Brief Announcement: Treewidth Modulator: Emergency Exit for DFVS

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Abstract

In the Directed Feedback Vertex Set (DFVS) problem, we are given as input a directed graph D and an integer k, and the objective is to check whether there exists a set S of at most k vertices such that $F = D - S$ is a directed acyclic graph (DAG). Determining whether DFVS admits a polynomial kernel (parameterized by the solution size) is one of the most important open problems in parameterized complexity. In this article, we give a polynomial kernel for DFVS parameterized by the solution size plus the size of any treewidth-η modulator, for any positive integer η. We also give a polynomial kernel for the problem, which we call Vertex Deletion to treewidth-η DAG, where given as input a directed graph D and a positive integer k, the objective is to decide whether there exists a set of at most k vertices, say S, such that $D - S$ is a DAG and the treewidth[1] of $D - S$ is at most η.

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1 Introduction and Overview of Our Results

In the Directed Feedback Vertex Set (DFVS) problem, the input consists of a directed graph D on n vertices, and an integer k. The parameter is k, and the objective is to check whether there exists a set of at most k vertices, say S, such that $F = D - S$ is a directed acyclic graph (DAG). The question whether DFVS is fixed-parameter tractable was posed as an open problem in the first few papers on fixed-parameter tractability (FPT) [8, 9].

[1] Throughout the article, by treewidth of a directed graph we mean the treewidth of its underlying undirected graph.

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This remained an open problem for over a decade until, in a breakthrough paper, DFVS was shown to be fixed-parameter tractable by Chen et al. [5] in 2008. Specifically, they gave an algorithm that runs in time $\mathcal{O}(4^k \cdot k! \cdot k^4 \cdot n^4)$. Following the resolution of the fixed-parameter tractability status of DFVS, one of the most natural follow-up questions in parameterized complexity, that has been raised several times, and has become one of the most fundamental questions, is "does DFVS admit a polynomial kernel?" A polynomial kernel is essentially a polynomial-time preprocessing algorithm that transforms the given instance of the problem into an equivalent one whose size is bounded polynomially in the specified parameter. Whenever the parameter is not specified, it is implied that the parameter is the solution size (in our case, the integer k in the input of the DFVS problem). In an attempt to develop an understanding on what makes this problem hard, and to move closer to answering this open question, several routes have been taken. These include:

1. enriching the parameterization to encompass not only solution size but also additional structural parameters,
2. restricting the input instances,
3. restricting the structure of the resulting DAG (F).

In this article, we give two results concerning DFVS that contribute to progress along all these three routes. We begin by first stating our results formally. For a directed graph D, a subset $M \subseteq V(D)$ is called a *treewidth η-modulator* if $D - M$ has treewidth at most η. For a fixed positive integer $\eta > 0$, let $\mathcal{F}_\eta$ be the family of digraphs of treewidth at most η. Formally, our first problem is the following.

DFVS/DFVS+Treewidth-η Modulator (DFVS/DFVS+Tw-η Mod) **Parameter:** $k + \ell$
Input: A digraph D, an integer k, $M \subseteq V(D)$ such that $|M| = \ell$ and $D - M \in \mathcal{F}_\eta$.
Question: Does there exist $S \subseteq V(D)$ such that $|S| \leq k$ and $D - S$ is a DAG?

Our first result is the following.

▶ **Theorem 1.** DFVS/DFVS+Tw-η Mod *admits a polynomial kernel of size* $(k \cdot \ell)^{\mathcal{O}(\eta^2)}$.

Our second problem is the following.

Vertex Deletion to treewidth-η DAG **Parameter:** k
Input: A digraph D, an integer k.
Question: Does there exist $S \subseteq V(D)$ such that $|S| \leq k$ and $D - S$ is a DAG and $D - S \in \mathcal{F}_\eta$?

The next theorem states our second result.

▶ **Theorem 2.** *For any fixed positive integer η,* Vertex Deletion to treewidth-η DAG *has polynomial kernel.*

Let us now see how both our results make progress along all the three routes described above. Along the first route, Bergougnoux et al. [4] studied DFVS parameterized by the feedback vertex set (**fvs**) number of the underlying undirected graph, and gave a polynomial kernel for this problem. Our first result gives a polynomial kernel for DFVS when the parameter is solution size (k) plus the size of any treewidth-η modulator in D (say ℓ), for any fixed positive integer η. Note that the parameter $k + \ell$ is not only upper bounded by $\mathcal{O}(\textbf{fvs})$, where **fvs** is the feedback vertex set number of the underlying undirected graph of D, but it can be arbitrarily smaller than **fvs**. Thus, studying such a parameter brings us closer to the problem of the existence of a polynomial kernel for DFVS. (Note that the question of the existence of a polynomial kernel for DFVS parameterized by the size of a treewidth-η modulator, for $\eta \geq 2$, alone has a negative answer because VERTEX COVER parameterized by the size of any treewidth-2 modulator cannot have a polynomial kernel, unless $\mathsf{NP} \subseteq \frac{\text{co-NP}}{\text{poly}}$ [6].) Moreover, the ideas harnessed during the construction of our polynomial kernel utilizes the tool of important separators in a novel fashion. To the best of our knowledge, this is the first time that the power of important separators has been harnessed to develop a polynomial kernel. Furthermore, to derive this result, we need to embed this tool in state-of-the-art machinery such as the use of protrusion replacers where the replacement is a minor of the part of the graph that is replaced.

Along the second route (that is, studying DFVS by restricting the input instance), there have been several results for polynomial kernels for DFVS when the input graph is a tournament or some generalization of it (like a bipartite tournament etc.) [1, 3, 7, 10]. However, the existence of a polynomial kernel for DFVS is open even when the input digraph is a planar digraph. From our first result (Theorem 1), we can conclude that we have a polynomial kernel for DFVS when the treewidth of the input graph is polynomial in the solution size ($k^{\mathcal{O}(1)}$).

Along the third route, Mnich and van Leeuwen [11] studied the problem, where they considered DFVS with an additional restriction on the output DAG rather than the input instance. They inspected this question by considering k vertex deletion to the classes of out-forests, out-trees and (directed) pumpkins. They obtained polynomial kernels for all these problems. Observe that for all these classes, the treewidth of the graphs in these classes is constant (at most 2). In a follow-up paper [2], the kernel sizes given by Mnich and van Leeuwen [11] were reduced. Our second result generalizes this approach by demanding that the resulting DAG has bounded treewidth (bounded by any fixed constant η).

Our Methods. We now give a very brief overview of the methods used to prove our results. In fact, here, we only focus on our first result.

Proof Idea of Theorem 1. Our kernelization algorithm can be divided into three main phases. Recall that the input is a directed graph D, an integer k and a treewidth-η modulator M of size ℓ. In the first phase, we decompose the graph into $\mathcal{O}(k\ell^2)$ parts (which we call *zones*), each of which have constant treewidth and a "controlled" neighbourhood in the rest of the graph. In the next step, we mark $(k\ell)^{\mathcal{O}(\eta^2)}$ vertices inside each zone, which have the property that in the case of a YES-instance, there is also a solution that does not use any of the unmarked vertices in these zones. Getting such a set of marked vertices of polynomial size is the core of our algorithm. Having such a set of marked vertices at our disposal, we then design reduction rules that partition the unmarked vertices in each zone into disjoint protrusions that are then replaced by constant size graphs, such that the modulator M remains a treewidth-η modulator in the resulting graph too. Note that this is done (over a standard protrusion replacement) to ensure that our parameter does not increase in the resulting instance. If one does not care about the parameter increase in the resulting instance, then some of the steps in this algorithm can be simplified.

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