*, computes a value* $\alpha \in \mathbb{Q} \cup \{-\infty\}$ *such that* $\alpha - \frac{1}{n} \leqslant \rho(\Gamma) \leqslant \alpha + \frac{1}{n}$*.*

Proof. The proof uses the main result of [7]. This result is originally stated for min-plus automata using only positive weights. These automata are defined over the *min-plus semiring* $(\mathbb{Z} \cup \{+\infty\}, \min, +, +\infty, 0)$. By using the morphism from the min-plus to the max-plus semiring that associates k to $-k$, we can state the result of [7] in the max-plus case: there is an algorithm $\mathfrak{A}$ that, given a max-plus automaton $\mathcal{A}$ over an alphabet Σ using only nonpositive weights and $n \in \mathbb{N} \setminus \{0\}$, computes a value $\alpha \in \mathbb{Q} \cup \{+\infty\}$ such that: $\alpha - \frac{1}{n} \leqslant \inf_{w \in \Sigma^+} \frac{[\![\mathcal{A}]\!](w)}{|w|} \leqslant \alpha + \frac{1}{n}$.

Now, let us exhibit an algorithm that gives an approximation of the joint spectral radius of any finite set of matrices with only nonpositive entries. Consider a finite set of matrices Γ with only nonpositive entries, and a max-plus automaton $\mathcal{A}$ such that $\Gamma = \Gamma_{\mathcal{A}}$. From [7], $\mathfrak{A}$ also gives an approximation of the joint spectral radius of Γ, since we have:

$$\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \|M_1 \cdots M_\ell\|_\infty \,\middle|\, M_1, \ldots, M_\ell \in \Gamma \right\} = \inf_{\ell > 0} \left\{ \frac{1}{\ell} [\![\mathcal{A}]\!](w) \,\middle|\, w \in \Sigma^\ell \right\}$$
$$= \inf_{w \in \Sigma^+} \frac{[\![\mathcal{A}]\!](w)}{|w|} \,.$$

Consider now a finite set of matrices Γ with arbitrary entries. Let k denote the greatest entry that appears in at least one of the matrices of Γ. Construct the set $\Gamma' = -k \odot \Gamma$. The set Γ' is then a finite set of matrices with only nonpositive entries, on which we can apply $\mathfrak{A}$. We then get an approximation of the joint spectral radius of Γ by adding k to the value given by the algorithm. ◀

This implies, in particular, that the joint spectral radius of every finite set of matrices is a computable real number.

Remarks about the complexity. The algorithm of [7] is EXPSPACE in the size of the automaton and in n. Moreover the problem is PSPACE-hard by reduction from the universality problem of a nondeterministic automaton: *Given a nondeterministic finite automaton* $\mathcal{A}$ *over a* 2*-letter alphabet* Σ*, the problem to determine whether the language accepted by* $\mathcal{A}$ *is* Σ^+ *is PSPACE-complete.* A precise statement of the reduction is given in the following lemma.

▶ **Lemma 11.** *Given a nondeterministic finite automaton* $\mathcal{A}$ *over a* 2*-letter alphabet* Σ*, one can construct in polynomial time a set of* 3 *matrices* Γ *with entries in* $\{-\infty, 0\}$ *such that* $\mathcal{A}$ *accepts* Σ^+ *if and only if the joint spectral radius of* Γ *is equal to* 0*. Otherwise, the joint spectral radius of* Γ *is equal to* $-\infty$*.*

Proof. Consider a nondeterministic finite automaton $\mathcal{A}$ over a 2-letter alphabet Σ. We construct a max-plus automaton $\mathcal{A}'$ from $\mathcal{A}$ by weighting the transitions by 0. Then, $\mathcal{A}$ accepts Σ^+ if and only if $[\![\mathcal{A}']\!] = 0$ (otherwise there is a word w such that $[\![\mathcal{A}']\!](w) = -\infty$). By Lemma 3, one can construct a max-plus automaton $\mathcal{B}$ over a 3-letter alphabet such that every state of $\mathcal{B}$ is both initial and final, $\mathcal{B}$ has only weight 0 on its transitions, and $[\![\mathcal{A}']\!] \geqslant 0$ if and only if $[\![\mathcal{B}]\!] \geqslant 0$. Hence, $[\![\mathcal{A}']\!] = 0$ if and only if $[\![\mathcal{B}]\!] = 0$. By Lemma 7, $[\![\mathcal{B}]\!] \geqslant 0$ if and only if the joint spectral radius of $\Gamma_{\mathcal{B}}$ is nonnegative. Since, $\Gamma_{\mathcal{B}}$ contains only matrices with entries in $\{0, -\infty\}$, it implies that $[\![\mathcal{B}]\!] = 0$ if and only if the joint spectral radius of $\Gamma_{\mathcal{B}}$ is equal to 0. All the constructions are polynomial. ◀

Notice that Lemma 11 also proves that $\texttt{JSR}^3(\{0, -\infty\})$ is PSPACE-hard. A result in [1] implies that $\texttt{JSR}^k(\mathbb{Z}^- \cup \{-\infty\})$ is also PSPACE, where $\mathbb{Z}^-$ denotes the set of nonpositive integers. Hence, Problem $\texttt{JSR}^3(\{0, -\infty\})$ is also PSPACE-complete.

4.5 Restriction to finite entries

Let us consider the restriction to matrices with only finite entries. In terms of automata, it means that for all letters a, there is a transition labelled by a between any pair of states. In this case, the joint spectral radius and ultimate rank are computable.

▶ **Theorem 12.** *There are PSPACE algorithms to compute the joint spectral radius and the ultimate rank of any finite set of matrices with finite entries. In particular, in this case, the joint spectral radius is a rational number. Moreover, $\texttt{JSR}^3(\{0, -1\})$ and $\texttt{UR}^3(\{0, -1\})$ are PSPACE-complete.*

PSPACE algorithm. To prove that the problems are PSPACE, the key point is the following lemma:

▶ **Lemma 13** ([13]). *Let $\Gamma \subset \mathcal{M}^d(\{-b, \dots, b\})$ for some nonnegative integers b and d. Then for all matrices $M \in \langle\Gamma\rangle$ and all indices i, j, the quantity $M_{i,j} - M_{1,1}$ belongs to $\{-2b, \dots, 2b\}$.*

Let Γ be a finite set of matrices with entries in $\{-b, \dots, b\}$. By Lemma 13, the set $\Lambda = \{-M_{1,1} \odot M | \, M \in \langle\Gamma\rangle\}$ contains at most $(4b + 1)^{d^2-1}$ matrices.

Moreover, since the operation of adding the same constant to all the entries of a matrix commutes with the product of matrices, Λ is the set of matrices $-M_{1,1} \odot M$ such that M is a product of at most $(4b + 1)^{d^2-1}$ matrices of Γ. Finally, the ultimate rank of Γ is the minimum of the ultimate rank of the matrices in Λ, which can be computed by the following algorithm in NPSPACE. Start with a matrix $M = M_1 \in \Gamma$ and a counter ℓ with value 1. At each (nondeterministic) step, either compute $\mathrm{urk}(M)$ and stop, or increase ℓ by one and multiply M by some matrix $M_\ell \in \Gamma$. If $\ell = (4b + 1)^{d^2-1}$, then compute $\mathrm{urk}(M)$ and stop.

Since the maximum value of ℓ is simply exponential in the size $|\Gamma|d^2 \log(b)$ of the input, both ℓ and the size of the entries of $M = M_1 \cdots M_\ell$ are simply exponential and thus can be stored in polynomial space. Since the product of matrices and ultimate rank of one matrix can be computed in P the algorithm is in NPSPACE=PSPACE.

For the joint spectral radius, let us prove that

$$\rho(\Gamma) = \min_{\substack{\ell \leqslant (4b+1)^d \\ M_1, \dots, M_\ell \in \Gamma}} \left\{ \frac{1}{\ell} \rho(M_1 \cdots M_\ell) \right\} \tag{3}$$

and conclude in the same way. Let us consider a product $M_1 \cdots M_\ell$ of matrices in Γ and the orbit of the vector with all entries equal to 0 under the action of $M_1, M_2, \dots, M_\ell, M_1, M_2, \dots$. By Lemma 13, this orbit projectively has size at most $(4b + 1)^d$. Hence, it cycles after t steps for some $t \leqslant (4b + 1)^d$ and has a period $p \leqslant (4b + 1)^d$. Each time the orbit goes back to the same vector projectively, all coordinates have increased by some value, which is the spectral radius of $M_{(t+1) \bmod \ell} M_{(t+2) \bmod \ell} \cdots M_{(t+p) \bmod \ell}$. Indeed, for matrices with only finite entries, the spectral radius is the only eigenvalue. Finally, we get $\frac{1}{\ell}\rho(M_1 \cdots M_\ell) = \frac{1}{p}\rho(M_{(t+1) \bmod \ell} M_{(t+2) \bmod \ell} \cdots M_{(t+p) \bmod \ell})$, which proves (3).

PSPACE-hardness. Let Γ be a finite set of matrices with entries in $\{0, -\infty\}$. Let Γ' be the set Γ where every entry with value $-\infty$ has been replaced by -1. The joint spectral radius of Γ is equal to 0 if and only if the joint spectral radius of Γ' is equal to 0. Otherwise, $\rho(\Gamma) = -\infty$ and $\rho(\Gamma')$ is strictly negative. By Lemma 11, $\texttt{JSR}^3(\{0, -\infty\})$ is PSPACE-hard, and thus $\texttt{JSR}^3(\{0, -1\})$ is PSPACE-hard.

Now, let us reduce $\texttt{JSR}^3([\{0, -1\})$ to $\texttt{UR}^3(\{0, -1\})$. From any matrix $M \in \mathcal{M}^d(\{0, -1\})$, with $d \in \mathbb{N} \setminus \{0\}$, one can build the matrix: $\widetilde{M} = \begin{bmatrix} M & (-1) \\ (-1) & 0 \end{bmatrix} \in \mathcal{M}^{d+1}(\{0, -1\})$, where (-1) is the vector with appropriate size whose entries are all -1.

It is then clear that, for any finite family of matrices Γ, the semigroup generated by $\widetilde{\Gamma} = \left\{ \widetilde{M} \middle| M \in \Gamma \right\}$ is $\left\langle \widetilde{\Gamma} \right\rangle = \left\{ \widetilde{M} \middle| M \in \langle \Gamma \rangle \right\}$.

Moreover, note that if $\rho(M) < 0$, then the critical graph of $\widetilde{M}$ is the loop over the last vertex, so that $\mathrm{urk}(\widetilde{M}) = 1$. Otherwise, $\rho(M) = 0$, and the critical graph is the union of this loop and the critical graph of M, so that $\mathrm{urk}(\widetilde{M}) = 1 + \mathrm{urk}(M)$. We deduce that the ultimate rank of $\widetilde{\Gamma}$ is greater than or equal to 2 if and only if $\rho(\Gamma) \geqslant 0$.

▶ Remark. Lemma 13 is implicitely used in [12, Corollary 2] to prove that the functions computed by max-plus automata with rational entries whose linear representation generates a so-called primitive semigroup (which includes matrices with finite entries) can be computed by a deterministic automaton.

It is also shown (as Corollary 4) that the minimal growth rate of a deterministic automaton, *i.e.* the joint spectral radius of its linear representation, can be computed as the spectral radius of one matrix whose indices are the states of the automaton.

This gives another algorithm to compute the joint spectral radius of a finite set of matrices with finite integers, but not a PSPACE one, since the size of the deterministic automaton is only bounded by $(4b + 1)^{d^2}$, while Equation (3) allows to compute only matrices of size d without storing them.

5 Conclusion and open questions

In this paper, we have proved that the joint spectral radius and the ultimate rank of a finite set of matrices over the tropical semiring are not computable (from the proof it can be seen that they are actually computably-enumerable-complete). To this end, we have proved the undecidability of the comparison of max-plus automata in restricted cases: when all the states are both initial and final and when the number of states is bounded.

As for the restriction on the number of states, we proved that comparison is undecidable when restricted to 553 states. Now, the question is to understand what happens between 2 and 552 states. Even when restricted to 2 states, it seems quite a difficult question to answer. Moreover, the various proofs highlight a link between several universal models: diophantine equations and two-counter machines. Having better size bounds on these models would give a better bound for our undecidable problem, but conversely, getting the decidability of comparison for max-plus automata with at most a certain number of states could lead to improve the known lower bounds on the size of these universal objects.

As for the joint spectral radius, one could ask if it is always rational or if, on the opposite, the set of joint spectral radii of finite families of matrices admits some computability-theoretic characterization. With respect to complexity, the main open question is whether it is PSPACE to approximate the joint spectral radius.

References

1. Shaull Almagor, Udi Boker, and Orna Kupferman. What's decidable about weighted automata? In *ATVA 2011*, pages 482–491. Springer-Verlag, oct 2011.
2. Yu. A. Al'pin. Bounds for joint spectral radii of a set of nonnegative matrices. *Mathematical Notes*, 87(1):12–14, 2010. `doi:10.1134/S0001434610010025`.
3. François Louis Baccelli, Geert Jan Olsder, Jean-Pierre Quadrat, and Guy Cohen. *Synchronization and linearity. An algebra for discrete event systems.* Chichester: Wiley, 1992.
4. Vincent D. Blondel, Stéphane Gaubert, and John N. Tsitsiklis. Approximating the spectral radius of sets of matrices in the max-algebra is np-hard. *Automatic Control, IEEE Transactions on*, 45(9):1762–1765, Sep 2000. `doi:10.1109/9.880644`.
5. Peter Butkovič. *Max-linear systems. Theory and algorithms.* London: Springer, 2010. `doi:10.1007/978-1-84996-299-5`.
6. Thomas Colcombet. On distance automata and regular cost function. Presented at the Dagstuhl seminar "Advances and Applications of Automata on Words and Trees", 2010.
7. Thomas Colcombet and Laure Daviaud. Approximate comparison of distance automata. In Natacha Portier and Thomas Wilke, editors, *STACS*, volume 20 of *LIPIcs*, pages 574–585. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013. `doi:10.4230/LIPIcs.STACS.2013.574`.
8. Thomas Colcombet, Laure Daviaud, and Florian Zuleger. Size-change abstraction and max-plus automata. In Erzsébet Csuhaj-Varjú, Martin Dietzfelbinger, and Zoltán Ésik, editors, *Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part I*, volume 8634 of *Lecture Notes in Computer Science*, pages 208–219. Springer, 2014. `doi:10.1007/978-3-662-44522-8_18`.
9. Ludwig Elsner and P. van den Driessche. Bounds for the perron root using max eigenvalues. *Linear Algebra and its Applications*, 428(8):2000–2005, 2008. `doi:10.1016/j.laa.2007.11.014`.
10. Stéphane Gaubert and Ricardo Katz. Reachability problems for products of matrices in semirings. *International Journal of Algebra and Computation*, 16(3):603–627, jun 2006. URL: `http://arxiv.org/abs/math/0310028`, `arXiv:0310028`, `doi:10.1142/S021819670600313X`.
11. Stéphane Gaubert and Jean Mairesse. Task resource models and (max, +) automata. In *Idempotency (Bristol, 1994)*, volume 11 of *Publ. Newton Inst.*, pages 133–144. Cambridge Univ. Press, Cambridge, 1998. `doi:10.1017/CBO9780511662508.009`.
12. Stéphane Gaubert. Performance evaluation of (max, +) automata. *IEEE Trans. Automat. Control*, 40(12):2014–2025, 1995. `doi:10.1109/9.478227`.
13. Stéphane Gaubert. On the Burnside problem for semigroups of matrices in the (max,+) algebra. *Semigroup Forum*, 52(1):271–294, 1996. `doi:10.1007/BF02574104`.
14. Stéphane Gaubert and Jean Mairesse. Modeling and analysis of timed Petri nets using heaps of pieces. *IEEE Trans. Automat. Control*, 44(4):683–697, 1999. `doi:10.1109/9.754807`.
15. Pierre Guillon, Zur Izhakian, Jean Mairesse, and Glenn Merlet. The ultimate rank of semi-groups of tropical matrices. *Journal of Algebra*, 437:222–248, September 2015. `doi:10.1016/j.jalgebra.2015.02.026`.
16. Bernd Heidergott, Geert Jan Oldser, and Jacob van der Woude. *Max plus at work. Modeling and analysis of synchronized systems: a course on max-plus algebra and its applications.* Princeton, NJ: Princeton University Press, 2006.
17. James P. Jones. Universal Diophantine equation. *J. Symbolic Logic*, 47(3):549–571, 1982. `doi:10.2307/2273588`.

18 Raphaël Jungers. *The joint spectral radius*, volume 385 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 2009. Theory and applications. `doi:10.1007/978-3-540-95980-9`.

19 Jui-Yi Kao, Narad Rampersad, and Jeffrey Shallit. On NFAs where all states are final, initial, or both. *Theoretical Computer Science*, 410(47):5010–5021, 2009. `doi:10.1016/j.tcs.2009.07.049`.

20 Daniel Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. In *Automata, languages and programming (Vienna, 1992)*, volume 623 of *Lecture Notes in Comput. Sci.*, pages 101–112. Springer, Berlin, 1992. `doi:10.1007/3-540-55719-9_67`.

21 Sylvain Lombardy and Jean Mairesse. Max-plus automaton. In *Handbook of Automata*. European Mathematical Society, To appear.

22 Glenn Merlet. Semigroup of matrices acting on the max-plus projective space. *Linear Algebra and its Applications*, 432(8):1923–1935, 2010. `doi:10.1016/j.laa.2009.03.029`.

23 Marvin L. Minsky. Recursive unsolvability of Post's problem of "tag" and other topics in theory of Turing machines. *Ann. of Math. (2)*, 74:437–455, 1961.

24 Marvin L. Minsky. *Computation: finite and infinite machines.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967. Prentice-Hall Series in Automatic Computation.

25 Marcel-Paul Schützenberger. On the definition of a family of automata. *Information and Control*, 4:245–270, 1961.