The Completeness of BCD for an Operational Semantics

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Abstract

We give a completeness theorem for the BCD theory of intersection types in an operational semantics based on logical relations.

2012 ACM Subject Classification Theory of computation → Computability, Theory of computation → Lambda calculus

Keywords and phrases intersection types, operational semantics, Beth model, logical relations, forcing

Digital Object Identifier 10.4230/LIPIcs.TYPES.2016.15

1 Introduction

The theorem of Coppo, Dezani, and Pottinger ([3], [6])states that an untyped lambda term is strongly normalizable if and only if it provably has an intersection type. Here we consider which terms have which types.

We define an operational semantics for the collection of intersection types which assigns to every intersection type A a set of strongly normalizable terms [[A]]. We show that the theory of intersection types BCD (Barendregt, Coppo, Dezani) proves X : A for an untyped term X if and only if $X : [[A]]$ for all interpretations of $[[,]]$ in the operational semantics. Here we shall use the notation ":" for both the formal statement that X has type A, and also set theoretic membership.

Our view of what operational semantics should be begins with Tait style proofs ([8]) of strong normalization. These proofs consider a complete lattice of sets S of strongly normalizable untyped terms([2] 9.3). Not all such sets are considered but the lattice operations are union and intersection. We require that S is closed under reduction, and possibly some other conditions,such as head expansion with strong normalizable arguments, depending on the variant. The operation → is then introduced

$$S \to T = \{X | \text{for all } Y : S \Rightarrow (XY) : T\}.$$

This is certainly familiar from the theory of logical relations ([2] 3.3) for the simple typed case, positive recursive types, and our principal concern in this note; intersection types. Given an intersection type A, if the atoms (atomic types) of A are evaluated among the sets S then A has a value among the sets S. This will be the interpretation $[[A]]$.

2 Beth Models

SN is the set of strongly normalizable untyped terms. Here, we do not distinguish beta from beta-eta strong normalizability since they are equivalent. A Beth model consists of a pair (O, E) where O is a poset with partial order [, and E is a monotone map from O x Atoms

22nd International Conference on Types for Proofs and Programs (TYPES 2016).
Editors: Silvia Ghilezan, Herman Geuvers, and Jelena Ivetić; Article No. 15; pp. 15:1–15:5
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

into the subsets of SN closed under beta reduction and we shall assume that $E(p, a)$ is non empty except possibly when p is the [smallest element of O, should this exist.

For λ terms X we define the "forcing relation" $\models$ by

$p \models X : a$ if for all $q]p$ there exists $r]q$ $s.t.$ $X : E(r, a)$

$p \models X : A/\backslash B$ if $p \models X : Ap \models X : B$

$p \models X : A \to B$ if whenever $q]p$ and $q \models U : A$ there exists $r]q$ such that $r \models (XU) : B$

and we assume that E satisfies the generalized monotonicity property if $[Y/x]X : E(p, a), q]p$, and $q \models Y : A$ then there exists $r]q$ such that $(\backslash xXY) : E(r, a)$ where $[Y/x]$ is the the substitution operation (the term Y for the variable x).

► **Definition 1.** An O chain (linearly ordered subset) W is generic if

(i) for any X and atom a there exists $p : W$ such that either $X : E(p, a)$ or there is no $q]p$ such that $q \models X : a$, and

(ii) for each $A \to B$ there exists $p : W$ such that either $p \models X : A \to B$ or there exists U and $q : W$ such that $q]p$ and $q \models U : A$ but there is no $r]q$ such that $r \models (XU) : B$. We could just as easily use directed subsets of O instead of chains but chains suffice.
For what follows the reader should consult the definition of BCD in [2] which appears on pages 582-583,but without the top element (U_{top}). When we wish to include the top element, U_{top}, together with its axiom ([2] page 583), we will write $BCD + U_{top}$. Especially, the reader should look at the definitions of equality and the ordering of types on page 582. These are reproduced in the appendix.

Facts

1. if $p \models X : A$ and $q]p$ then $q \models X : A$
2. $p \models X : A$ iff for each $q]p$ there exists $r]q$ $s.t.$ $r \models X : A$
3. if $p \models X : A$ and

   $A < B$ or $A = B$ in BCD (page 582)

   then $p \models X : B$
4. if W is generic then for any X and atom a there exists $p : W$ such that for all $q]p$ we have $q \models X : a$ or there is no $q]p$ $s.t.$ $q \models X : a$
5. if W is generic then for any X and $A/\backslash B$ there exists a $p : W$ such that $p \models X : A/\backslash B$ or there is no $q]p$ such that $q \models X : A/\backslash B$
6. if W is an O chain with a maximal element then there exists a generic O chain extending W.

► **Proposition 2.** *Let W be a generic O chain and set $R(A) = \{X|$ there exists $p : W$ such that $p \models X : A\}$. Let $X : SN$ then*

(i) $X : R(a)$ iff *there exists* $p : W$ *s.t.* $X : E(p, a)$

(ii) $X : R(A \to B)$ iff *for each*
$U : R(A)$ *we have* $(XU) :$ R (B)

(iii) $X : R(A/\backslash B)$ iff $X : R(A)$ & $X : R(B)$

Proof. by induction on A. The basis case (i) is by definition. Induction step; Case (ii) $\Rightarrow$. Suppose that we have a $p : W$ such that $p \models X : A \Rightarrow B$ and we have $U : R(A)$. Thus there exists $q : W$ such that $q \models U : A$. By fact (1) we may assume that $q]p$. Now for any $r]q$

there exists $t]r$ such that $t \models (XU) : B$ but W is generic so there must be an $r : W$ such that $r \models (XU) : B$. That is $(XU) : R(B)$. $\Leftarrow$. Suppose that for each $U : R(A)$ we have $(XU) : R(B)$. Now if there is no $p : W$ such that $p \models X : A \to B$, since W is generic, there exists $p : W$ and aU such that $p \models U : A$ but there is no $q]p$ such that $q \models (XU) : B$. But by fact (1) this contradicts the hypothesis. Case (iii) similar to case (ii). ◀

The proposition clearly states that if the atoms a are evaluated $\{X|$ for some $p : W$ we have $X : E(p, a)\}$ then the value of the type A mentioned in the introduction is $R(A)$

▶ **Example 3** (finite sets)**.** In this case we let O be the collection of finite sets of SN terms closed under beta-eta reduction and ordered by inclusion. We set $E(p, a) = p$. Suppose that $A = A(1) \to (...(A(t) \to)...)$ and $\sim (p \models X : A)$. Then there exists $q]p$ and $Y(1), ..., Y(t)$ s, t $q \models Y(i) : A(i)$ for $i = 1, ..., t$ but there is no $r]q$ with $XY(1)...Y(t) : r$. But this can only be the case if $XY(1)...Y(t)$ is not SN. Thus we can find a generic W such that $X : R(A)$ or there exists $Y(1), ..., Y(t)$ $s.t.$ $Y(i) : R(A(i))$ for $i = 1, ..., t$ and $XY(1)...Y(t)$ is not SN.

▶ **Example 4.** In this case we consider sets $S : O$ of closed beta-eta normal terms for which there exists an integer n such that $X : S$ iff every path in the Bohm tree of X has at most n lambdas and every node in the Bohm tree ([1] pg 212) of X has at most n descendants. Then for any partial recursive function f which is total on S and maps S to S there exists $M : R(S \to S)$ such that for any $N : S$ we have $MN = f(N)$ modulo beta-eta conversion.

▶ **Proposition 5.** *Suppose that O has a smallest element 0. Then, $0 \models X : A$ iff for every generic W we have $X : R(A)$.*

Proof. Immediate by facts (1)–(6). ◀

We next consider the theory BCD with its provability relation $\vdash$ as described in [2] and reproduced in the appendix. A basis F is a map from a finite set of lambda calculus variables, $dom(F)$, to the set of types. Below we shall often conflate F with the finite set

$$\{x : F(x) | x : dom(F)\}.$$

Let O, E be as above and W generic.

▶ **Proposition 6** (soundness)**.** *Suppose that @ is a substitution and F is a base such that for all $x : dom(F), @(x) : R(F(x))$. Then if in BCD*

$$F \vdash X : A$$

we have $@(X) : R(A)$.

Now let O be the set of bases partially ordered by $F[G$ iff $dom(F)$ is contained in $dom(G)$ and for each $x : dom(F)$ we have $G(x)$ and $F(x)$ are equal types in BCD. Now define $E(a)$ by $X : E(F, a)$ if $FV(X)$ is contained in $dom(F)$ and $F \vdash X : a$. Clearly E is [monotone. In addition, $E(F, a)$ is closed under beta-eta reduction. However, generally $E(F, a)$ is not closed under beta head expansion for reasons similar to the case of BCD. In particular this happens when $(\backslash uUV)$ reduces to X, u is not free in U and there is an $x : FV(V)/\backslash FV(U)$ such that the basis entry $x : F(x)$ prevents V from having a BCD type. Thus we have to verify the generalized monotonicity property to insure soundness.First, we observe that there is no difference between E and $\models$ at atomic types.

Fact 7. $F \models X : a$ iff $X : E(F, a)$

Proof. If $FV(X)$ is contained in $dom(F)$ then the equivalence follows from the monotonicity of E and the weakening rule of BCD ([2] page 585). Otherwise suppose that $F \models X : a$. For each $x : FV(X) - dom(F)$ add a new atom $a(x)$ and extend F to G by $G(x) = a(x)$. Then $G \models X : a$ so by the previous argument $G \vdash X : a$ in BCD. But we may substitute U_{top} for each $a(x)$ and $(\backslash x.xx)(\backslash x.xx)$ for each x. So in $BCD + U_{top}$ we have

$$F \vdash [..., (\backslash x.xx)(\backslash x.xx)/x, ...]X : a$$

and this contradicts the fact that if a term has a BCD type (U_{top} free) in $BCD + U_{top}$ then it is strongly normalizable (theorem 17.2.15 (i) [2]). ◀

▶ **Lemma 7.** *Suppose that $FV(X)$ is contained in $dom(F)$.*

$$F \models X : A \text{ iff } F \vdash X : A \text{ in } BCD$$

Proof. this is proved by induction on A. The basis case is by fact 7. For the induction step the case $A = B/\backslash C$ is obvious. We consider the case $A = B \to C$. $\Rightarrow$. Suppose that $F \models X : B \to C$. Let z be a new variable and extend F by G by with $G(z) = B$. Since $G \models z : B$ by induction hypothesis $G \models z : B$ thus there exists $H]G$ such that $H \models Xz : C$. Again by induction hypothesis $H \vdash Xz : C$ in BCD. Reasoning in BCD, $H - \{z : B\} \vdash \backslash z(Xz) : B \to C$. Now by hypothesis $FV(X)$ is contained in $dom(F)$, so by weakening, $F \vdash \backslash z(Xz) : B \to C$. Hence by subject reduction for eta ([2] page 621)

$$F \vdash X : B \to C.$$

Conversely,suppose that $F \vdash X : A$. Let $G]F$ and $G \models U : B$. By induction hypothesis $G \vdash U : B$ in BCD Thus by induction hypothesis $G \models (XU) : C$. Hence

$$F \models X : B \to C.$$ ◀

▶ **Corollary 8.** *Generalized monotonicity holds and we have a Beth model.*

From the lemma we get the completeness theorem.

▶ **Theorem 9.** *Let M be closed. Then $BCD \vdash M : A$ iff for every Beth model (O, E) and generic $W, M : R(A)$.*

Proof. $\Rightarrow$. This is the soundness proposition. $\Leftarrow$. Consider the Beth model defined by the conditions above. By proposition 2 $0 \models M : A$. Hence by the lemma $\vdash M : A$ in BCD. ◀

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A Appendix

(1) terms and types
variables $x, y, z, \ldots$ are terms
if X and Y are terms then so are (XY) and $\backslash xX$
atoms $a, b, c, \ldots$ are types
if A and B are types then so are $A/\backslash B$ and $A \rightarrow B$

(2) (quasi) order on types
A less than or equal A
$A/\backslash B$ less than or equal A
$A/\backslash B$ less than or equal B
$(A \rightarrow B)/\backslash(A \rightarrow C)$ less than or equal $A \rightarrow (B/\backslash C)$
if C less than or equal A and C less than or equal B
then C less than or equal $A/\backslash B$
if C less than or equal B and B less than or equal A
then C less than or equal A
if A less than or equal C and D less than or equal B
then $C \rightarrow D$ less than or equal $A \rightarrow B$
A equals B if A less than or equal B and
B less than or equal A

(3) axioms and rules of BCD
$F \vdash x : A$ if $(x : A)$ belongs to F
if $F, x : A \vdash X : B$ then $F \vdash \backslash xX : A \rightarrow B$
if $F \vdash X : A \rightarrow B$ and $F \vdash Y : A$ then $F \vdash (XY) : B$
if $F \vdash X : A$ and $F \vdash X : B$ then $F \vdash X : A/\backslash B$
if $F \vdash X : A$ and A less then or equal B in BCD
then $F \vdash X : B$