Erdős-Pósa Property of Obstructions to Interval Graphs

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Abstract

The duality between packing and covering problems lies at the heart of fundamental combinatorial proofs, as well as well-known algorithmic methods such as the primal-dual method for approximation and win/win-approach for parameterized analysis. The very essence of this duality is encompassed by a well-known property called the Erdős-Pósa property, which has been extensively studied for over five decades. Informally, we say that a class of graphs $\mathcal{F}$ admits the Erdős-Pósa property if there exists f such that for any graph G, either G has k vertex-disjoint "copies" of the graphs in $\mathcal{F}$, or there is a set $S \subseteq V(G)$ of $f(k)$ vertices that intersects all copies of the graphs in $\mathcal{F}$. In the context of any graph class $\mathcal{G}$, the most natural question that arises in this regard is as follows – do obstructions to $\mathcal{G}$ have the Erdős-Pósa property? Having this view in mind, we focus on the class of interval graphs. Structural properties of interval graphs are intensively studied, also as they lead to the design of polynomial-time algorithms for classic problems that are NP-hard on general graphs. Nevertheless, about one of the most basic properties of such graphs, namely, the Erdős-Pósa property, nothing is known. In this paper, we settle this anomaly: we prove that the family of obstructions to interval graphs – namely, the family of chordless cycles and ATs – admits the Erdős-Pósa property. Our main theorem immediately results in an algorithm to decide whether an input graph G has k vertex-disjoint ATs and chordless cycles, or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all ATs and chordless cycles.

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1 Introduction

Packing and covering problems are ubiquitous in both graph theory and computer science. The duality between packing and covering problems lies at the heart of not only fundamental combinatorial proofs, but also well-known algorithmic methods such as the primal-dual method for approximation and win/win-approach for parameterized analysis. The very essence of this duality is encompassed by a well-known property called the Erdős-Pósa property. This property, being both simple and powerful, has been extensively studied for over five decades. In the context of any graph class $\mathcal{G}$, the most natural question that arises in this regard is as follows – do obstructions to $\mathcal{G}$ have the Erdős-Pósa property? Having this view in mind, we focus on the class of interval graphs. Arguably, this is the most basic class of graphs that can be viewed as geometric inputs – indeed, an interval graph is the intersection graph of a family of intervals on the real lines. Interval graphs are among the most well-studied classes of graphs in the literature. In particular, the usage of interval graphs as models is relevant to a wide variety of applications, ranging from resource allocation in operations research and scheduling theory to assembling contiguous subsequences in DNA mapping. From an algorithmic point of view, the structural properties of interval graphs are also intensively studied as they allow to design polynomial-time algorithms for well-known problems in computer science, such as Independent Set and Hamiltonian Path, that are NP-hard on general graphs. Nevertheless, about one of the most basic properties of such graphs, namely, the Erdős-Pósa property, nothing is known! Our main contribution settles this anomaly: we prove that obstructions to interval graphs admit the Erdős-Pósa property.

Before we turn to consider our contribution in more detail, we present a gentle introduction to the rich realm of studies of Erdős-Pósa properties. For this purpose, we first define packing and covering problems. Let $\preceq$ be a containment relation (of a graph into another graph), and let $\mathcal{F}$ be a family of graphs. For example, we can define the containment relationship $\preceq$ as follows: for graphs G and H, $H \preceq G$ if and only if H is an induced subgraph/subgraph/minor/topological minor of G. In this setting, $(\mathcal{F}, \preceq)$-Packing is the problem whose input consists of a graph G and an integer k, and the objective is to decide if G has k vertex-disjoint subsets, $S_1, S_2, \ldots, S_k \subseteq V(G)$, where for each $i \in [k]$, there exists $F \in \mathcal{F}$ such that $F \preceq G[S_i]$. For example, if $\mathcal{F} = \{F\}$ and the relation refers to induced subgraphs, then we simply ask whether G has k vertex-disjoint "exact copies" of F. The $(\mathcal{F}, \preceq)$-Covering problem has the same input, but its objective is to decide if there is a set $S \subseteq V(G)$ of size at most k such that there does not exist $F \in \mathcal{F}$ that satisfies $F \preceq G - S$. Some well-known examples of packing problems (and their corresponding covering problems) are Maximum Matching (Vertex Cover), Vertex-Disjoint s-t Paths (s-t-Separator), Cycle Packing (Feedback Vertex Set), P_3-Packing (Cluster Vertex Deletion), and Triangle Packing (Triangle Free Deletion).

Kőnig's and Menger's theorems are cornerstones of Graph Theory in general, and of the study of packing and covering problems in particular, which have also found a wide variety of applications in computer science. For example, Menger's theorem is particularly relevant to survivable network design (see, e.g., [5, 46]) and combinatorial optimization (see, e.g., [43, 19]). Formally, Kőnig's theorem states that in bipartite graphs, the maximum size of a matching *equals* the minimum size of a vertex cover [30, 13]. Menger's theorem also exhibits an equality – it states that for a given graph G and a pair of vertices s and t, either G has k vertex-disjoint paths between s and t or there is a set $S \subseteq V(G) \setminus \{s, t\}$ of size k such that $G - S$ has no path between s and t [33, 13]. Both theorems relate a packing problem to a covering problem[1], by exhibiting equality between the size of a maximum packing and the size of a minimum covering. However, most natural packing and covering problems are not known to exhibit such an equality; in fact, frequently such an equality is *proven* not to exist. By relaxing the notion of equality, we enter the rich realm of the Erdős-Pósa properties.

The Erdős-Pósa Property. A celebrated theorem by Erdős and Pósa [14] states that for any graph G, either there is a set of k vertex-disjoint cycles in G, or there is a set $S \subseteq V(G)$ of $f(k) = \mathcal{O}(k \log k)$ vertices that intersects (covers) all cycles of G.[2] Notably, Erdős and Pósa [14] also showed that there exists a constant c and infinitely many pairs (G, k) such that G has neither k vertex-disjoint cycles nor a set $S \subseteq V(G)$ of $ck \log k$ vertices that covers all cycles of G. That is, not only equality cannot be expected, but also any function $f(k) = o(k \log k)$. We remark that later, Simonovits [45] provided concrete examples which realize the lower bound. The result of Erdős and Pósa [14] initiated a flurry of extensive study of the so called "Erdős-Pósa property" for various families of graphs as well as containment relationships. Formally, a family of graphs $\mathcal{F}$ and a containment relation $\preceq$ are said to admit the Erdős-Pósa property if there exists a function $f(\cdot)$ such that given a graph G and an integer k, either there are k vertex-disjoint subsets $S_1, \ldots S_k \subseteq V(G)$ so that for each $i \in [k]$, there is $F \in \mathcal{F}$ satisfying $F \preceq G[S_i]$, or there is a set $S \subseteq V(G)$ of size at most $f(k)$ so that there is no $F \in \mathcal{F}$ satisfying $F \preceq G - S$. Here, the first question that comes to mind is – do all families of graphs $\mathcal{F}$ and containment relationships $\preceq$ exhibit the Erdős-Pósa property?

The answer to this question is negative. For example, consider a fixed graph H, and let $\mathcal{F}(H)$ be the family of graphs that contain H as a minor. Robertson and Seymour [42] showed that $\mathcal{F}(H)$ with the containment relation referring to subgraphs admits the Erdős-Pósa property if and only if H is a planar graph. This result generalizes the result in [14]. However, the function $f(\cdot)$ given by [42] is exponential – can it be made polynomial? A few years ago, the bound was improved to $\mathcal{O}(k \log^c k)$ by Chekuri and Chuzhoy [12] following a more general approach which is applicable to other families as well. A well-known example of a different flavor conerns odd cycles. Specifically, for $\mathcal{F}$ being the family of odd length cycles, Reed [39] showed that $\mathcal{F}$ (for subgraphs and induced subgraphs) does not admit the Erdős-Pósa property.

Since the emergence of the result of Erdős and Pósa [14], a multitude of studies on the Erdős-Pósa property have appeared in the literature for several combinatorial objects beyond graphs. This includes extensions to digraphs [32, 44, 40, 22, 20], rooted graphs [9, 26, 35, 24], labeled graphs [29], signed graphs [23, 3], hypergraphs [1, 6, 7], matroids [16], helly-type theorems [21], H-minors [41], H-immersions [17, 31], and H-butterfly directed minors [2] (also see [38]). This list is not comprehensive but rather illustrative. We refer to surveys such as [37] for more information. Even for subfamilies of cycles alone, there is a vast literature devoted to the Erdős-Pósa property. Studies of the Erdős-Pósa property for subfamilies of cycles include, for example, long cycles (subgraphs) [4, 34], directed cycles (subgraphs and induced subgraphs) [40, 20], chordless cycles (induced subgraphs) [25] and cycles intersecting a prescribed vertex set [27, 35]. Not all subfamiles of cycles admit the Erdős-Pósa property. For example, recall the result stated earlier regarding the family of odd cycles [39]. For this subfamily of cycles alone, there has been a sequence of research about finding classes of

[1] For example, Kőnig's theorem addresses the class $\mathcal{F} = \{F\}$ such that F is the graph on a single edge, where $\preceq$ refers to induced subgraphs/subgraphs.

[2] In the terminology of packing and covering, we address the class $\mathcal{F}$ of all cycles, where $\preceq$ refers to induced subgraphs/subgraphs.

graphs for which the family of odd cycles (subgraphs and induced subgraphs) admits the Erdős-Pósa property. This includes planar graphs [15], or graphs with certain connectivity constraints [48, 36, 28, 24]. Not only the family of odd cycles does not admit the Erdős-Pósa property, but also subfamilies such as the family of chordless cycles of length at least 5 [25].

A large number of the results above can be viewed as the question of packing or covering obstructions to a class of graphs. In some of these papers, this view is explicitly stated as the motivation behind the conducted studies. For example, the classic result by Erdős and Pósa [14] regards the question of packing and covering obstructions to forests. The results concerning odd cycles address obstructions to bipartite graphs. The setting of the work about packing and covering chordless cycles, as presented by [25], addresses obstructions to chordal graphs. Furthermore, Kőnig's theorem relates to obstructions to edgeless graphs, and the work by Robertson and Seymour [42] relates to obstructions to subfamilies of minor free graphs. We remark that other results can also be interpreted in this manner. Given that the class of interval graphs is among the most basic, well-studied families of graphs, we find it important to study the Erdős-Pósa [14] property with respect to it. Let $\mathcal{F}$ be the family of chordless cycles and asteroidal triples (ATs), see Section 2. It is well known that the class of interval graphs is precisely the class of graphs that exclude every graph in $\mathcal{F}$ as an induced subgraph [18, 8]. Given this clean characterization, the following question naturally arises:

> Does the family of chordless cycles and ATs – that is, obstructions to interval graphs – admit the Erdős-Pósa property?

Our Contribution. We provide an affirmative answer to the question above. Moreover, the dependency of the size of the covering set on k in our result is only $\mathcal{O}(k^2 \log k)$.[3] Specifically, we obtain the following theorem, where from now on, "obstructions" refer to ATs and chordless cycles.

▶ **Theorem 1.** *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph.*

As a consequence of our main theorem, we also derive an algorithm to decide whether an input graph G has k vertex-disjoint obstructions (to interval graphs), or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all such obstructions.

We conclude the introduction with a high-level (informal) overview of our proof. We begin by easily "getting rid" of all chordless cycles due to the work by [25], as well as all small ATs. Now, the heart of our proof consists of two main components. First, we exhibit the Erdős-Pósa property of the family of ATs on graphs that have a clique caterpillar (that is, a tree decomposition that is a caterpillar, where every bag is a clique). Second, we show how this result can be utilized to derive our main theorem by analyzing "conflict-free sets" (defined below) with respect to a modular tree decomposition of the graph. Let us now elaborate on each component.

To analyze the case of a clique caterpillar, we present a procedure that at each iteration, finds an AT $\mathbb{O}$ with specific properties, inserts a set S of $\mathcal{O}(k)$ new vertices into a set $S^\star$ initialized to be empty, and removes the vertices in S from the graph (only for the sake of the execution of the procedure). Specifically, the set S consists of the terminals, centers and

[3] In fact, all of our arguments achieve the dependency $\mathcal{O}(k^2)$, but we gain an extra $\log k$ factor due to an invocation of a result by Kwon and Kim [25].

a few base vertices of $\mathbb{O}$, as well as all of the vertices of a "small" separator between the non-shallow terminals of $\mathbb{O}$ that we push as much as possible to the right of the caterpillar. The procedure terminates once the graph becomes an interval graph. Hence, it is clear that if at most $\mathcal{O}(k)$ iterations take place, then $S^\star$ is a set of size $\mathcal{O}(k^2)$ that intersects all ATs, which implies that our job is done. Otherwise, we require an intricate analysis to establish the existence of k vertex-disjoint ATs. Roughly, the two main components here are (1) from the sequence of ATs encountered by our procedure, we can extract a sequence of the same length (of possibly *different* ATs) where each AT has the property that the subpath of its base that lies after the separator does not intersect any AT positioned after it in the sequence, and (2) from the modified sequence, we can extract a *subsequence* of ATs where disjointess is also guaranteed with respect to base vertices that lie before the separator.

Towards the proof of the second item, we first show that for every sequence "resembling" the one encountered by our procedure, and for all ATs $\mathbb{O}$ and $\mathbb{O}'$ in that sequence such that $\mathbb{O}'$ comes before $\mathbb{O}$, we have the following property: only the leftmost terminal and base vertex of $\mathbb{O}$ can belong to the base path of $\mathbb{O}'$ that lies before the separator associated with $\mathbb{O}'$, and even that is only possible under certain conditions. This result then allows us to further argue about the relation between every *three* ATs in the sequence with respect to the "left sides of separators". Having established this relation, the argument about a complete sequence is derived. Towards the proof of the first item, we first show that for any AT $\mathbb{O}$ in the sequence, we can find a path between a vertex in the separator associated with $\mathbb{O}$ and the right terminal of $\mathbb{O}$ that avoids all ATs coming after $\mathbb{O}$ in the sequence. Then, by relying on structural results by Cao and Marx [11], we argue that this path can be used to replace part of $\mathbb{O}$ so that the result is yet another AT.

Let us now turn to our analysis of the general case – specifically, we explain how it is reduced to instances of the case of a clique caterpillar. We define "problematic" nodes in the modular tree decomposition of the input graph as the nodes associated with subgraphs that contain at least one AT that is not present in any of the subgraphs associated with their children. This definition also immediately gives rise to an association between nodes and ATs, so that each AT is associated with exactly one node. We observe that maximal modules of problematic nodes are vertex disjoint, and that each problematic node has "many" children. It is also easily shown that the set of all problematic nodes can be partitioned into two sets that have no "conflict" – that is, on the unique path between every two nodes of one set, there exists a node of the other set. The point in analyzing each conflict-free set P separately is that for each problematic node in such a set, we prove that there exist at least k vertices in the subgraph associated with that node that do not belong to any subgraph associated with its problematic descendants from P. In particular, this allows us to examine each problematic node *individually*, and associate an instance of the clique caterpillar case with it (the construction of the caterpillar decomposition itself partially follows from structural results by Cao and Marx [11]). Specifically, we are able to collect the sets of ATs found in each instance, and argue that (after some modification) all of these ATs across all the sets are in fact vertex disjoint. This result then allows us to handle the "packing perspective" of the proof. We remark that although we can create $\mathcal{O}(k)$ instances of the clique caterpillar case, and each individual instance can create a gap of $\mathcal{O}(k^2)$, we eventually get a gap of only $\mathcal{O}(k^2)$ rather than $\mathcal{O}(k^3)$ as we argue that the sum of the contributions to the gap of all individual instances is $\mathcal{O}(k^2)$.

Due to lack of space, proofs of statements marked by "*" were omitted.

2 Preliminaries

For $n \in \mathbb{N}$, we use $[n]$ as a shorthand for $\{1, 2, \ldots, n\}$. Given a function $f : A \to B$ and a subset $A' \subset A$, we use $f|_{A'}$ to denote the restriction of f to A'.

We refer to standard terminology from the book of Diestel [13] for graph-related terms that are not explicitly defined here. When the graph G is clear from context, denote $n = |V(G)|$ and $m = |E(G)|$. We say that a vertex v in G is *simplicial* if $N_G(v)$ induces a clique. A *caterpillar* is a tree T for which there exists a subpath P of T, called a *central path*, such that the removal of the vertices of P from T results in an edgeless graph. Given a rooted tree T and a vertex $v \in V(T)$, we use $T|_v$ to denote the subtree of T rooted at v. Moreover, $\mathsf{child}(v)$ denotes the set of children of v in T. We do not treat a vertex as a descendant of itself. A *chordal graph* is a graph with no chordless cycle on at least four vertices.

Interval Graphs. An *interval graph* is a graph G that does not contain any of the following graphs, called *obstructions*, as an induced subgraph.

- **Long Claw.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_3\}, \{t, b_2\}, \{c, b_1\}, \{c, b_2\}, \{c, b_3\}\}$.
- **Whipping Top (or Umbrella).** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_2\}, \{c, t\}, \{c, b_1\}, \{c, b_2\}, \{b_3, t_\ell\}, \{b_3, b_1\}, \{b_3, c\}, \{b_3, b_2\}, \{b_3, t_r\}\}$.
- **†-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c), \{t_\ell, b_1\}, \{t_r, b_z\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 2$. A †-AW where $z = 2$ is called a *net*.
- **‡-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c_1, c_2\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c_1\}, \{t, c_2\}, \{c_1, c_2\}, \{t_\ell, b_1\}, \{t_r, b_z\}, \{t_\ell, c_1\}, \{t_r, c_2\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ is called a *tent*.
- **Hole.** A chordless cycles on at least four vertices.

Long claws and whipping tops are also called ATs, but we shall reserve this name for †-AWs and ‡-AWs (AW stand for Asteroidal Witness).[4] An obstruction $\mathbb{O}$ is *minimal* if there does not exist an obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subset V(\mathbb{O})$. In each of the first four obstructions, the vertices t_ℓ, t_r, and t are called *terminals*, the vertices c, c_1, and c_2 are called *centers*, and the other vertices are called *base vertices*. To simplify notation, when we consider a †-AW, we use c_1 and c_2 to refer to c (this allows us to refer to a †-AW and a ‡-AW in a unified manner). Furthermore, the vertex t is called the *shallow terminal*. The induced path on the set of base vertices is called the *base* of the AT, and it is denoted by $\mathsf{base}(\mathbb{O})$. Moreover, we say that the induced path on the set of base vertices, t_ℓ and t_r is the *extended base* of the AT, and it is denoted by $P(\mathbb{O})$. Given a graph G, a vertex v is *shallow in G* if G has at least one AT where v is the shallow terminal.

Tree Decomposition. For a tree decomposition (T, β) of a graph G, if T is a path, then (T, β) is also called a *path decomposition*, and if T is a caterpillar then (T, β) is also called a *caterpillar decomposition*. A *clique path* (*clique caterpillar*) of a graph G is a path decomposition (resp. *caterpillar decomposition*) of G where every bag is a distinct maximal clique. We remark that not every graph admits a clique caterpillar.

[4] Like other papers on this topic, we abuse the standard usage of the term AT in the literature, which refers to a triple of vertices such that each pair is joined by a path that avoids the neighborhood of the other vertex. Our usage and the standard one are "almost equivalent" (see, e.g., [10]).

Modules. Let G be a graph. A subset $M \subseteq V(G)$ is a *module* if for all $u, w \in M$ and $v \in V(G) \setminus M$, either both u and w are adjacent to v or both u and w are not adjacent to v. A module is *nontrivial* if neither $V(M) = \emptyset$ nor $V(M) = V(G)$.

A *modular tree decomposition* of a graph $G = (V, E)$ is a linear-size representation of all its modules. It consists of a rooted tree T, a function $f : V(T) \to 2^{V(G)}$ and a function $g : V(T) \to \{0, 1\}$, which in particular satisfy the following properties:

1. M is a module of G if and only if there is a node $v \in V(T)$ for which, either $M = f(v)$, or both $g(v) = 1$ and there is a subset U of the set of children of v such that $M = \bigcup_{u \in U} f(u)$.
2. Every $v, u \in V(T)$ that have the same parent in T satisfy $f(v) \cap f(u) = \emptyset$.
3. For every $v \in V(T)$, $\bigcup_{u \in \mathsf{child}(v)} f(u) = f(v)$.
4. $|V(T)| \leq 2n - 1$.

Furthermore, no node in T has exactly one child. Every graph G admits a modular tree decomposition, which can be constructed in $O(n^2)$ time and $O(n)$ space [47].

Hitting Chordless Cycles and Small Obstructions. We first state the following corollary of the results of Kim and Kwon [25].

▶ **Corollary 2.** *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices.*

3 The Case of a Clique Caterpillar

This section analyzes the Erdős-Pósa Property of ATs on graphs with a clique caterpillar. Let us begin with a definition.

▶ **Definition 3.** Let G be a graph. A clique caterpillar (T, β) of G is *nice* if every shallow vertex belongs to the bag of only one node of T and that node is a leaf.

The objective of this section is to prove the following lemma.

▶ **Lemma 4.** *Let $k \in \mathbb{N}$, and let G be a graph with a nice clique caterpillar (T, β), such that G is chordal and has no obstruction on at most ten vertices.*[5] *Then, at least one of the following conditions holds:* **(i)** *G has k vertex-disjoint ATs;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ is an interval graph.*

To simplify statements in this section, let us fix $k \in \mathbb{N}$ and a chordal graph G with a nice clique caterpillar (T, β), which has no obstruction on at most ten vertices. Thus, whenever we discuss an obstruction in G, that obstruction is necessarily an AT on more than ten vertices. Moreover, let us fix a central path of T, and call it P. We denote $P = p_1 - p_2 - \cdots - p_d$ for $d = |V(P)|$. We think of P as a path oriented from p_1 to p_d. For a vertex $v \in V(G)$, we let $\mathsf{first}(v)$ be the first node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define $\mathsf{first}(v) = \mathsf{nil}$), and we let $\mathsf{last}(v)$ be the last node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define $\mathsf{last}(v) = \mathsf{nil}$). The notation $p_i < p_j$ means that $i < j$ (similarly, we define $\leq$). Note that as non-terminal vertices of an AT have non-adjacent neighbors, we have the following observation.

[5] We remark that the existence of the clique caterpillar already implies that G is chordal [18, 8].

▶ **Observation 5.** *Let $\mathbb{O}$ be an AT in G. For every non-terminal vertex v of $\mathbb{O}$, there exists $p \in V(P)$ such that $v \in \beta(p)$.*

Observation 5 implies that the notation presented next is well defined. In what follows, when we consider an AT $\mathbb{O}$, we index the base vertices $b_1^{\mathbb{O}}, b_2^{\mathbb{O}}, \ldots, b_{\eta^{\mathbb{O}}}^{\mathbb{O}}$ such that $\mathsf{first}(b_1^{\mathbb{O}}) \leq \mathsf{first}(b_{\eta^{\mathbb{O}}}^{\mathbb{O}})$. When $\mathbb{O}$ is clear from context, we simplify the notation, also in the context of terminal and center vertices.[6] Note that $\eta \geq 5$, as G does not have ATs on at most ten vertices (we use this observation implicitly throughout, e.g. to assume that $b_1, b_2, b_{\eta-2}, b_{\eta-1}$ and b_η are distinct vertices). We remark that clearly, for all $i \in \{2, 3, \ldots, \eta-1\}$, $\mathsf{first}(b_i) \leq \mathsf{last}(b_{i-1}) < \mathsf{first}(b_{i+1})$ (also stated as Proposition 8.4 in [11]).

Our analysis relies on a notion of a special type of obstruction, defined by Cao and Marx [11], to exploit the "almost linear nature" of a caterpillar. To this end, we have the following notation. Given an AT $\mathbb{O}$, $\widehat{N}(\mathbb{O})$ denotes the set of vertices $v \in V(G)$ such that v is adjacent to every vertex in $\mathsf{base}(\mathbb{O})$. We also need to give three definitions.

▶ **Definition 6** ([11]).

(i) An AT $\mathbb{O}$ in G is *minimal* if there does not exist an AT $\mathbb{O}'$ such that $\mathsf{last}(b_1) \leq \mathsf{last}(b'_1) \leq \mathsf{first}(b'_{\eta'}) \leq \mathsf{first}(b_\eta)$, and $\mathsf{last}(b_1) < \mathsf{last}(b'_1)$ or $\mathsf{first}(b'_{\eta'}) < \mathsf{first}(b_\eta)$.

(ii) An AT $\mathbb{O}$ in G is *short* if $P(\mathbb{O})$ is a shortest path between t_ℓ and t_r in $G[\beta(p_i) \cup \beta(p_{i+1}) \cup \ldots \cup \beta(p_j) \cup \{t_\ell, t_r\}] - \widehat{N}(\mathsf{base}(\mathbb{O}))$, where $p_i = \mathsf{last}(b_1)$ and $p_j = \mathsf{first}(b_\eta)$.

(iii) An AT $\mathbb{O}$ in G is *first* if there does not exist an AT $\mathbb{O}'$ such that $\mathsf{first}(b'_{\eta'}) < \mathsf{first}(b_\eta)$.

We say that an AT is *good* if it is first, minimal and short. The following proposition asserts that a good AT exists. In this context, recall that we implicitly assume that G is not an arbitrary graph, but in particular it is a graph that has a nice clique caterpillar.

▶ **Proposition 7** (Lemma 8.8 & Proof of Theorem 2.4 (Page 31) [11]). *If G is not an interval graph, then it has a good AT.*

▶ **Proposition 8** (Claim 5 [11]). *Let $\mathbb{O}$ be a good AT. For any vertex $v \in (\beta(p_1) \cup \beta(p_2) \cup \ldots \beta(p_i)) \setminus \widehat{N}(\mathbb{O})$, where $p_i = \mathsf{first}(b_{\eta-2})$, it holds that v is not adjacent to any vertex that is shallow in G.*

Procedure `SeparateProcedure`. Let us consider the following procedure, which we call `SeparateProcedure`. Initialize $G^1 = G$ and $i = 1$. Now, as long as G^i is not an interval graph, we execute the following procedure:

1. Let $\mathbb{O}^i$ be a good AT in G^i, whose existence is guaranteed by Proposition 7.
2. Denote $p_j = \mathsf{first}(b^i_{\eta^i-2})$ and $p_q = \mathsf{last}(b^i_1)$. For all $\delta \in [d]$, denote $\beta^i(p_\delta) = \beta(p_\delta) \cap V(G^i)$. Let $\gamma^i = \gamma$ be the index in $\{q, q+1, \ldots, j-1\}$ such that,
   - there does not exist an index $\delta \in \{\gamma+1, \gamma+2, \ldots, j-1\}$ such that $|(\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$, and
   - $|(\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$.

   If such an index γ does not exist, define $\gamma = \mathsf{nil}$. Intuitively, γ is the largest index of a "small" separator in $G^i \setminus \widehat{N}(\mathbb{O})$ between b^i_1 and $b^i_{\eta-2}$.
3. Denote $S^i = (\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1})) \setminus \widehat{N}(\mathbb{O}^i)$ if $\gamma \neq \mathsf{nil}$, and $S^i = \emptyset$ otherwise.
4. Define $G^{i+1} = G^i - ((V(\mathbb{O}^i) \setminus \mathsf{base}(\mathbb{O}^i)) \cup \{b^i_1, b^i_2, b^i_3, b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}, b^i_{\eta^i}\} \cup S^i)$.
5. Increment i by 1.

[6] For example, if we consider an AT denoted by $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}^i$, then we use b_1 (b_η), b'_1 ($b'_{\eta'}$), b^i_1 ($b^i_{\eta^i}$) to refer to the first (last) base vertex of $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}^i$, respectively.

Let $i^\star$ denote the last index i considered by `SeparateProcedure`. In particular, $G^{i^\star}$ is an interval graph. Let us denote $S^\star = V(G) \setminus V(G^{i^\star})$. Then, $G - S^\star$ is an interval graph. Furthermore, note that $|S^\star| = \mathcal{O}(i^\star \cdot k)$.

▶ **Observation 9.** *If $i^\star \leq 2k$, then $S^\star \subseteq V(G)$ is a set of size $\mathcal{O}(k^2)$ such that $G - S^\star$ is an interval graph.*

Thus, to prove Lemma 4, it is sufficient to prove the following claim.

▶ **Lemma 10.** *If $i^\star > 2k$, then G has k vertex-disjoint obstructions.*

In what follows, we suppose that $i^\star > 2k$. To prove this lemma, we first need to introduce the following definitions.

▶ **Definition 11.** Let $i \in [2k]$. We say that an AT $\mathbb{O}$ in G is *i-relevant* if it is an AT in G^i, $t = t^i$, $t_\ell = t^i_\ell$, $t_r = t^i_r$, $c_1 = c^i_1$, $c_2 = c^i_2$, $b_1 = b^i_1$, $b_2 = b^i_2$, $b_{\eta-2} = b^i_{\eta^i-2}$, $b_{\eta-1} = b^i_{\eta^i-1}$ and $b_\eta = b^i_{\eta^i}$.[7] If in addition $b_3 = b^i_3$ and $b_{\eta-3} = b^i_{\eta-3}$, then we say that $\mathbb{O}$ is *highly i-relevant.*

▶ **Definition 12.** A tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ is *relevant* if for all $i \in [2k]$, $\widehat{\mathbb{O}}^i$ is i-relevant.

We further need the following notation. For every $i \in [2k]$, $\mathsf{before}(i) = \beta(p_1) \cup \beta(p_2) \cup \ldots \cup \beta(p_{\gamma^i})$ if $\gamma^i \neq \mathsf{nil}$ and $\mathsf{before}(i) = \emptyset$ otherwise. The heart of the proof of Lemma 10 is given by two statements. Towards the first one, let us first prove the following claim.

▶ **Lemma 13 (*).** *For all $i, i' \in [2k]$ where $i > i'$, i-relevant AT $\mathbb{O}$ and i'-relevant AT $\mathbb{O}'$, it holds that,* (1) *$V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i') \subseteq \{t_\ell, b_1\}$;* (2) *$|V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')| \leq 1$; and* (3) *if $b_1 \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')$ then $t_\ell \notin (\bigcup_{i\in[d]} \beta(p_i))$.*

We now present the first statement that lies at the heart of the proof.

▶ **Lemma 14 (*).** *For all $i, i', \widehat{i} \in [2k]$ such that $i > i' > \widehat{i}$, i-relevant AT $\mathbb{O}$, i'-relevant AT $\mathbb{O}'$ and $\widehat{i}$-relevant AT $\widehat{\mathbb{O}}$, for at least one index $j \in \{i', \widehat{i}\}$, the following condition holds: $V(\mathbb{O}) \cap V(\mathbb{O}^j) \cap \mathsf{before}(j) = \emptyset$.*

▶ **Corollary 15 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. There exist k indices, $i_1 < i_2 < \ldots < i_k$, so that for every two indices $x, y \in \{i_1, i_2, \ldots, i_k\}$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \mathsf{before}(x) = \emptyset$.*

Towards the statement of the second lemma that lies at the heart of our proof, let us first state an immediate observation and one additional lemma.

▶ **Observation 16.** *An AT in G can contain at most four vertices of a clique in G.*

▶ **Lemma 17 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. For all $i \in [2k-1]$, there exists a path in $G^i - \widehat{N}(\mathbb{O}^i)$ from a vertex in $S^i \cup \{b^i_1\}$ to $b^i_{\eta-2}$ that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.*

We are now ready to prove the second statement central to the proof of Lemma 10.

▶ **Lemma 18 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. For all $i \in [2k-1]$ such that $\mathbb{O}^i$ is a highly i-relevant good AT in G^i, there exists an i-relevant AT $\mathbb{O}'$ such that the following condition holds: the base path of $\mathbb{O}'$ has a subpath Q from a vertex in $S^i \cup \{b^i_1\}$ to b^i_η that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.*

[7] That is, $\mathbb{O}$ and the AT $\mathbb{O}^i$ considered in the i-th iteration of `SeparateProcedure` have the same terminals, centers and two first and three last base vertices.

▶ **Corollary 19 (*).** *There exists a relevant tuple* $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \dots, \widehat{\mathbb{O}}^{2k})$ *such that for all* $i \in [2k]$*, the following condition holds: the base path of* $\widehat{\mathbb{O}}^i$ *has a subpath* Q *from a vertex in* $S^i \cup \{b^i_1\}$ *to* b^i_η *that does not contain any of the vertices of the ATs* $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \dots, \widehat{\mathbb{O}}^{2k}$*.*

We are now ready to prove Lemma 10.

Proof of Lemma 10. By Corollary 19, there exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \dots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b^i_1\}$ to b^i_η that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \dots, \widehat{\mathbb{O}}^{2k}$. By Corollary 15, there exist k indices, $i_1 < i_2 < \dots < i_k$, such that for every two indices $x, y \in \{i_1, i_2, \dots, i_k\}$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \mathsf{before}(x) = \emptyset$. Without loss of generality, suppose that $i_1 = 1, i_2 = 2, \dots, i_k = k$ (the arguments to follow hold for any $i_1 < i_2 < \dots < i_k$).

We claim that $\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \dots, \widehat{\mathbb{O}}^k$ are vertex disjoint, which would complete the proof. To prove this claim, we arbitrarily choose $i, j \in [k]$ such that $i < j$. First note that as $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are i-relevant and j-relevant, we have that the terminals and centers of $\widehat{\mathbb{O}}^i$ do not belong to $\widehat{\mathbb{O}}^j$. Moreover, the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex $v^\star$ in $S^i \cup \{b^i_1\}$ to $b^i_{\eta^i}$ that has no vertex of $\widehat{\mathbb{O}}^j$. Let W denote the subpath of the base path of $\widehat{\mathbb{O}}^i$ from b^i_1 to $v^\star$. Hence, to conclude that $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are vertex disjoint, it remains to show that no vertex of W belongs to $\widehat{\mathbb{O}}^j$. Notice that $V(W) \subseteq V(\widehat{\mathbb{O}}^i) \cap \mathsf{before}(i)$. By our choice of $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \dots, \widehat{\mathbb{O}}^k)$, it holds that $V(\widehat{\mathbb{O}}^i) \cap \mathsf{before}(i)$ does not have any vertex of $\widehat{\mathbb{O}}^j$. Thus, the proof is complete. ◀

4 Decomposition of Modules

Let us begin with the following simple observation, on which we rely implicitly in our arguments, and which follows immediately from the definition of a modular tree decomposition. For simplicity, we use the abbreviations $f|_v = f|_{V(T|_v)}$ and $g|_v = g|_{V(T|_v)}$.

▶ **Observation 20.** *Let* G *be a graph with a modular tree decomposition* (T, f, g)*, and let* $v \in V(T)$*. Then,* $(T|_v, f|_v, g|_v)$ *is a modular tree decomposition of* $G[f(v)]$*.*

We proceed by introducing the definition of a *problematic set* and a *problematic node*.

▶ **Definition 21.** Let G be a graph with a modular tree decomposition (T, f, g). The *set of problematic obstructions* of a node $v \in V(T)$, denoted by $\mathsf{prob}_G(v)$, is the set of all obstructions $\mathbb{O}$ in $G[f(v)]$ such that for every child u of v in T, $\mathbb{O}$ is not an obstruction in $G[f(u)]$, that is, $V(\mathbb{O}) \setminus f(u) \neq \emptyset$. When G is clear from context, it is omitted.

▶ **Definition 22.** Let G be a graph with a modular tree decomposition (T, f, g). A node $v \in V(T)$ is *problematic* if $\mathsf{prob}(v) \neq \emptyset$. The set of problematic nodes is denoted by $\mathsf{prob}_G(T)$. When G is clear from context, it is omitted.

▶ **Observation 23.** *Let* G *be a graph with a modular tree decomposition* (T, f, g)*. The sets* $\mathsf{prob}(v)$*,* $v \in \mathsf{prob}(T)$*, define a partition of the set of obstructions of* G*. That is, for all* $u, v \in V(T)$*,* $\mathsf{prob}(u) \cap \mathsf{prob}(v) = \emptyset$*, and the set of obstructions of* G *is precisely* $\bigcup_{v \in \mathsf{prob}(T)} \mathsf{prob}(v)$*.*

We argue that nodes assigned 1 by g are non-problematic. And further, a problematic node should have "many" children.

▶ **Lemma 24 (*).** *G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let* (T, f, g) *be a modular tree decomposition of* G*, and let* $v \in V(T)$ *such that* $g(v) = 1$*. Then,* v *is not a problematic node.*

▶ **Lemma 25 (*).** *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node. Then, v has at least* $\max\{2k, 10\} + 1$ *children in T.*

In order to proceed, we need the following definition and notation.

▶ **Definition 26.** Let G be a graph with a modular tree decomposition (T, f, g). A subset $P \subseteq \mathsf{prob}(T)$ has a conflict if there exist $u, v \in P$ such that v is a descendant of u in T and on the (unique) path between u and v in T no vertex belongs to $\mathsf{prob}(T) \setminus P$.

▶ **Definition 27.** Let G be a graph with a modular tree decomposition (T, f, g). For a node $v \in V(T)$, $\mathsf{pack}_G(v)$ is the maximum number of vertex-disjoint obstructions in $\mathsf{prob}(v)$. When G is clear from context, it is omitted.

Note that a problematic node is precisely a node such that $\mathsf{pack}(v) \geq 1$.

▶ **Lemma 28 (*).** *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices, and which does not have k vertex-disjoint obstructions. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathsf{prob}(T)$ with no conflicts. Then, for each $v \in P$ and each child u of v in T such that u has a problematic descendant, there exist at least k vertices in $f(u)$ that do not belong to $\bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of u in T.*

▶ **Lemma 29 (*).** *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathsf{prob}(T)$ with no conflicts. Then, G has* $\min\{k, \sum_{v \in P} \mathsf{pack}(v)\}$ *vertex-disjoint obstructions.*

We also show that $\mathsf{prob}(T)$ can be divided into two sets with no conflicts.

▶ **Lemma 30 (*).** *Let G be a graph with a modular tree decomposition (T, f, g). There exists a partition (P_1, P_2) of $\mathsf{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict.*

Specific classes of graphs, called *reduced graphs* and *nice interval graphs*, were defined by Cao and Marx as follows.

▶ **Definition 31** ([11])**.** A graph G is *reduced* if it satisfies the following properties: (i) Every non-trivial module of G is a clique, and (ii) G does not have any obstruction on at most ten vertices.

▶ **Definition 32** ([11])**.** A graph G is *nice* if it satisfies the following properties: (i) G is chordal; (ii) G does not have any obstruction on at most ten vertices; and (iii) every vertex in G that is a shallow terminal of at least one obstruction is simplicial.

These definitions were in particular used to derive the following results.

▶ **Proposition 33** (Theorem 2.1 [11])**.** *Let G be a reduced graph. Every vertex in G that is a shallow terminal of at least one obstruction is simplicial.*

▶ **Proposition 34** (Proposition 8.3 [11])**.** *Any nice graph has a nice clique caterpillar (T, β).*

▶ **Corollary 35.** *Any chordal reduced graph has a nice clique caterpillar (T, β).*

Let us derive a consequence of Corollary 35 with respect to a modular tree decomposition.

▶ **Lemma 36 (*).** *Let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node such that for every child u of v in T, $G[f(u)]$ is a clique. Then, $G[f(v)]$ has a nice clique caterpillar.*

Towards the proof of the main result of this section, we need one additional notation.

▶ **Definition 37.** Let G be a graph with a modular tree decomposition (T, f, g), and let $v \in V(T)$. Then, $\mathsf{clique}(G, v)$ denotes the graph obtained from G by turning each $G[f(u)]$, $u \in \mathsf{child}(v)$, into a clique. That is, $V(\mathsf{clique}(G, v)) = V(G)$ and $E(\mathsf{clique}(G, v)) = E(G) \cup (\bigcup_{u \in \mathsf{child}(v)} \{\{x, y\} : x, y \in f(u)\})$.

▶ **Lemma 38 (*).** *Let G be a graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$. Then, the set of obstructions in $\mathsf{clique}(G, v)[f(v)]$ is precisely $\mathsf{prob}_G(v)$.*

▶ **Lemma 39 (*).** *Let $k \in \mathbb{N}$, and let G be a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$. Then, at least one of the following conditions holds:* **(i)** *$\mathsf{pack}(v) \geq k$;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ that intersects the vertex set of every obstruction in $\mathsf{prob}(v)$.*

We are now ready to prove the main result of this section.

▶ **Lemma 40.** *Let $k \in \mathbb{N}$, and let G be a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. Then, at least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ is an interval graph.*

Proof. Suppose that G does not have k vertex-disjoint obstructions, else we are done. By Lemma 30, there exists a partition (P_1, P_2) of $\mathsf{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict. By Lemma 29, for each $i \in [2]$, G has $\sum_{v \in P_i} \mathsf{pack}(v)$ vertex-disjoint obstructions. Thus, by Observation 23, for each $i \in [2]$, $|\sum_{v \in P_i} \mathsf{pack}(v)| < k$. This means that $\sum_{v \in \mathsf{prob}(T)} \mathsf{pack}(v) < 2k$.

By Lemma 39, for all $v \in \mathsf{prob}(T)$, there exists a subset $D_v \subseteq V(G)$ of size $\mathcal{O}((\mathsf{pack}(v)+1)^2)$ that intersects the vertex set of every obstruction in $\mathsf{prob}(v)$. Denote $D = \bigcup_{v \in \mathsf{prob}(T)} D_v$. Then, $|D| = \mathcal{O}(\sum_{v \in \mathsf{prob}(T)} (\mathsf{pack}(v) + 1)^2)$. By Observation 23, we have that $G - D$ is an interval graph. Thus, to conclude the proof, it remains to show that $\sum_{v \in \mathsf{prob}(T)} (\mathsf{pack}(v) + 1)^2 = \mathcal{O}(k^2)$. Since for all $v \in \mathsf{prob}(T)$, $\mathsf{pack}(v) \geq 1$, it is sufficient to show that $\sum_{v \in \mathsf{prob}(T)} (\mathsf{pack}(v))^2 = \mathcal{O}(k^2)$. Recall that $\sum_{v \in \mathsf{prob}(T)} \mathsf{pack}(v) < 2k$. Thus, $\sum_{v \in \mathsf{prob}(T)} (\mathsf{pack}(v))^2 \leq (\sum_{v \in \mathsf{prob}(T)} \mathsf{pack}(v)) \cdot (\sum_{v \in \mathsf{prob}(T)} \mathsf{pack}(v)) < 2k \cdot 2k = \mathcal{O}(k^2)$. This completes the proof. ◀

5 Putting It All Together

Finally, we are ready to prove our main theorem.

Proof of Theorem 1. By Corollary 2, at least one of the following conditions hold: **(i)** G has k vertex-disjoint obstructions; **(ii)** there exists a subset $D' \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D'$ is a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. In the first case, our proof is complete, and thus we next suppose that the second case applies. Then, by Lemma 40, at least one of the following conditions hold: **(i)** $G - D'$ has k vertex-disjoint obstructions; **(ii)** there exists a subset $\widehat{D} \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $(G - D') - \widehat{D}$ is an interval graph. In the first case, our proof is complete. In the second case, we have that $D = D' \cup \widehat{D}$ is a set of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph, which again completes the proof. ◀

Before we turn to prove a corollary of our main theorem, we need one more proposition.

▶ **Proposition 41** ([10]). *There exists an $\mathcal{O}(nm)$-time algorithm that, given a graph G, outputs an integer d' such that the following conditions hold:* **(i)** *there exists a subset $D' \subseteq V(G)$ of size at most d' such that $G - D'$ is an interval graph;* **(ii)** $d' \leq 8d$ *for the integer d that is the minimum size of a subset $D \subseteq V(G)$ such that $G - D$ is an interval graph.*

As a consequence of Theorem 1 and Proposition 41, we derive the following corollary.

▶ **Corollary 42** (*). *There exist a constant $c \in \mathbb{N}$ and an $\mathcal{O}(nm)$-time algorithm that, given a graph G and an integer $k \in \mathbb{N}$, correctly concludes which one of the following conditions holds:*[8] **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $ck^2 \log k$ such that $G - D$ is an interval graph.*

References

1. Noga Alon. Covering a hypergraph of subgraphs. *Discrete Mathematics*, 257(2-3):249–254, 2002.
2. Saeed Akhoondian Amiri, Ken-ichi Kawarabayashi, Stephan Kreutzer, and Paul Wollan. The erdos-posa property for directed graphs. *CoRR*, abs/1603.02504, 2016. `arXiv:1603.02504`.
3. Julio Aracena, Jacques Demongeot, and Eric Goles. Positive and negative circuits in discrete neural networks. *IEEE Transactions on Neural Networks*, 15(1):77–83, 2004.
4. Etienne Birmelé, J Adrian Bondy, and Bruce A Reed. The Erdős–Pósa property for long circuits. *Combinatorica*, 27(2):135–145, 2007.
5. John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*, volume 290. Citeseer, 1976.
6. Nicolas Bousquet. *Hitting sets: VC-dimension and Multicut.* PhD thesis, Université Montpellier II-Sciences et Techniques du Languedoc, 2013.
7. Nicolas Bousquet and Stéphan Thomassé. Vc-dimension and Erdős–Pósa property. *Discrete Mathematics*, 338(12):2302–2317, 2015.
8. Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey.* SIAM, 1999.
9. Henning Bruhn, Felix Joos, and Oliver Schaudt. Long cycles through prescribed vertices have the Erdős–Pósa property. *Journal of Graph Theory*, 2017.
10. Yixin Cao. Linear recognition of almost interval graphs. In Robert Krauthgamer, editor, *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 1096–1115. SIAM, 2016. `doi:10.1137/1.9781611974331.ch77`.
11. Yixin Cao and Dániel Marx. Interval deletion is fixed-parameter tractable. *ACM Trans. Algorithms*, 11(3):21:1–21:35, 2015. `doi:10.1145/2629595`.
12. Chandra Chekuri and Julia Chuzhoy. Large-treewidth graph decompositions and applications. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC)*, pages 291–300, 2013.
13. Reinhard Diestel. *Graph Theory.* Springer-Verlag, Heidelberg, 4th edition, 2010.
14. P Erdős and L Pósa. On independent circuits contained in a graph. *Canad. J. Math.*, 17:347–352, 1965.
15. Samuel Fiorini, Nadia Hardy, Bruce Reed, and Adrian Vetta. Approximate min–max relations for odd cycles in planar graphs. *Mathematical programming*, 110(1):71–91, 2007.

[8] In particular, at least one condition holds, and if both conditions hold, then the algorithm can choose either of the two conditions.

16 Jim Geelen and Kasper Kabell. The Erdős–Pósa property for matroid circuits. *Journal of Combinatorial Theory, Series B*, 99(2):407–419, 2009.

17 Archontia C. Giannopoulou, O-joung Kwon, Jean-Florent Raymond, and Dimitrios M. Thilikos. Packing and covering immersion-expansions of planar sub-cubic graphs. *Eur. J. Comb.*, 65:154–167, 2017. `doi:10.1016/j.ejc.2017.05.009`.

18 Martin Charles Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, 1980.

19 Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012.

20 Bertrand Guenin and Robin Thomas. Packing directed circuits exactly. *Combinatorica*, 31(4):397–421, 2011.

21 András Gyárfás and Jenö Lehel. A helly-type problem in trees. In *Colloquia mathematica Societatis Jànos Bolyai*, volume 4, pages 571–584, 1970.

22 Frédéric Havet and Ana Karolinna Maia. On disjoint directed cycles with prescribed minimum lengths. *Research Report RR-8286, INRIA*, 2013.

23 Winfried Hochstättler, Robert Nickel, and Britta Peis. Two disjoint negative cycles in a signed graph. *Electronic Notes in Discrete Mathematics*, 25:107–111, 2006.

24 Felix Joos. Parity linkage and the erdős-pósa property of odd cycles through prescribed vertices in highly connected graphs. *Journal of Graph Theory*, 85(4):747–758, 2017. `doi:10.1002/jgt.22103`.

25 O joung Kwon and Eun Jung Kim. Erdős–Pósa property of chordless cycles and its application. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, To Appear*, 2018.

26 Naonori Kakimura and Ken-ichi Kawarabayashi. Fixed-parameter tractability for subset feedback set problems with parity constraints. *Theoretical Computer Science*, 576:61–76, 2015.

27 Naonori Kakimura, Ken-ichi Kawarabayashi, and Dániel Marx. Packing cycles through prescribed vertices. *Journal of Combinatorial Theory, Series B*, 101(5):378–381, 2011.

28 Ken-ichi Kawarabayashi and Bruce Reed. Highly parity linked graphs. *Combinatorica*, 29(2):215–225, 2009.

29 Ken-ichi Kawarabayashi and Paul Wollan. Non-zero disjoint cycles in highly connected group labelled graphs. *Journal of Combinatorial Theory, Series B*, 96(2):296–301, 2006.

30 Dénes König. Über graphen und ihre anwendung auf determinantentheorie und mengenlehre. *Mathematische Annalen*, 77(4):453–465, 1916.

31 Chun-Hung Liu. Packing and covering immersions in 4-edge-connected graphs. *CoRR*, abs/1505.00867, 2015. URL: `https://arxiv.org/abs/1505.00867`.

32 Claudio L Lucchesi and DH Younger. A minimax theorem for directed graphs. *Journal of the London Mathematical Society*, 2(3):369–374, 1978.

33 Karl Menger. Zur allgemeinen kurventheorie. *Fund. Math.*, 10(4):96–115, 1927.

34 Frank Mousset, Andreas Noever, Nemanja Škorić, and Felix Weissenberger. A tight Erdős–Pósa function for long cycles. *Journal of Combinatorial Theory, Series B*, 125:21–32, 2017.

35 Matteo Pontecorvi and Paul Wollan. Disjoint cycles intersecting a set of vertices. *Journal of Combinatorial Theory, Series B*, 102(5):1134–1141, 2012.

36 Dieter Rautenbach and Bruce Reed. The Erdős–Pósa property for odd cycles in highly connected graphs. *Combinatorica*, 21(2):267–278, 2001.

37 Jean-Florent Raymond and Dimitrios M Thilikos. Recent techniques and results on the Erdős–Pósa property. *Discrete Applied Mathematics*, 2017.

38 Bruce Reed. *Tree Width and Tangles: A New Connectivity Measure and Some Applications*, page 87–162. Cambridge University Press, 1997.

39 Bruce Reed. Mangoes and blueberries. *Combinatorica*, 19(2):267–296, 1999.

40 Bruce Reed, Neil Robertson, Paul Seymour, and Robin Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

41 Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Comb. Theory, Ser. B*, 62(2):323–348, 1994. doi:10.1006/jctb.1994.1073.

42 Neil Robertson and P.D Seymour. Graph minors. v. excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.

43 Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2002.

44 Paul D. Seymour. Packing circuits in eulerian digraphs. *Combinatorica*, 16(2):223–231, 1996.

45 Miklós Simonovits. A new proof and generalizations of a theorem of Erdős–Pósa on graphs without k+ 1 independent circuits. *Acta Mathematica Hungarica*, 18(1-2):191–206, 1967.

46 Mechthild Stoer. *Design of survivable networks*. Springer, 2006.

47 Marc Tedder, Derek G. Corneil, Michel Habib, and Christophe Paul. Simpler linear-time modular decomposition via recursive factorizing permutations. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfsdóttir, and Igor Walukiewicz, editors, *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Tack A: Algorithms, Automata, Complexity, and Games*, volume 5125 of *Lecture Notes in Computer Science*, pages 634–645. Springer, 2008. doi:10.1007/978-3-540-70575-8_52.

48 Carsten Thomassen. The Erdős–Pósa property for odd cycles in graphs of large connectivity. *Combinatorica*, 21(2):321–333, 2001.