The $\mathbb{Z}_2$-Genus of Kuratowski Minors

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Abstract

A drawing of a graph on a surface is *independently even* if every pair of nonadjacent edges in the drawing crosses an even number of times. The $\mathbb{Z}_2$*-genus* of a graph G is the minimum g such that G has an independently even drawing on the orientable surface of genus g. An unpublished result by Robertson and Seymour implies that for every t, every graph of sufficiently large genus contains as a minor a projective $t \times t$ grid or one of the following so-called t*-Kuratowski graphs*: $K_{3,t}$, or t copies of K_5 or $K_{3,3}$ sharing at most 2 common vertices. We show that the $\mathbb{Z}_2$-genus of graphs in these families is unbounded in t; in fact, equal to their genus. Together, this implies that the genus of a graph is bounded from above by a function of its $\mathbb{Z}_2$-genus, solving a problem posed by Schaefer and Štefankovič, and giving an approximate version of the Hanani–Tutte theorem on orientable surfaces.

2012 ACM Subject Classification Mathematics of computing → Graphs and surfaces

Keywords and phrases Hanani–Tutte theorem, genus of a graph, $\mathbb{Z}_2$-genus of a graph, Kuratowski graph

Digital Object Identifier 10.4230/LIPIcs.SoCG.2018.40

Related Version A full version of this paper is available at `https://arxiv.org/abs/1803.05085`

Acknowledgements The research was partially performed during the BIRS workshop "Geometric and Structural Graph Theory" (17w5154) in August 2017 and during a workshop on topological combinatorics organized by Arnaud de Mesmay and Xavier Goaoc in September 2017. We thank Zdeněk Dvořák, Xavier Goaoc and Pavel Paták for helpful discussions. We also thank Bojan Mohar, Paul Seymour, Gelasio Salazar, Jim Geelen and John Maharry for information about their unpublished results related to Claim 5.

1 Introduction

The *genus* $\mathrm{g}(G)$ of a graph G is the minimum g such that G has an embedding on the orientable surface M_g of genus g. We say that two edges in a graph are *independent* (also

[1] Supported by Austrian Science Fund (FWF): M2281-N35.
[2] Supported by project 16-01602Y of the Czech Science Foundation (GAČR), by the Czech-French collaboration project EMBEDS II (CZ: 7AMB17FR029, FR: 38087RM) and by Charles University project UNCE/SCI/004.

34th International Symposium on Computational Geometry (SoCG 2018).
Editors: Bettina Speckmann and Csaba D. Tóth; Article No. 40; pp. 40:1–40:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

nonadjacent) if they do not share a vertex. The $\mathbb{Z}_2$*-genus* $\mathrm{g}_0(G)$ of a graph G is the minimum g such that G has a drawing on M_g with every pair of independent edges crossing an even number of times. Clearly, every graph G satisfies $\mathrm{g}_0(G) \leq \mathrm{g}(G)$.

The Hanani–Tutte theorem [13, 24] states that $\mathrm{g}_0(G) = 0$ implies $\mathrm{g}(G) = 0$. The theorem is usually stated in the following form, with the optional adjective "strong".

▶ **Theorem 1** (The (strong) Hanani–Tutte theorem [13, 24])**.** *A graph is planar if it can be drawn in the plane so that no pair of independent edges crosses an odd number of times.*

Theorem 1 gives an interesting algebraic characterization of planar graphs that can be used to construct a simple polynomial algorithm for planarity testing [21].

Pelsmajer, Schaefer and Stasi [17] extended the strong Hanani–Tutte theorem to the projective plane, using the list of minimal forbidden minors. Colin de Verdière et al. [7] recently provided an alternative proof, which does not rely on the list of forbidden minors.

▶ **Theorem 2** (The (strong) Hanani–Tutte theorem on the projective plane [7, 17])**.** *If a graph G has a drawing on the projective plane such that every pair of independent edges crosses an even number of times, then G has an embedding on the projective plane.*

Whether the strong Hanani–Tutte theorem can be extended to some other surface than the plane or the projective plane has been an open problem. Schaefer and Štefankovič [22] conjectured that $\mathrm{g}_0(G) = \mathrm{g}(G)$ for every graph G and showed that a minimal counterexample to the extension of the strong Hanani–theorem on any surface must be 2-connected. Recently, a counterexample has been found on the orientable surface of genus 4 [11].

▶ **Theorem 3** ([11])**.** *There is a graph G with $\mathrm{g}(G) = 5$ and $\mathrm{g}_0(G) \leq 4$. Consequently, for every positive integer k there is a graph G with $\mathrm{g}(G) = 5k$ and $\mathrm{g}_0(G) \leq 4k$.*

Schaefer and Štefankovič [22] also posed the following natural question.

▶ **Problem 1** ([22])**.** *Is there a function f such that $\mathrm{g}(G) \leq f(\mathrm{g}_0(G))$ for every graph G?*

We give a positive answer to Problem 1 for several families of graphs, which we conjectured to be "unavoidable" as minors in graphs of large genus. Recently we have found that a similar Ramsey-type statement is a folklore unpublished result in the graph-minors community. Together, these results would imply a positive solution to Problem 1 for all graphs. We state the results in detail in Sections 3 and 4 after giving necessary definitions in Section 2.

2 Preliminaries

2.1 Graphs on surfaces

We refer to the monograph by Mohar and Thomassen [16] for a detailed introduction into surfaces and graph embeddings. By a *surface* we mean a connected compact 2-dimensional topological manifold. Every surface is either *orientable* (has two sides) or *nonorientable* (has only one side). Every orientable surface S is obtained from the sphere by attaching $g \geq 0$ *handles*, and this number g is called the *genus* of S. Similarly, every nonorientable surface S is obtained from the sphere by attaching $g \geq 1$ *crosscaps*, and this number g is called the *(nonorientable) genus* of S. The simplest orientable surfaces are the sphere (with genus 0) and the torus (with genus 1). The simplest nonorientable surfaces are the projective plane (with genus 1) and the Klein bottle (with genus 2). We denote the orientable surface of genus g by M_g, and the nonorientable surface of genus g by N_g.

Let $G = (V, E)$ be a graph or a multigraph with no loops, and let S be a surface. A *drawing* of G on S is a representation of G where every vertex is represented by a unique point in S and every edge e joining vertices u and v is represented by a simple curve in S joining the two points that represent u and v. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts. We assume that in a drawing no edge passes through a vertex, no two edges touch, every edge has only finitely many intersection points with other edges and no three edges cross at the same inner point. In particular, every common point of two edges is either their common endpoint or a crossing.

A drawing of G on S is an *embedding* if no two edges cross. A *face* of an embedding of G on S is a connected component of the topological space obtained from S by removing all the edges and vertices of G. A 2*-cell embedding* is an embedding whose each face is homeomorphic to an open disc. The *facewidth* (also called *representativity*) $\mathrm{fw}(\mathcal{E})$ of an embedding $\mathcal{E}$ on a surface S of positive genus is the smallest nonnegative integer k such that there is a closed noncontractible curve in S intersecting $\mathcal{E}$ in k vertices.

The *rotation* of a vertex v in a drawing of G on an orientable surface is the clockwise cyclic order of the edges incident to v. We will represent the rotation of v by the cyclic order of the other endpoints of the edges incident to v. The *rotation system* of a drawing is the set of rotations of all vertices.

The *Euler characteristic* of a surface S of genus g, denoted by $\chi(S)$, is defined as $\chi(S) = 2 - 2g$ if S is orientable, and $\chi(S) = 2 - g$ if S is nonorientable. Equivalently, if v, e and f denote the numbers of vertices, edges and faces, respectively, of a 2-cell embedding of a graph on S, then $\chi(S) = v - e + f$. The *Euler genus* $\mathrm{eg}(S)$ of S is defined as $2 - \chi(S)$. In other words, the Euler genus of S is equal to the genus of S if S is nonorientable, and to twice the genus of S if S is orientable.

An edge in a drawing is *even* if it crosses every other edge an even number of times. A drawing of a graph is *even* if all its edges are even. A drawing of a graph is *independently even* if every pair of independent edges in the drawing crosses an even number of times. In the literature, the notion of $\mathbb{Z}_2$*-embedding* is used to denote both an even drawing [5] and an independently even drawing [22].

The *genus* $\mathrm{g}(G)$ and the $\mathbb{Z}_2$*-genus* $\mathrm{g}_0(G)$ of a graph G have been defined in the introduction, as parameters related to drawings on orientable surfaces. The following two "Euler" analogues involve drawings on both orientable and nonorientable surfaces. The *Euler genus* $\mathrm{eg}(G)$ of G is the minimum g such that G has an embedding on a surface of Euler genus g. The *Euler* $\mathbb{Z}_2$*-genus* $\mathrm{eg}_0(G)$ of G is the minimum g such that G has an independently even drawing on a surface of Euler genus g.

The *embedding scheme* of a drawing $\mathcal{D}$ on a surface S consists of the rotation system and a signature $+1$ or -1 assigned to every edge, representing the parity of the number of crosscaps the edge is passing through. If S is orientable, the embedding scheme can be given just by the rotation system. The following weak analogue of the Hanani–Tutte theorem was proved by Cairns and Nikolayevsky [5] for orientable surfaces and then extended by Pelsmajer, Schaefer and Štefankovič [18] to nonorientable surfaces.

▶ **Theorem 4** (The weak Hanani–Tutte theorem on surfaces [5, Lemma 3], [18, Theorem 3.2])**.** *If a graph G has an even drawing $\mathcal{D}$ on a surface S, then G has an embedding on S that preserves the embedding scheme of $\mathcal{D}$.*

A simple closed curve γ in a surface S is 1*-sided* if it has a small neighborhood homeomorphic to the Möbius strip, and 2*-sided* if it has a small neighborhood homeomorphic to

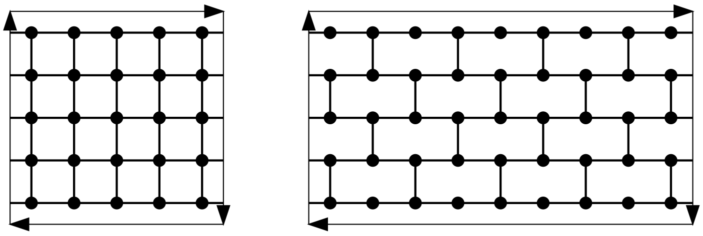

Figure 1 Left: a projective 5×5 grid. Right: a projective 5-wall.

the cylinder. We say that γ is *separating* in S if the complement $S \setminus \gamma$ has two components, and *nonseparating* if $S \setminus \gamma$ is connected. Note that on an orientable surface every simple closed curve is 2-sided, and every 1-sided simple closed curve (on a nonorientable surface) is nonseparating.

2.2 Special graphs

2.2.1 Projective grids and walls

For a positive integer n we denote the set $\{1, \dots, n\}$ by $[n]$. Let $r, s \geq 3$. The *projective* $r \times s$ *grid* is the graph with vertex set $[r] \times [s]$ and edge set

$$\{\{(i,j),(i',j')\}; |i-i'|+|j-j'|=1\} \ \cup \ \{\{(i,1),(r+1-i,s)\}; i \in [r]\}.$$

In other words, the projective $r \times s$ grid is obtained from the planar $r \times (s+1)$ grid by identifying pairs of opposite vertices and edges in its leftmost and rightmost column. See Figure 1, left. The projective $t \times t$ grid has an embedding on the projective plane with facewidth t. By the result of Robertson and Vitray [20], [16, p. 171], the embedding is unique if $t \geq 4$. Hence, for $t \geq 4$ the genus of the projective $t \times t$ grid is equal to $\lfloor t/2 \rfloor$ by the result of Fiedler, Huneke, Richter and Robertson [10], [16, Theorem 5.8.1].

Since grids have vertices of degree 4, it is more convenient for us to consider their subgraphs of maximum degree 3, called walls. For an odd $t \geq 3$, a *projective* t*-wall* is obtained from the projective $t \times (2t-1)$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for i odd and $1 \leq j \leq t-1$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for i even and $1 \leq j \leq t$. Similarly, for an even $t \geq 4$, a *projective* t*-wall* is obtained from the projective $t \times 2t$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for i odd and $1 \leq j \leq t$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for i even and $1 \leq j \leq t$. The projective t-wall has maximum degree 3 and can be embedded on the projective plane as a "twisted wall" with inner faces bounded by 6-cycles forming the "bricks", and with the "outer" face bounded by a $(4t-2)$-cycle for t odd and a $4t$-cycle for t even. See Figure 1, right. This embedding has facewidth t and so again, for $t \geq 4$ the projective t-wall has genus $\lfloor t/2 \rfloor$. It is easy to see that the projective 3-wall has genus 1 since it contains a subdivision of $K_{3,3}$ and embeds on the torus.

2.2.2 Kuratowski graphs

A graph is called a t*-Kuratowski* graph [23] if it is one of the following:

(a) $K_{3,t}$,
(b) a disjoint union of t copies of K_5,
(c) a disjoint union of t copies of $K_{3,3}$,
(d) a graph obtained from t copies of K_5 by identifying one vertex from each copy to a single common vertex,

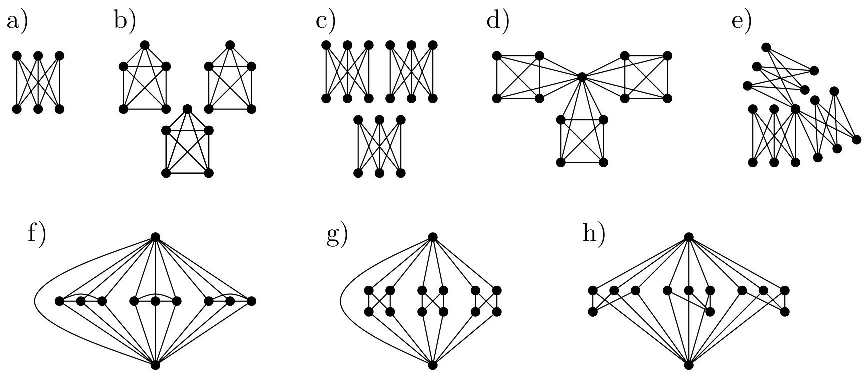

Figure 2 The eight 3-Kuratowski graphs.

(e) a graph obtained from t copies of $K_{3,3}$ by identifying one vertex from each copy to a single common vertex,
(f) a graph obtained from t copies of K_5 by identifying a pair of vertices from each copy to a common pair of vertices,
(g) a graph obtained from t copies of $K_{3,3}$ by identifying a pair of adjacent vertices from each copy to a common pair of vertices,
(h) a graph obtained from t copies of $K_{3,3}$ by identifying a pair of nonadjacent vertices from each copy to a common pair of vertices.

See Figure 2 for an illustration.

The genus of each of the t-Kuratowski graphs is known precisely. The genus of $K_{3,t}$ is $\lceil (t-2)/4 \rceil$ [4, 19], which coincides with the lower bound from Euler's formula. The genus of t copies of K_5 or $K_{3,3}$ sharing a vertex is t by the additivity of genus over blocks [1]. Finally, from a general formula by Decker, Glover and Huneke [9] it follows that the genus of t copies of K_5 or $K_{3,3}$ sharing a pair of adjacent or nonadjacent vertices is $\lceil t/2 \rceil$ if $t > 1$: cases f) and g) follow from their proof of Corollary 0.2, case h) follows from their Corollary 2.4 after realizing that $\mu(K_{3,3}) = 3$ if x, y are nonadjacent in $K_{3,3}$.

3 Ramsey-type results

The following Ramsey-type statement for graphs of large Euler genus is a folklore unpublished result.

▶ **Claim 5** (Robertson–Seymour [2, 23], unpublished)**.** *There is a function g such that for every $t \geq 3$, every graph of Euler genus $g(t)$ contains a t-Kuratowski graph as a minor.*

For 7-connected graphs, Claim 5 follows from the result of Böhme, Kawarabayashi, Maharry and Mohar [2], stating that for every positive integer t, every sufficiently large 7-connected graph contains $K_{3,t}$ as a minor. Böhme et al. [3] later generalized this to graphs of larger connectivity and $K_{a,t}$ minors for every fixed $a > 3$.

Richter and Salazar [6] proved a similar statement for graph-like continua.

We obtain an analogous Ramsey-type statement for graphs of large genus as an almost direct consequence of Claim 5.

▶ **Theorem 6.** *Claim 5 implies that there is a function h such that for every $t \geq 3$, every graph of genus $h(t)$ contains, as a minor, a t-Kuratowski graph or the projective t-wall.*

We give a detailed proof of Theorem 6 in the full version of this paper [12].

4 Our results

As our main result we complete a proof that the $\mathbb{Z}_2$-genus of each t-Kuratowski graph and the projective t-wall grows to infinity with t; in fact, the $\mathbb{Z}_2$-genus of each of these graphs is equal to their genus. Schaefer and Štefankovič [22] proved this for those t-Kuratowski graphs that consist of t copies of K_5 or $K_{3,3}$ sharing at most one vertex. For the projective t-wall, the result follows directly from the weak Hanani–Tutte theorem on orientable surfaces [5, Lemma 3]: indeed, all vertices of the projective t-wall have degree at most 3, therefore pairs of adjacent edges crossing oddly in an independently even drawing can be redrawn in a small neighborhood of their common vertex so that they cross evenly, and the weak Hanani–Tutte theorem can be applied. Thus, the remaining cases are t-Kuratowski graphs of type a), f), g) and h).

▶ **Theorem 7.** *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each t-Kuratowski graph of type a), f), g) and h) is equal to its genus. In particular,*

(a) $g_0(K_{3,t}) \geq \lceil (t-2)/4 \rceil$, *and*

(b) *if G consists of t copies of K_5 or $K_{3,3}$ sharing a pair of adjacent or nonadjacent vertices, then* $g_0(G) \geq \lceil t/2 \rceil$.

Combining Theorem 7 with the result of Schaefer and Štefankovič [22] and the simple argument for the projective t-wall we obtain the following result.

▶ **Corollary 8.** *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each t-Kuratowski graph and the projective t-wall is equal to its genus.*

Combining Corollary 8 with Theorem 6 we get the following implication.

▶ **Corollary 9.** *Claim 5 implies a positive answer to Problem 1.*

5 Lower bounds on the $\mathbb{Z}_2$-genus

In this section we prove Theorem 7 for the t-Kuratowski graphs of type a), f), g) and h).

The fact that the $\mathbb{Z}_2$-genus of $K_{3,t}$ or the other t-Kuratowski graphs is unbounded when t goes to infinity is not obvious at first sight. The traditional lower bound on the genus of $K_{3,t}$ relies on Euler's formula and the notion of a face. However, there is no analogue of a "face" in an independently even drawing, and the rotations of vertices no longer "matter". We thus need different tools to compute the $\mathbb{Z}_2$-genus.

5.1 $\mathbb{Z}_2$-homology of curves

We refer to Hatcher's textbook [14] for an excellent general introduction to homology theory. Unfortunately, we were unable to find a more compact treatment of the homology theory for curves on surfaces in the literature, thus we sketch here the main aspects that are most important for us.

We will use the $\mathbb{Z}_2$-homology of closed curves on surfaces. That is, for a given surface S, we are interested in its first homology group with coefficients in $\mathbb{Z}_2$, denoted by $H_1(S;\mathbb{Z}_2)$. It is well-known that for each $g \geq 0$, the first homology group $H_1(M_g;\mathbb{Z}_2)$ of M_g is isomorphic to $\mathbb{Z}_2^{2g}$ [14, Example 2A.2. and Corollary 3A.6.(b)]. This fact was crucial in establishing the weak Hanani–Tutte theorem on M_g [5, Lemma 3].

To every closed curve γ in M_g one can assign its homology class $[\gamma] \in H_1(S;\mathbb{Z}_2)$, and this assignment is invariant under continuous deformation (homotopy). In particular, the homology class of each contractible curve is 0. More generally, the homology class of each separating curve in M_g is 0 as well. Moreover, if γ is obtained by a composition of γ_1 and γ_2, the homology classes satisfy $[\gamma] = [\gamma_1] + [\gamma_2]$. The assignment of homology classes to closed curves is naturally extended to formal integer combinations of the closed curves, called *cycles*, and so $[\gamma]$ can be considered as a set of cycles. Since we are interested in homology with coefficients in $\mathbb{Z}_2$, it is sufficient to consider cycles with coefficients in $\mathbb{Z}_2$, which may also be regarded as finite sets of closed curves.

If γ_1 and γ_2 are cycles in M_g that cross in finitely many points and have no other points in common, we denote by $\mathrm{cr}(\gamma_1, \gamma_2)$ the number of their common crossings. We use the following well-known fact, which may be seen as a consequence of the Jordan curve theorem.

▶ **Fact 10.** *Let $\gamma_1' \in [\gamma_1]$ and $\gamma_2' \in [\gamma_2]$ be a pair of cycles in M_g such that the intersection number $\mathrm{cr}(\gamma_1', \gamma_2')$ is defined and is finite. Then*

$$\mathrm{cr}(\gamma_1', \gamma_2') \equiv \mathrm{cr}(\gamma_1, \gamma_2) \pmod 2.$$

Fact 10 allows us to define a group homomorphism (which is also a bilinear form)

$$\Omega_{M_g} : H_1(M_g; \mathbb{Z}_2) \times H_1(M_g; \mathbb{Z}_2) \to \mathbb{Z}_2$$

such that

$$\Omega_{M_g}([\gamma_1], [\gamma_2]) = \mathrm{cr}(\gamma_1, \gamma_2) \bmod 2$$

whenever $\mathrm{cr}(\gamma_1, \gamma_2)$ is defined and is finite. Cairns and Nikolayevsky [5] call Ω_{M_g} the *intersection form* on M_g. Clearly, Ω_{M_g} is symmetric and $\Omega_{M_g}([\gamma], [\gamma]) = 0$ for every cycle γ, since simple closed curves in M_g are 2-sided, and every closed curve with finitely many self-intersections is a composition of finitely many simple closed curves.

We have the following simple observation about intersections of disjoint cycles in independently even drawings.

▶ **Observation 11** ([22, Lemma 1]). *Let $\mathcal{D}$ be an independently even drawing of a graph G on M_g. Let C_1 and C_2 be vertex-disjoint cycles in G, and let γ_1 and γ_2 be the closed curves representing C_1 and C_2, respectively, in $\mathcal{D}$. Then $\mathrm{cr}(\gamma_1, \gamma_2) \equiv 0 \pmod 2$, which implies that $\Omega_{M_g}([\gamma_1], [\gamma_2]) = 0$.*

5.2 Combinatorial representation of the $\mathbb{Z}_2$-homology of drawings

Schaefer and Štefankovič [22] used the following combinatorial representation of drawings of graphs on M_g. First, every drawing of a graph on M_g can be considered as a drawing on the nonorientable surface N_{2g+1}, since M_g minus a point is homeomorphic to an open subset of N_{2g+1}. The surface N_{2g+1} minus a point can be represented combinatorially as the plane with $2g + 1$ *crosscaps*. A crosscap at a point x is a combinatorial representation of a Möbius strip whose boundary is identified with the boundary of a small circular hole centered in x. Informally, the main “objective” of a crosscap is to allow a set of curves intersect transversally at x without counting it as a crossing.

Every closed curve γ drawn in the plane with $2g + 1$ crosscaps is assigned a vector $y_\gamma \in \{0, 1\}^{2g+1}$ such that $(y_\gamma)_i = 1$ if and only if γ passes an odd number of times through the ith crosscap. When γ comes from a drawing on M_g, then y_γ has an even number of coordinates equal to 1. The vectors y_γ represent the elements of the homology group $H_1(M_g; \mathbb{Z}_2)$, and the value of the intersection form $\Omega_{M_g}([\gamma], [\gamma'])$ is equal to the scalar product

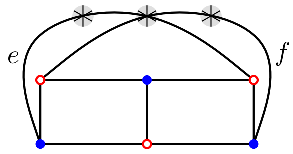

Figure 3 An embedding of $K_{3,3}$ on the torus represented as a drawing in the plane with three crosscaps. The nonzero vectors assigned to the edges are $y_e = (1,1,0)$ and $y_f = (0,1,1)$.

$y_\gamma^\top y_{\gamma'}$ over $\mathbb{Z}_2$. Analogously, we assign a vector y_e to every curve e representing an edge in a drawing of a graph in this model. See Figure 3.

We use the following two lemmata by Schaefer and Štefankovič [22].

▶ **Lemma 12** ([22, Lemma 5]). *Let G be a graph that has an independently even drawing $\mathcal{D}$ on M_g and let F be a forest in G. Then G has a drawing $\mathcal{E}$ in the plane with $2g+1$ crosscaps such that*

(1) *every pair of independent edges has an even number of common crossings outside the crosscaps, and*

(2) *every edge f of F passes through each crosscap an even number of times; that is, $y_f = 0$.*

Moreover, $\mathcal{E}$ can be obtained from $\mathcal{D}$ by a sequence of continuous deformations of edges and neighborhoods of vertices, so the homology classes of all cycles are preserved between the two drawings.

▶ **Lemma 13** ([22, Lemma 3]). *Let G be a graph that has a drawing in the plane with $2g+1$ crosscaps with every pair of independent edges having an even number of common crossings outside the crosscaps. Let d be the dimension of the vector space generated by the set $\{y_e;\ e \in E(G)\}$. Then G has an independently even drawing on $M_{\lfloor d/2 \rfloor}$.*

Lemma 12 and Lemma 13 imply the following corollary generalizing the strong Hanani–Tutte theorem. The proof appears in the full version of this paper.

▶ **Corollary 14.** *Let G be a connected graph with an independently even drawing on M_g such that each cycle in the drawing is homologically zero (that is, the homology class of the corresponding closed curve is 0). Then G is planar.*

Corollary 14 can be further strengthened using Lemma 12 as follows.

▶ **Lemma 15.** *Let G be a connected graph with an independently even drawing $\mathcal{D}$ on M_g. Let F be a spanning tree of G. If G is nonplanar, then there are independent edges $e, f \in E(G) \setminus E(F)$ such that the closed curves γ_e and γ_f representing the fundamental cycles of e and f, respectively, satisfy $\Omega_{M_g}([\gamma_e], [\gamma_f]) = 1$.*

Proof. Let $\mathcal{E}$ be a drawing of G from Lemma 12. By the strong Hanani–Tutte theorem, there are two independent edges e and f in G that cross an odd number of times in $\mathcal{E}$. Moreover, conditions 1) and 2) of Lemma 12 imply that none of the edges e and f is in F and so e and f cross an odd number of times in the crosscaps. This means that $y_e^\top y_f = 1$, which is equivalent to $\Omega_{M_g}([\gamma_e], [\gamma_f]) = 1$. ◀

5.3 Proof of Theorem 7a)

We will show three lower bounds on $\mathrm{g}_0(K_{3,t})$, in the order of increasing strength and complexity of their proof.

We will adopt the following notation for the vertices of $K_{3,t}$. The vertices of degree t forming one part of the bipartition are denoted by a, b, c, and the remaining vertices by $u_0, u_1, \dots, u_{t-1}$. Let $U = \{u_0, u_1, \dots, u_{t-1}\}$. For each $i \in [t-1]$, let C_i be the cycle au_ibu_0 and C'_i the cycle au_icu_0.

The first lower bound, $\mathrm{g}_0(K_{3,t}) \geq \Omega(\log\log\log t)$, follows from Ramsey's theorem and the weak Hanani–Tutte theorem on surfaces. The proof appears in the full version of this paper.

The second lower bound is based on the pigeonhole principle and Corollary 14 from the previous subsection.

▶ **Proposition 16.** *We have* $\mathrm{g}_0(K_{3,t}) \geq \Omega(\log t)$.

Proof. Let $\mathcal{D}$ be an independently even drawing of $K_{3,t}$ on M_g. By the pigeonhole principle, there is a subset $I_b \subseteq [t-1]$ of size at least $(t-1)/2^{2g}$ such that all the cycles C_i with $i \in I_b$ have the same homology class in $\mathcal{D}$. Analogously, there is a subset $I_c \subseteq I_b$ of size at least $|I_b|/2^{2g}$ such that all the cycles C'_i with $i \in I_c$ have the same homology class in $\mathcal{D}$. Suppose that $t \geq 2 \cdot 16^g + 2$. Then $|I_b| \geq 2 \cdot 4^g + 1$ and $|I_c| \geq 3$. Let $i, j, k \in I_c$ be three distinct integers. We now consider the subgraph H of $K_{3,t}$ induced by the vertices a, b, c, u_i, u_j, u_k, isomorphic to $K_{3,3}$, and show that all its cycles are homologically zero. Indeed, the cycle space of H is generated by the four cycles au_ibu_j, au_ibu_k, au_icu_j and au_icu_k, and each of them is the sum (mod 2) of two cycles of the same homology class: $au_ibu_j = C_i + C_j$, $au_ibu_k = C_i + C_k$, $au_icu_j = C'_i + C'_j$ and $au_icu_k = C'_i + C'_k$. Corollary 14 now implies that H is planar, but this is a contradiction. Therefore $t \leq 2 \cdot 16^g + 1$. ◀

To prove the lower bound in Theorem 7a), we use the same general idea as in the previous proof. However, we will need a more precise lemma about drawings of $K_{3,3}$, strengthening Corollary 14 and Lemma 15. We also replace the pigeonhole principle with a linear-algebraic trick.

▶ **Lemma 17.** *Let $\mathcal{D}$ be an independently even drawing of $K_{3,3}$ on M_g. For $i \in \{1, 2\}$, let γ_i and γ'_i be the closed curves representing the cycles C_i and C'_i, respectively, in $\mathcal{D}$. The intersection numbers of their homology classes satisfy*

$$\Omega_{M_g}([\gamma_1], [\gamma'_2]) + \Omega_{M_g}([\gamma'_1], [\gamma_2]) = 1.$$

Lemma 17 is a consequence of Corollary 23. In the full version of this paper we include a direct proof using a different method.

▶ **Proposition 18.** *We have* $\mathrm{g}_0(K_{3,t}) \geq \lceil (t-2)/4 \rceil$.

Proof. Let $\mathcal{D}$ be an independently even drawing of $K_{3,t}$ on M_g. For every $i \in [t-1]$, let γ_i and γ'_i be the closed curves representing the cycles C_i and C'_i, respectively, in $\mathcal{D}$. For every $i, j \in [t-1]$, $i < j$, we apply Lemma 17 to the drawing of $K_{3,3}$ induced by the vertices a, b, c, u_0, u_i, u_j in $\mathcal{D}$. Let A be the $(t-1) \times (t-1)$ matrix with entries

$$A_{i,j} = \Omega_{M_g}([\gamma_i], [\gamma'_j]).$$

Lemma 17 implies that $A_{i,j} + A_{j,i} = 1$ whenever $i \neq j$; in other words, A is a *tournament matrix* [8]. Repeating the argument by de Caen [8], it follows that $A + A^\top$, with the addition mod 2, is the matrix with zeros on the diagonal and 1-entries elsewhere. This implies that the rank of A over $\mathbb{Z}_2$ is at least $(t-2)/2$. Hence, the rank of Ω_{M_g} is at least $(t-2)/2$, which implies $2g \geq (t-2)/2$. ◀

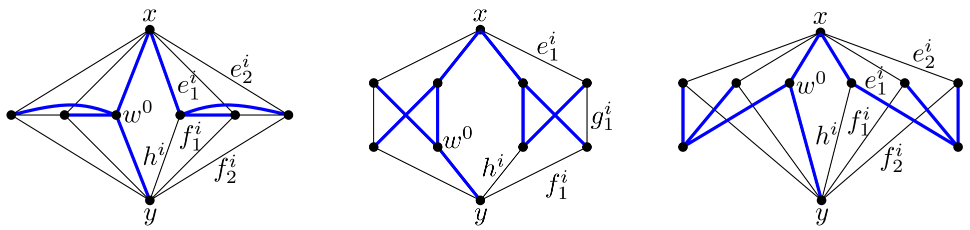

Figure 4 2-amalgamations of two Kuratowski xy-wings. The spanning tree T is drawn bold.

5.4 Proof of Theorem 7b)

Before proving Theorem 7b) we first show an asymptotic $\Omega(\log t)$ lower bound on the $\mathbb{Z}_2$-genus for a more general class of graphs that includes the t-Kuratowski graphs of types f), g) and h).

Let H be a 2-connected graph and let x, y be two nonadjacent vertices of H. Let t be a positive integer. The 2-*amalgamation* of t copies of H (with respect to x and y), denoted by $\amalg_{x,y} tH$, is the graph obtained from t disjoint copies of H by gluing all t copies of x into a single vertex and gluing all t copies of y into a single vertex. The two vertices obtained by gluing are again denoted by x and y.

An xy-*wing* is a 2-connected graph H with two nonadjacent vertices x and y such that the subgraph $H - x - y$ is connected, and the graph obtained from H by adding the edge xy is nonplanar. Clearly, the graphs $K_5 - e$ and $K_{3,3} - e$, where $e = xy$, are xy-wings, and similarly $K_{3,3}$, with nonadjacent vertices x and y, is an xy-wing. The t-Kuratowski graphs of types f) and g) are obtained from $\amalg_{x,y} t(K_5 - e)$ and $\amalg_{x,y} t(K_{3,3} - e)$, respectively, by adding the edge xy, whereas the t-Kuratowski graph of type h) is exactly the 2-amalgamation $\amalg_{x,y} t(K_{3,3})$. See Figure 4 for an illustration of 2-amalgamations of two xy-wings.

Let H be an xy-wing. We will use the following notation. Let w be a vertex of H adjacent to y and let F' be a spanning tree of $H - x - y$. Let F be a spanning tree of $H - y$ extending F'. In the 2-amalgamation $\amalg_{x,y} tH$ we distinguish the ith copy of H, its vertices, edges, and subgraphs, by the superscript $i \in \{0, 1, \ldots, t-1\}$. In particular, for every $i \in \{0, 1, \ldots, t-1\}$, H^i is an induced subgraph of $\amalg_{x,y} tH$, F^i is a spanning tree of $H^i - y$ and x is a leaf of F^i. For a given t, let

$$T = yw^0 + \bigcup_{i=0}^{t-1} F_i$$

be a spanning tree of $\amalg_{x,y} tH$. For every edge $e \in E(\amalg_{x,y} tH) \setminus E(T)$, let C_e be the fundamental cycle of e with respect to T; that is, the unique cycle in $T + e$.

Enumerate the edges of $E(H) \setminus E(F)$ incident to x as $e_1, \ldots, e_k$, the edges of $E(H) \setminus E(F) \setminus \{yw\}$ incident to y as $f_1, \ldots, f_l$, and the edges of $E(H - x - y) \setminus E(F)$ as $g_1, \ldots, g_m$. Let h be the edge yw. Thus, for every $i \in [t-1]$, we have $E(H^i) \setminus E(T) = \{e_1^i, \ldots, e_k^i\} \cup \{f_1^i, \ldots, f_l^i\} \cup \{g_1^i, \ldots, g_m^i\} \cup \{h^i\}$.

If C and C' are cycles in $\amalg_{x,y} tH$, we denote by $C + C'$ the element of the cycle space of $\amalg_{x,y} tH$ obtained by adding C and C' mod 2. We also regard $C + C'$ as a subgraph of $\amalg_{x,y} tH$ with no isolated vertices. Note that if C and C' are fundamental cycles sharing at least one edge then $C + C'$ is again a cycle.

▶ **Observation 19.** *Let $i \in [t-1]$.*

(a) *For every $j \in [k]$, the cycle $C_{e_j^i}$ is a subgraph of $H^i - y$.*

(b) *For every $j \in [l]$, the cycle $C_{f_j^i} + C_{h^i}$ is a subgraph of $H^i - x$.*
(c) *For every $j \in [m]$, the cycle $C_{g_j^i}$ is a subgraph of $H^i - x - y$.*
The cycles $C_{e_j^i}$ with $j \in [k]$, $C_{f_j^i} + C_{h^i}$ with $j \in [l]$, and $C_{g_j^i}$ with $j \in [m]$ generate the cycle space of H^i; in particular, they are the fundamental cycles of H^i with respect to the spanning tree $F^i + yw^i$. ◀

▶ **Corollary 20.** *Let $i, i' \in [t-1]$ be distinct indices. Then the following pairs of cycles are vertex-disjoint, for all possible pairs of indices j, j':*
(a) $C_{e_j^i}$ *and* $C_{f_{j'}^{i'}} + C_{h^{i'}}$,
(b) $C_{f_j^i} + C_{h^i}$ *and* $C_{g_{j'}^{i'}}$,
(c) $C_{e_j^i}$ *and* $C_{g_{j'}^{i'}}$,
(d) $C_{g_j^i}$ *and* $C_{g_{j'}^{i'}}$.

Our first lower bound on the $\mathbb{Z}_2$-genus of 2-amalgamations of xy-wings is similar to Proposition 16, and combines the pigeonhole principle and Corollary 14.

▶ **Proposition 21.** *Let H be an xy-wing. Then $g_0(\amalg_{x,y} tH) \geq \Omega(\log t)$.*

Proof. Let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y} tH$ on M_g. For every $i \in [t-1]$ and $e \in E(H) \setminus E(F)$, let $\gamma(e^i)$ be the closed curve representing C_{e^i} in $\mathcal{D}$.

The homology class $[\gamma(e^i)]$ has one of 2^{2g} possible values in $H_1(M_g; \mathbb{Z}_2)$. Thus, if $t \geq 2^{2g(k+l+m+1)} + 2$, then there are distinct indices $i, i' \in [t-1]$ such that for every $e \in E(H) \setminus E(F)$ we have $[\gamma(e^i)] = [\gamma(e^{i'})]$. Combining this with Observation 11 and Corollary 20, for all possible pairs of indices j, j' we have

$$\Omega_{M_g}([\gamma(e_j^i)], [\gamma(f_{j'}^i)] + [\gamma(h^i)]) = \Omega_{M_g}([\gamma(e_j^i)], [\gamma(f_{j'}^{i'})] + [\gamma(h^{i'})]) = 0, \tag{1}$$

$$\Omega_{M_g}([\gamma(f_j^i)] + [\gamma(h^i)], [\gamma(g_{j'}^i)]) = \Omega_{M_g}([\gamma(f_j^i)] + [\gamma(h^i)], [\gamma(g_{j'}^{i'})]) = 0, \tag{2}$$

$$\Omega_{M_g}([\gamma(e_j^i)], [\gamma(g_{j'}^i)]) = \Omega_{M_g}([\gamma(e_j^i)], [\gamma(g_{j'}^{i'})]) = 0, \tag{3}$$

$$\Omega_{M_g}([\gamma(g_j^i)], [\gamma(g_{j'}^i)]) = \Omega_{M_g}([\gamma(g_j^i)], [\gamma(g_{j'}^{i'})]) = 0. \tag{4}$$

Let $H^{i,i'}$ be the union of the graph H^i with the unique xy-path $P^{i'}$ in $F^{i'} + yw^{i'}$. Since H is an xy-wing, the graph $H^{i,i'}$ is nonplanar. The graph $F^{i,i'} = F^i \cup P^{i'}$ is a spanning tree of $H^{i,i'}$, and $E(H^{i,i'}) \setminus E(F^{i,i'}) = E(H^i) \setminus E(T)$.

The fundamental cycle C'_{h^i} of h^i in $H^{i,i'}$ with respect to $F^{i,i'}$ is equal to $C_{h^i} + C_{h^{i'}}$. Since $[\gamma(h^i)] = [\gamma(h^{i'})]$, the cycle C'_{h^i} is homologically zero.

For every $j \in [k]$, the fundamental cycle of e_j^i in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{e_j^i}$ and its homology class in $\mathcal{D}$ is $[\gamma(e_j^i)]$.

For every $j \in [l]$, the fundamental cycle of f_j^i in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{f_j^i} + C_{h^{i'}}$ and its homology class is $[\gamma(f_{j'}^i)] + [\gamma(h^{i'})] = [\gamma(f_{j'}^i)] + [\gamma(h^i)]$.

For every $j \in [m]$, the fundamental cycle of g_j^i in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{g_j^i}$ and its homology class in $\mathcal{D}$ is $[\gamma(g_j^i)]$.

By (1)–(4), for every pair of independent edges in $E(H^{i,i'}) \setminus E(F^{i,i'})$, the homology classes of their fundamental cycles are orthogonal with respect to Ω_{M_g}. This is a contradiction with Lemma 15 applied to $H^{i,i'}$ and the spanning tree $F^{i,i'}$. Therefore, $t \leq 2^{2g(k+l+m+1)} + 1$. ◀

To prove the lower bound in Theorem 7b), we follow the idea of the previous proof and again replace the pigeonhole principle with a linear-algebraic argument. We will also need a stronger variant of the Hanani–Tutte theorem and Lemma 15 for the graphs K_5 and $K_{3,3}$.

▶ **Lemma 22** (Kleitman [15]). *In every drawing of K_5 and $K_{3,3}$ in the plane the total number of pairs of independent edges crossing an odd number of times is odd.*

▶ **Corollary 23.** *Let $G = K_5$ or $G = K_{3,3}$. Let F be a forest in G. Let $\mathcal{E}$ be a drawing of G from Lemma 12. Then there are an odd number of pairs of independent edges e, f in $E(G) \setminus E(F)$ such that $y_e^\top y_f = 1$.* ◀

The following simple fact is a key ingredient in the proof of Lemma 22.

▶ **Observation 24.** *The graph obtained from each of K_5 and $K_{3,3}$ by removing an arbitrary pair of adjacent vertices is a cycle; in particular, all of its vertices have an even degree.* ◀

An xy-wing H is called a *Kuratowski xy-wing* if H is one of the graphs $K_5 - e$ where $e = xy$, $K_{3,3} - e$ where $e = xy$, or $K_{3,3}$; see Figure 4. Observation 24 implies the following important property of Kuratowski xy-wings.

▶ **Observation 25.** *Let H be a Kuratowski xy-wing and let u be a vertex adjacent to x in H. Then $H - x - u$ is a cycle; in particular, y is incident to exactly two edges in $H - x - u$.* ◀

In the following key lemma we keep using the notation for the 2-amalgamation $\amalg_{x,y} tH$ established earlier in this subsection.

▶ **Lemma 26.** *Let $t \geq 2$, let H be a Kuratowski xy-wing and let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y} tH$ on M_g. Then for every $i \in [0, t-1]$ the graph H^i has two cycles C_1^i and C_2^i such that*

- *(C_1^i is a subgraph of $H^i - x$ and C_2^i is a subgraph of $H^i - y$) or C_2^i is a subgraph of $H^i - x - y$, and*
- *the closed curves γ_1^i and γ_2^i representing C_1^i and C_2^i, respectively, in $\mathcal{D}$ satisfy $\Omega_{M_g}([\gamma_1^i], [\gamma_2^i]) = 1$.*

Proof. For every $i \in [t-1]$, let $H^{i,0}$ be the union of the graph H^i with the unique xy-path P^0 in $F^0 + yw^0$. The graph $F^{i,0} = F^i \cup P^0$ is a spanning tree of $H^{i,0}$, and $E(H^{i,0}) \setminus E(F^{i,0}) = E(H^i) \setminus E(T)$.

Let $\mathcal{E}$ be a drawing of G from Lemma 12. If $H = K_{3,3}$, we apply Corollary 23 to $G = H^i$ and $F = F^i$. If $H = K_5 - e$ or $H = K_{3,3} - e$ where $e = xy$, we apply Corollary 23 to $G = H^i + e$, $F = F^i + e$, and the drawing of $H^i + e$ where e is drawn along the path P^0 in $\mathcal{E}$ (with self-crossings removed if necessary). In each of the three cases at least one of the following alternatives occurs:

(1) $y_{e_j^i}^\top y_{g_{j'}^i} = 1$ for some $j \in [k]$ and $j' \in [m]$,
(2) $y_{f_j^i}^\top y_{g_{j'}^i} = 1$ for some $j \in [l]$ and $j' \in [m]$,
(3) $y_{h^i}^\top y_{g_{j'}^i} = 1$ for some $j' \in [m]$,
(4) $y_{g_{j'}^i}^\top y_{g_{j''}^i} = 1$ for some $j', j'' \in [m]$,
(5) $y_{e_j^i}^\top (y_{f_{j'}^i} + y_{f_{j''}^i}) = 1$ for some $j \in [k]$ and $j', j'' \in [l]$,
(6) $y_{e_j^i}^\top (y_{f_{j'}^i} + y_{h^i}) = 1$ for some $j \in [k]$ and $j' \in [l]$.

Here we used Observation 25 for each $j \in [k]$ to pair the edges of $E(H^i) \setminus E(T)$ incident with y and independent from e_j^i. We note that in each of the six alternatives the edges on the left side of the scalar product can be required to be independent from the edges on the right side; however, we do not use this fact in further arguments.

To finish the proof of the lemma for $i \in [t-1]$, we use Observation 19 together with the additional fact that for every $j', j'' \in [l]$, the cycle $C_{f_{j'}^i} + C_{f_{j''}^i}$ is a subgraph of $H^i - x$. In particular, in case 1) we choose $C_1^i = C_{g^i_{j'}}$ and $C_2^i = C_{e^i_j}$, in case 2) we choose $C_1^i = C_{f^i_j}$ and $C_2^i = C_{g^i_{j'}}$, in case 3) we choose $C_1^i = C_{h^i}$ and $C_2^i = C_{g^i_{j'}}$, in case 4) we choose $C_1^i = C_{g^i_{j'}}$ and $C_2^i = C_{g^i_{j''}}$, in case 5) we choose $C_1^i = C_{f^i_{j'}} + C_{f^i_{j''}}$ and $C_2^i = C_{e^i_j}$, and in case 6) we choose $C_1^i = C_{f^i_{j'}} + C_{h^i}$ and $C_2^i = C_{e^i_j}$.

Finally, by exchanging the roles of H^1 and H^0 in $\amalg_{x,y}tH$ in the proof, we also obtain cycles C_1^0 and C_2^0 with the required properties. ◀

We are now ready to finish the proof of Theorem 7b).

▶ **Proposition 27.** *Let $t \geq 2$ and let H be a Kuratowski xy-wing. Then $\mathrm{g}_0(\amalg_{x,y}tH) \geq \lceil t/2 \rceil$.*

Proof. Let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y}tH$ on M_g. For every $i \in [0, t-1]$, let C_1^i and C_2^i be the cycles from Lemma 26 and let γ_1^i and γ_2^i, respectively, be the closed curves representing them in $\mathcal{D}$.

Without loss of generality, we assume that there is an $s \in [0, t-1]$ such that

- for every $i \in [0, s]$, C_1^i is a subgraph of $H^i - x$ and C_2^i is a subgraph of $H^i - y$, and
- for every $i \in [s+1, t-1]$, the cycle C_2^i is a subgraph of $H^i - x - y$.

It follows that for distinct $i, i' \in [0, t]$, the cycles C_1^i and $C_2^{i'}$ are vertex-disjoint whenever $i, i' \in [0, s]$, $i, i' \in [s+1, t-1]$, or $i \leq s < i'$.

Let A be the $t \times t$ matrix with entries

$$A_{i,i'} = \Omega_{M_g}([\gamma_1^i], [\gamma_2^{i'}]).$$

By Lemma 26, Observation 11 and the previous discussion, the matrix A has 1-entries on the diagonal and 0-entries above the diagonal. Thus, the rank of A over $\mathbb{Z}_2$ is t. Hence, the rank of Ω_{M_g} is at least t, which implies $2g \geq t$. ◀

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