Maximal Independent Sets and Maximal Matchings in Series-Parallel and Related Graph Classes

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Abstract

We provide combinatorial decompositions as well as asymptotic tight estimates for two maximal parameters: the number and average size of maximal independent sets and maximal matchings in series-parallel graphs (and related graph classes) with n vertices. In particular, our results extend previous results of Meir and Moon for trees [Meir, Moon: On maximal independent sets of nodes in trees, Journal of Graph Theory 1988]. We also show that these two parameters converge to a central limit law.

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1 Introduction

In this extended abstract we consider labelled, loopless and simple graphs only. For a graph $G = (V(G), E(G))$, a subset J of $V(G)$ is said to be *independent* if, for any pair of vertices x and y contained in J, the edge $\{x, y\}$ does not belong to $E(G)$. An independent set J of a graph G is said to be *maximal* if any other vertex of G that is not contained in J is adjacent to at least one vertex of J. A subset N of the edge set $E(G)$ is called a *matching* if every vertex x of G is incident to at most one edge of N. A matching N is called *maximal* if it cannot be extended to a bigger matching by adding an edge from $E(G) \setminus N$.

The purpose of this paper is to enumerate maximal independent sets and maximal matchings (by means of symbolic methods) and to study their size distribution (using complex analytic tools) in certain classes of graphs including trees, cactus graphs, outerplanar graphs and series-parallel graphs. For simplicity we will only consider vertex labelled graphs, thus making the combinatorial analysis as well as the analytic one considerably simpler. However, in principle it is also possible to consider unlabelled graphs. We use the concept of generating function in order to follow the classical connectivity-decomposition scheme, first starting with the rooted *blocks*, i.e. maximal 2-connected components, then going to the level of rooted connected graphs and finally to general (not necessarily connected and unrooted) graphs.

Let $\mathcal{G}$ denote a proper class of vertex labelled graphs, which means that the vertices of a graph with n vertices are labelled with the labels $\{1, 2, \ldots, n\}$. We denote by $\mathcal{G}_n$ the set of graphs in $\mathcal{G}$ with n vertices. For a graph $G \in \mathcal{G}$ we denote by $I(G)$ the set of maximal independent sets of G and by

$$\mathcal{I}_n = \bigcup_{G \in \mathcal{G}_n} I(G) \times \{G\}$$

the system of all maximal independent sets of graphs of size n. More precisely, every maximal independent set J is *indexed* by the corresponding graph, this is formally done by taking pairs (J, G). Similarly, we denote by $M(G)$ the set of maximal matchings of G and by

$$\mathcal{M}_n = \bigcup_{G \in \mathcal{G}_n} M(G) \times \{G\}$$

the system of all maximal matchings of graphs of size n.

In this extended abstract, we present precise enumerative results on $\mathcal{I}_n$ and $\mathcal{M}_n$. In particular, we will apply our method to two important graph families: Cayley trees and series-parallel graphs. In principle our results can be extended to other graph classes that have a so-called *subcritical analytic structure*, we will make this more precise in Subsection 2.3 (for instance, cactus graphs and outerplanar graphs also satisfy this analytic scheme). For the mentioned graph classes we have the following universal structure in the asymptotic enumeration formula for the number of graphs on n vertices, for n large enough:

$$g_n = |\mathcal{G}_n| \sim c\, n^{-5/2} \rho^{-n} n!,$$

where $c > 0$ and ρ is the radius of convergence of the (exponential) generating function $G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$ associated to the graph class under study. The first result is an asymptotic estimate for both $|\mathcal{I}_n|$ and $|\mathcal{M}_n|$:

▶ **Theorem 1.** *Let $\mathcal{G}$ either be the class of vertex labelled trees, cactus graphs, outerplanar graphs or series-parallel graphs, and let ρ be the radius of convergence of the generating function $G(x)$ associated to $\mathcal{G}$. Then we have*

$$|\mathcal{I}_n| \sim A_1\, n^{-5/2} \rho_1^{-n} n! \quad \text{and} \quad |\mathcal{M}_n| \sim A_2\, n^{-5/2} \rho_2^{-n} n!,$$

where A_1, A_2, ρ_1, ρ_2 are positive constants with $0 < \rho_1 < \rho$ and $0 < \rho_2 < \rho$.

As a direct corollary we obtain:

▶ **Corollary 2.** *Let $\mathcal{G}$ be as in Theorem 1 and let AI_n be the average number of maximal independent sets in a graph of size n in $\mathcal{G}$ and AM_n be the average number of matchings in a graph of size n in $\mathcal{G}$. Then it holds that*

$$AI_n = \frac{|\mathcal{I}_n|}{g_n} \sim C \cdot \alpha^n \quad \text{and} \quad AM_n = \frac{|\mathcal{M}_n|}{g_n} \sim D \cdot \beta^n,$$

where C, D, α, β are positive constants and α and β are larger than 1.

The second main result concerns the distribution of the respective size of maximal independent sets and matchings. The following theorem shows that the limiting distribution follows a central limit theorem with linear expectation and variance:

▶ **Theorem 3.** *Let $\mathcal{G}$ either be the class of vertex labelled trees, cactus graphs, outerplanar graphs or series-parallel graphs. Furthermore, let SI_n denote the size of a uniformly randomly chosen maximal independent set in $\mathcal{I}_n$ and SM_n the size of a uniformly randomly chosen matching in $\mathcal{M}_n$. Then,*

$$\mathbb{E}[SI_n] = \mu n + O(1), \quad \mathbb{V}\mathrm{ar}[SI_n] = \sigma_1^2 n + O(1),$$
$$\mathbb{E}[SM_n] = \lambda n + O(1), \quad \mathbb{V}\mathrm{ar}[SM_n] = \sigma_2^2 n + O(1),$$

for some constants $\mu, \lambda > 0$ and $\sigma_1^2, \sigma_2^2 > 0$. Moreover, SI_n and MI_n satisfy a central limit theorem:

$$\frac{SI_n - \mathbb{E}[SI_n]}{\sqrt{\mathbb{V}\mathrm{ar}[SI_n]}} \xrightarrow{d} \mathrm{N}(0,1) \quad \text{and} \quad \frac{SM_n - \mathbb{E}[SM_n]}{\sqrt{\mathbb{V}\mathrm{ar}[SM_n]}} \xrightarrow{d} \mathrm{N}(0,1).$$

Apart for constants C and D in Corollary 2, all the other appearing constants can be computed explicitly to any degree of precision. The following table lists some of them:

Family	α	μ	β	λ
Trees	1.273864	0.463922	1.313080	0.285910
Cactus graphs	1.282413	0.429472	1.371652	0.268268
Series-parallel graphs	1.430394	0.269206	1.470167	0.254122

Let us mention that in [13], Meir and Moon obtained the estimate of Theorem 1 and the expectation in Theorem 3 for maximal independent sets in Cayley trees, plane trees and binary trees. Our contribution generalises their work, providing a precise limiting distribution for the size of maximal independent sets in Cayley trees.

Finally, let us briefly discuss the extremal versions of those problems. In the literature, one can find two such directions. One of them, started by Wilf [17] who was motivated by the design of an algorithm to compute the chromatic number, consists in characterising the extremal instances of a given family of graphs containing the maximum number of maximal independent sets (see [9], [15] and [18]), as well as maximum independent sets (see [19] and [12]). Furthermore, the maximum number of both maximal matchings [10] and maximum matchings [11] have been treated. The other direction consists in bounding the size of a maximum matching in a graph [3]. However, the problems discussed in this extended abstract seem to be of a different nature.

It is worth noticing that in [3], the authors also give tight bounds on the size of a maximal matching in 3-connected planar graphs and in graphs with bounded maximum degree.

Structure of the extended abstract

Section 2 introduces the necessary background, namely the language of generating functions and how they apply to graph decompositions in terms of their connectivity, as well as the analytic concepts needed in the context of subcritical graph classes. Later, in Section 3 we obtain a system of functional equations encoding maximal independent sets in subcritical graph classes. We then analyse it using complex analytic tools in Subsection 3.2. And in Section 4 we apply our results to the families of Cayley trees and series-parallel graphs. The reader will finally find the analoguous scheme for maximal matchings in an appendix at the end of the extended abstract.

2 Preliminaries

2.1 Generating functions

We follow the notation from [6]. A *labelled combinatorial class* is a set $\mathcal{A}$ together with a size measure, such that if $n \geq 0$, then the set of elements of size n, denoted by $\mathcal{A}_n$, is finite. Each element a of $\mathcal{A}_n$ is built from n atoms (typically, vertices in graph classes) assembled in a certain way, the atoms bearing distinct labels in the set $\{1, \ldots, n\}$. We always assume that a combinatorial class is stable under graph isomorphism, namely, $a \in \mathcal{A}$ if and only if all graphs a' isomorphic to a are also elements of $\mathcal{A}$.

In enumerative problems, it is often useful to use the exponential generating function (shortly the EGF) associated to the labelled class $\mathcal{A}$:

$$A(x) := \sum_{n\geq 0} \frac{|\mathcal{A}_n|}{n!} x^n, \qquad [x^n]A(x) = \frac{|\mathcal{A}_n|}{n!}.$$

In our setting, we use the (exponential) variable x to encode vertices.

We can root the elements of a class $\mathcal{A}$ by distinguishing one of the items and discounting it, which means that we reduce the size function by 1. Since we assume that our combinatorial class is stable under graph isomorphism, this procedure can be performed by taking the item with the largest label as the root. The corresponding new rooted class will be denoted by $\mathcal{A}^\circ$. Since every element of $\mathcal{A}$ correponds uniquely to an element of $\mathcal{A}^\circ$, but the corresponding term $x^n/n!$ in the generating function is replaced by $x^{n-1}/(n-1)!$ (for an element of size n), the correponding generating function satisfies

$$A^\circ(x) = A'(x).$$

Similarly, we can consider a pointed structure $\mathcal{A}^\bullet$ by distinguishing one of the items without discounting it. Since there are n different ways of choosing an item (for an element of size n), the corresponding term $x^n/n!$ in the generating function is replaced by $nx^n/n! = x^n/(n-1)!$ which leads to the relation

$$A^\bullet(x) = xA'(x).$$

Finally, we will deal with the set construction of classes: given a labelled combinatorial structure $\mathcal{A}$, the set construction $\mathrm{Set}(\mathcal{A})$ takes all possible sets of elements in $\mathcal{A}$. The corresponding generating function is then $\exp((A(x))$, where $A(x)$ is the generating function associated to $\mathcal{A}$.

2.2 Graph decompositions

A *block* of a graph G is a maximal 2-connected subgraph of G. A graph class $\mathcal{G}$ is said to be *block-stable* if it contains the graph e, the unique connected graph with two labelled vertices, and satisfies that a connected graph G belongs to $\mathcal{G}$ if and only if any one of its blocks is in $\mathcal{G}$. The class $\mathcal{G}$ is also said to be *connected component-stable* when any graph G is in $\mathcal{G}$ if and only if all connected components of G belong to $\mathcal{G}$. For a graph class $\mathcal{G}$, we denote by $\mathcal{C}$ and $\mathcal{B}$ the families of connected and 2-connected graphs in $\mathcal{G}$, respectively. In particular, if $\mathcal{G}$ is a block-stable and connected-component stable class of graphs, then the following combinatorial decomposition holds:

$$\mathcal{G} = \mathrm{Set}(\mathcal{C}), \qquad \mathcal{C}^\bullet = \bullet \times \mathrm{Set}(\mathcal{B}^\circ \circ \mathcal{C}^\bullet).$$

The previous formulas read as follows: first, each graph in $\mathcal{G}$ is a set of elements in $\mathcal{C}$. Secondly, a pointed connected graph in $\mathcal{C}^\bullet$ can be decomposed as the root vertex, and a set of pointed blocks (the ones incident with the root vertex) where we substitute on each vertex a rooted connected graph. See [1, 4, 8] for details. These expressions translate into equations of EGF in the following way:

$$G(x) = \exp(C(x)), \qquad C^\bullet(x) = x\exp(B^\circ(C^\bullet(x))).$$

See [16] for further results on graph decompositions and connectivity on graphs.

2.3 Asymptotics for subcritical graph classes

We call a block-stable and vertex labelled graph class *subcritial* if $\eta B''(\eta) > 1$, where η denotes the radius of convergence of $B(x)$. In particular this is satisfied if $B''(x) \to \infty$ as $x \to \eta-$. Cayley trees, cactus graphs, outerplanar graphs and series-parallel graphs are subcritical. The main analytic property of subcritical graph classes is that they have many universal asymptotic behaviours, see [2, 5, 14, 7].

In our context, we will just use the fact that the property $\eta B''(\eta) > 1$ ensures that the functional equation $C^\bullet(x) = x\exp(B^\circ(C^\bullet(x)))$ has solution $C^\bullet(x)$ that has a squareroot singularty at its radius of convergence ρ and, thus, a local expansion of the form

$$C^\bullet(x) = xC'(x) = \underline{c}_0 + \underline{c}_1\left(1-\frac{x}{\rho}\right)^{1/2} + \underline{c}_2\left(1-\frac{x}{\rho}\right) + \underline{c}_3\left(1-\frac{x}{\rho}\right)^{3/2} + \cdots, \tag{1}$$

where ρ is given by $\rho = \underline{c}_0 e^{-B'(\underline{c}_0)}$ and $0 < \underline{c}_0 = C^\bullet(\rho) < \eta$ is given by the equation $\underline{c}_0 B''(\underline{c}_0) = 1$. Furthermore $\underline{c}_1 < 0$. Note that the singular behaviour of $B(x)$ at its radius of convergence η is irrelevant for the singular behaviour of $C^\bullet(x) = xC'(x)$, we only make use of the (analytic) behaviour of $B'(x)$ around $x = \underline{c}_0 < \eta$.

From (1), and if we assume that the class is also connected component-stable, it follows that $C(x)$ and $G(x) = e^{C(x)}$ have the following singular behaviour around their common radius of convergence ρ:

$$C(x) = c_0 + c_2\left(1-\tfrac{x}{\rho}\right) + c_3\left(1-\tfrac{x}{\rho}\right)^{3/2} + \cdots,$$
$$G(x) = g_0 + g_2\left(1-\tfrac{x}{\rho}\right) + g_3\left(1-\tfrac{x}{\rho}\right)^{3/2} + \cdots,$$

where c_3 and g_3 are positive. If we further assume that $x = \rho$ is the only singularity on the circle of convergence $|x| = \rho$ which is satisfied for all our cases, and for proper positive constants c', c'', it then follows that (see for instance [6])

$$|\mathcal{C}_n| = n!\,[x^n]\,C(x) \sim c' n^{-5/2} \rho^{-n} n! \quad \text{and} \quad |\mathcal{G}_n| = n!\,[x^n]\,C(x) \sim c'' n^{-5/2} \rho^{-n} n!.$$

3 Counting in block-stable graph classes

In this section, we consider block-stable vertex labelled graph classes and set up functional equations for counting maximal independent subsets and maximal matchings. We use the notation $\mathcal{B}$ for the family of 2-connected blocks in a block-stable graph class $\mathcal{G}$ and $\mathcal{C}$ for the family of connected graphs in $\mathcal{G}$.

3.1 Maximal independent sets in block-stable graph classes

A *coloured block* is a pair (I, b) consisting of a block $b \in \mathcal{B}$ together with a distinguished independent set I of b (note that I can be any independent set of b and not only a maximal one). Let $B(x, y_0, y_1, y_2)$ be the generating function enumerating coloured-blocks, where the variable x marks vertices. The extra variables encodes the following: y_0 corresponds to vertices of I, y_1 corresponds to vertices adjacent to a vertex in I (i.e. at distance one from I), and y_2 corresponds to all other vertices, that is to vertices at distance at least two from I.

Similarly, a *pointed coloured block* is a pair (I, b°) consisting of a pointed block $b^\circ \in \mathcal{B}^\circ$ together with a distinguished independent set I of b. Let $B_i = B_i(x, y_0, y_1, y_2)$ be the generating function counting pointed coloured blocks, where the pointed vertex is at distance *exactly* i from I, for $i \in \{0, 1\}$, and at distance *at least* 2 (case $i = 2$). In those cases, the pointed-vertex must neither be encoded by x or by any y_i, for $i \in \{0, 1, 2\}$. Hence,

$$B_i = \frac{1}{x} \cdot \frac{\partial B}{\partial y_i}, \text{ for } i \in \{0, 1, 2\}.$$

A *coloured graph* (J, g) is a pair consisting of a connected graph $g \in \mathcal{C}$ and of a *maximal* independent set J of g. We can define pointed coloured graphs similarly to coloured blocks. Let $C = C(x, y_0, y_1)$ be the generating function counting coloured-graphs, where y_0 and y_1 have the same meaning as in coloured blocks. For $i \in \{0, 1\}$, let $C_i = C_i(x, y_0, y_1)$ be the generating functions enumerating pointed coloured-graphs, for which the pointed vertex is at distance *exactly* i from J. Those two generating functions are given by

$$C_i = \frac{1}{x} \cdot \frac{\partial C}{\partial y_i}, \text{ for } i \in \{0, 1\}. \tag{2}$$

We finally need an auxiliary class. A *special pointed coloured-graph* is a pair (J, g°) where J is an independent set of g which becomes maximal when adding the pointed vertex to J. In other words, a special pointed coloured-graph is obtained from a coloured-graph pointed at a vertex in J by removing it from J. We denote the corresponding counting formula by $C_2(x, y_0, y_1)$. Finally, observe that given a coloured-graph (J, g), the independent set J together with the vertices of g at distance one from J define a partition of $V(g)$. Hence, the following equalities hold:

$$\frac{\partial C}{\partial x} = \frac{y_0}{x} \frac{\partial C}{\partial y_0} + \frac{y_1}{x} \frac{\partial C}{\partial y_1} = y_0 C_0 + y_1 C_1. \tag{3}$$

Obviously we also have

$$G(x, y_0, y_1) = \exp(C(x, y_0, y_1)),$$

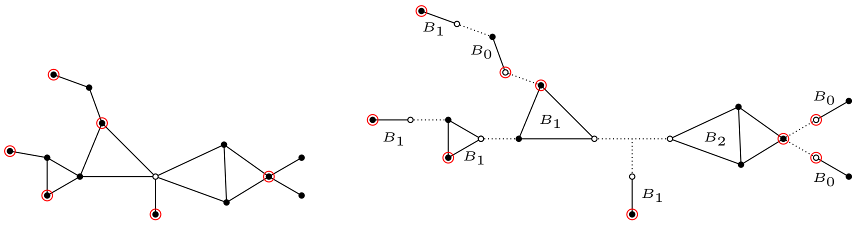

Figure 1 Left is a connected series-parallel graph with a maximal independent set I (vertices circled in red) and pointed at a vertex at distance one from I. Right is its block-decomposition. Pointed vertices are coloured in white.

where $G(x, y_0, y_1)$ denotes the corresponding generating function of coloured graphs in $\mathcal{G}$. The following lemma describes connected structures in terms of their block-decomposition (see Figure 1 for an example). Thus, if we know $B(x, y_0, y_1, y_2)$ (or just $B_i(x, y_0, y_1, y_2)$, for each $i \in \{0, 1, 2\}$), then we can determine $\frac{\partial C}{\partial x}(x, y_0, y_1)$ and consequently $C(x, y_0, y_1)$ and $G(x, y_0, y_1)$.

▶ **Lemma 4.** *With the above notations, the following system of equations holds:*

$$\begin{aligned} C_0 &= \exp(B_0(x, y_0C_0, y_1(C_1 + C_2), y_1C_1)),\\ C_1 &= (\exp(B_1(x, y_0C_0, y_1(C_1 + C_2), y_1C_1)) - 1) \cdot C_2,\\ C_2 &= \exp(B_2(x, y_0C_0, y_1(C_1 + C_2), y_1C_1)). \end{aligned} \tag{4}$$

Proof. Let us start by finding an expression for C_0 and let (I, g°) be a pointed coloured-graph whose pointed vertex is in I. Following the decomposition of graphs into blocks, observe that the pointed vertex of g° determines a set of pointed coloured-blocks (J_i, b_i°) (with $i = 1, \ldots, k$ for a certain k) for which the root of each b_i° belongs to J_i, i.e. coloured-blocks with the pointed vertex in J_i (and hence, counted by B_0). Observe that the independent sets J_i can be extended to I by pasting pointed coloured-graphs on each of their vertices (and completing the graph to g°).

Without loss of generality, let us now fix a $j \in \{1, \ldots, k\}$ and analyse the pair (J_j, b_j°). First, to every vertex of b_j° in J_j must be attached a coloured-graph (L, h°) whose root is in L, i.e. a coloured-graph counted by C_0. In terms of generating functions, this translates to the substitution of y_0 by y_0C_0. Second, to each vertex of b_j° at distance one from J_j, the root of the pointed coloured-graph (L, h°) attached to it can either be at distance one or more from L. This then translates to the substitution of y_1 by $y_1(C_1 + C_2)$. Finally, if a vertex of b_j° is at distance at least two from J_j, then the root of the coloured-graph (L, h°) attached to it must be at distance one from L, as we need to extend the independent set to a maximal one. This translates to the substitution of y_2 by y_1C_1 and the first equation of (4) holds. The study of C_2 is obtained following the exact same arguments as in C_0.

Let us finally discuss the equation for C_1. Assume that (I, g°) is a pointed coloured-graph and that (J_i, b_i°) (for $i = 1, \ldots, k$) are the pointed coloured-blocks incident with the pointed vertex of g°. In particular, for each $i \in \{1, \ldots, k\}$ the pointed vertex of b_i° is either at distance one or at least two from J_i. Nevertheless, observe that there exists at least one of the pointed-blocks (J_j, b_j°) whose pointed vertex is at distance one from J_j. This gives us that

$$\begin{aligned} C_1 &= \exp_{\geq 1}(B_1(x, y_0C_0, y_1(C_1 + C_2), y_1C_1)) \cdot \exp(B_2(x, y_0C_0, y_1(C_1 + C_2), y_1C_1))\\ &= C_2\left(\exp(B_1(x, y_0C_0, y_1(C_1 + C_2), y_1C_1)) - 1\right). \end{aligned}$$

Which concludes the argument. ◀

3.2 Asymptotic Analysis

We study next the analytic structure of the solutions of the systems (4) and (9) provided that the functions B_i behave in a *proper way* that is similar to the behaviour of $B(x)$ in the case of sub-critical graph classes. Under these hypothesis (see Lemma 5) it is then very easy to prove Theorems 1 and 3 which will be done at the end of this subsection. For the sake of brevity we only discuss the system (4), the analysis of (9) runs along the same lines.

First we note that the functions $B_i(x, y_0, y_1, y_2)$ are actually functions in three variables since a monomial $x^n y_0^{k_0} y_1^{k_1} y_2^{k_2}$ can only appear if $k_0 + k_1 + k_2 = n$, that is, we have $B_i(x, y_0, y_1, y_2) = B_i(1, xy_0, xy_1, xy_2)$ or equivalently $B_i(x, y_0, y_1, y_2)$ $= B_i(xy_2, y_0/y_2, y_1/y_2, 1)$. However, it is more convenient to work with all four variables x, y_0, y_1, y_2. If y_0, y_1, y_2 are positive real numbers then the function $x \mapsto B(x, y_0, y_1, y_2)$ is a power series with non-negative coefficients. Hence the radius of convergence of this function coincides with its dominant singularity in x. We will denote this radius of convergence by $R(y_0, y_1, y_2)$. Similarly for the solution functions C_0, C_1, C_2 of the System (4) we denote by $\rho_i(y_0, y_1)$, $i = 0, 1, 2$, the radius of convergence with respect to x when y_0, y_1 are positive real numbers.

▶ **Lemma 5.** *Suppose that the function $R(y_0, y_1, y_2)$ extends to an analytic function $R(y_0, y_1, y_2)$ for a sufficiently small neighbourhood around the positive real numbers. Furthermore assume that for all positive real numbers y_0, y_1, y_2 we have*

$$\lim_{x \to R(y_0,y_1,y_2)-} \frac{\partial^2 B}{\partial y_i^2}(x, y_0, y_1, y_2) = \infty \tag{5}$$

for at least one of the $i \in \{0, 1, 2\}$. Then the solutions C_0, C_1, C_2 of the system (4) have the property that the functions $\rho_i(y_0, y_1)$, $i = 0, 1, 2$, coincide and extend to an analytic function $\rho(y_0, y_1)$ for a sufficiently small neighbourhood around the positive real numbers. Moreover, the dominant singularity is of squareroot type and we have a local expansion of the form

$$\begin{aligned} C_i(x, y_0, y_1) = \quad & c_{i,0}(y_0, y_1) + c_{i,1}(y_0, y_1) \left(1 - \tfrac{x}{\rho_1(y_0,y_1)}\right)^{1/2} \\ & + c_{i,2}(y_0, y_1) \left(1 - \tfrac{x}{\rho_1(y_0,y_1)}\right) + \cdots , \end{aligned} \tag{6}$$

where $c_{i,1}(y_0, y_1) < 0$ (for positive real y_0, y_1) and that extends to sufficiently small neighbourhood in x, y_0, y_1 around the positive real numbers.

Proof. We recall some basic facts on (positive) systems of functional equations that are taken from [4]. Suppose that we have a system of three equations of the form

$$\begin{aligned} C &= F(x, C, D, E), \\ D &= G(x, C, D, E), \\ E &= H(x, C, D, E), \end{aligned}$$

in unknown functions $C = C(x)$, $D = D(x)$, $E = E(x)$, where F, G, H are power series with non-negative coefficients. We also assume that the system is strongly connected which means that no subsystem can be solved before solving the whole system. We set

$$\Delta = \begin{vmatrix} 1 - F_C & -F_D & -F_E \\ -G_C & 1 - G_D & -G_E \\ -H_C & -H_D & 1 - H_E \end{vmatrix}$$

the functional determinant of the system $\{C - F = 0,\ D - G = 0,\ E - H = 0\}$ and let r be the spectral radius of the Jacobian matrix of the right hand-side of the system of equations.

Note that $r = 1$ implies that $\Delta = 0$. We also assume that there is a unique non-negative solution $C(0)$, $D(0)$, $E(0)$ for $x = 0$ with the property that $r < 1$, which also shows that $\Delta \neq 0$. Thus by iteration, the solution for $x = 0$ extends to power series solutions $C(x)$, $D(x)$, $E(x)$ with non-negative coefficients and a positive radius of convergence. By the strongly connectedness assumption, this radius of convergence ρ is the same for all three solutions functions $C(x)$, $D(x)$, $E(x)$. By the theory given in [4], this radius of convergence is determined by the condition $r = 1$ provided that we are still working within the region of convergence of F, G, and H. The condition $r = 1$ can be also witnessed by the condition $\Delta = 0$ or equivalently by the condition

$$\tfrac{F_D G_E H_C + F_E G_C H_C}{(1-F_C)(1-G_D)(1-H_E)} + \tfrac{G_E H_D}{(1-G_D)(1-H_E)} + \tfrac{F_E H_C}{(1-F_C)(1-H_E)} + \tfrac{F_D G_C}{(1-F_C)(1-G_D)} = 1. \tag{7}$$

Note that the left hand-side is smaller than 1 for $x = 0$ and $C = C(0)$, $D = D(0)$, $E = E(0)$ and is strictly increasing in x. Thus, in order to find ρ we just have to find the value for which the left hand-side hits the value 1. If we are still inside the region of convergence of F, G, and H, then it follows that the solution functions $C(x)$, $D(x)$, $E(x)$ have a squareroot singularity of the form (1) at $x = \rho_1$.

In our special situation all the above assumptions concerning positivity, strongly connectedness etc. are satisfied. Now let us also observe that $\frac{\partial^2 B}{\partial y_0^2} \to \infty$ implies that $F_C \to \infty$, since $F(x) = \exp(B_0(x, y_0 C, y_1(D+E), y_1 D)$ and $B_0 = \frac{1}{x}\frac{\partial B}{\partial y_0}$ (note the two different meanings of y_0). Similar observations hold for G_D and H_E. Thus, it is clear that (7) is satisfied inside the region of convergence of F, G and H. We recall the fact that the left hand-side of (7) is smaller than 1 for $x = 0$ and strictly increasing in x. ◀

Finally we show that under the hypothesis of Lemma 5, it is immediate to deduce our main results Theorem 1 and Theorem 3: from (6) and (3) it follows that $C(x, y_0, y_1)$ can be represented as

$$C(x, y_0, y_1) = c_0(y_0, y_1) + c_2(y_0, y_1)\left(1 - \frac{x}{\rho_1(y_0, y_1)}\right) + c_3(y_0, y_1)\left(1 - \frac{x}{\rho_1(y_0, y_1)}\right)^{3/2} + \cdots,$$

where $c_3(y_0, y_1) > 0$ for positive real y_0, y_1. Thus, if we set $y_0 = y_1 = 1$ and $\rho_1(1,1) = \rho_1$, then we have

$$C(x, 1, 1) = c_0(1,1) + c_2(1,1)\left(1 - \frac{x}{\rho_1}\right) + c_3(1,1)\left(1 - \frac{x}{\rho_1}\right)^{3/2} + \cdots,$$

and consequently

$$\begin{aligned} G(x,1,1) &= \sum_{n\geq 0} |\mathcal{I}_n| \frac{z^n}{n!} = \exp(C(x,1,1)) \\ &= g_0(1,1) + g_2(1,1)\left(1 - \frac{x}{\rho_1}\right) + g_3(1,1)\left(1 - \frac{x}{\rho_1}\right)^{3/2} + \cdots. \end{aligned}$$

This directly implies Theorem 1 for the case of maximal independent sets by standard singularity analysis (see [6]). We just have to observe that $x_0 = \rho = \rho(1,1)$ is the only singularity on the circle of convergence. However, this follows from the fact that there exists graphs of all sizes $n \geq 1$.

Finally, if we set $y_1 = 1$ then we have

$$G(x, y_0, 1) = \sum_{n\geq 0} \mathbb{E}[y_0^{SI_n}]\,|\mathcal{I}_n|\,\frac{z^n}{n!} = \exp(C(x, y_0, 1))$$

$$= g_0(y_0, 1) + g_2(y_0, 1)\left(1 - \frac{x}{\rho_1(y_0, 1)}\right) + g_3(1, 1)\left(1 - \frac{x}{\rho_1(y_0, 1)}\right)^{3/2} + \cdots$$

Hence, a direct application of [4, Theorem 2.35] implies a central limit theorem of the proposed form, as well as the asymptotic expansions for the expected value and variance. This proves Theorem 3 for the case of maximal independent sets.

What remains is to check condition (5). We work this out in details for trees and series-parallel graphs in Section 4. The other cases (cactus graphs and outerplanar graphs) can be handled in a similar way and this will be covered in the paper version of this extended abstract.

4 Applications

Our first application concerns the most basic subcritical graph class, namely Cayley trees. We note that the case of maximal independent sets was already discussed in [13]. We will then deal with the class of series-parallel graphs.

4.1 Maximal independent sets in trees

In both structures (maximal independent sets and maximal matchings), we proceed following the block-decomposition of trees, and we explicitly give the generating functions B_0, B_1 and B_2. Notice that in a tree, blocks are reduced to single edges. The computations of the constants given in Table 1 are obtained by computing the branch point of the corresponding system, using the explicit expressions for B_0, B_1 and B_2.

We first give the generating functions counting the rooted blocks carrying an independent set. From the possible choices of an independent set in a single edge, namely $B(x, y_0, y_1, y_2) = \frac{x^2}{2}\left(2y_0y_1 + y_2^2\right)$, we obtain that

$$B_0 = xy_1, \qquad B_1 = xy_0, \qquad B_2 = xy_2.$$

Thus, the following property holds:

$$\lim_{x\to\infty} \frac{\partial^2 B}{\partial y_2^2} = \lim_{x\to\infty} x = \infty.$$

So Lemma 5 applies in the case of maximal independent sets in trees, which completes the proof.

4.2 Maximal independent sets in series-parallel graphs

We are now concerned with the generating functions of the labelled series-parallel graphs carrying a maximal independent set. As above, the vertices of the graphs carrying an independent set I are said to be of type i ($i \in \{0, 1\}$), when they are at distance i from I, and of type 2 otherwise. We will now explicit the classical decomposition of graphs in terms of networks.

Series-parallel networks

A *series-parallel network* D_{ij} is a labelled graph with an oriented edge ij that is distinguished and whose endpoints, called the *poles*, are unlabeled and respectively of type i and j. Observe that by symmetry $D_{ij} = D_{ji}$, so we can restrict the range of the pairs of indexes ij to the set $\{00, 01, 02, 11, 12, 22\}$. The network D_{ij} is either the single rooted edge e_{ij}, where $e_{01} = e_{22} = y$ and $e_{ij} = 0$ otherwise, a series network counted by the generating function S_{ij}, or a parallel network counted by the generating function P_{ij}. We then specify those generating functions via the following positive system of 18 equations and 18 unknowns:

$$\begin{aligned}
&D_{ij} = e_{ij} + S_{ij} + P_{ij},\\
&S_{ij} = D_{i0}xy_0(D_{0j} - S_{0j}) + (D_{i1} + D_{i2})xy_1(D_{1j} - S_{1j}) + (D_{i1}y_1 + D_{i2}y_2)x(D_{2j} - S_{2j}),\\
&P_{00} = \exp_{\geq 2}(S_{00}),\\
&P_{01} = y\exp_{\geq 1}(S_{01} + S_{02}) + \exp_{\geq 2}(S_{01}) + \exp_{\geq 1}(S_{01})\exp_{\geq 1}(S_{02}),\\
&P_{02} = \exp_{\geq 2}(S_{02}),\\
&P_{11} = \exp_{\geq 2}(S_{11}) + \exp_{\geq 1}(S_{11})(y\exp(2S_{12} + S_{22}) + \exp_{\geq 1}(2S_{12} + S_{22}))\\
&\qquad + (1 + y)\exp_{\geq 1}(S_{12})^2\exp(S_{22}),\\
&P_{12} = y\exp_{\geq 1}(S_{12})\exp(S_{22}) + \exp_{\geq 2}(S_{12}) + \exp_{\geq 1}(S_{12})\exp_{\geq 1}(S_{22}),\\
&P_{22} = y\exp_{\geq 1}(S_{22}) + \exp_{\geq 2}(S_{22}).
\end{aligned}$$

In order to proceed further, we eliminate D_{ij} from this system to obtain a positive and strongly connected system of equations for $S_{ij} = S_{ij}(x, y, y_0, y_1, y_2)$ and $P_{ij} = P_{ij}(x, y, y_0, y_1, y_2)$, where the right hand-side consists of entire functions (note that for the equations defining S_{ij}, the term $D_{ij} - S_{ij} = e_{ij} + P_{ij}$, which makes the whole system positive). Thus, all functions have a common singular behaviour that is (again) of squareroot type:

$$S_{ij}(x, y, y_0, y_1, y_2) = s_{0;ij}(y, y_0, y_1, y_2) + s_{1;ij}(y, y_0, y_1, y_2)\left(1 - \frac{x}{\rho(y, y_0, y_1, y_2)}\right)^{1/2} + \cdots$$

and

$$P_{ij}(x, y, y_0, y_1, y_2) = p_{0;ij}(y, y_0, y_1, y_2) + p_{1;ij}(y, y_0, y_1, y_2)\left(1 - \frac{x}{\rho(y, y_0, y_1, y_2)}\right)^{1/2} + \cdots,$$

where $s_{1;ij}(y, y_0, y_1, y_2) < 0$ and $p_{1;ij}(y, y_0, y_1, y_2) < 0$ for positive y, y_0, y_1, y_2.

2-connected series-parallel graphs

The next step is to relate these network generating functions with the generating function $B(x, y, y_0, y_1, y_2)$ of independent sets in 2-connected series parallel graphs. Note that an added variable y takes into account the number of edges. In the (usual) counting procedure for series parallel graphs, we have the property that $\frac{\partial B}{\partial y} = \frac{x^2}{2}\exp(S(x, y))$, where $S(x, y)$ denotes the generating function of series networks (similarly to the above). The combinatorial property behind this relation is that an edge-rooted series-parallel graph (that corresponds to the generating function $\frac{\partial B}{\partial y}$) can be seen as a series-parallel network between the two vertices of the root-edge, consisisting of this edge and a collection of series-networks between the two vertices.

In our present situation we have a similar property, namely

$$\frac{\partial B}{\partial y} = x^2 y_0 y_1 \exp(S_{01} + S_{02}) + \frac{x^2}{2} y_2^2 \exp(S_{22}) + x^2 y_1 y_2 \exp_{\geq 1}(S_{12}) \exp(S_{22})$$
$$+ \frac{x^2}{2} y_1^2 \left(\exp(S_{11} + 2S_{12} + S_{22}) - 2\exp(S_{12} + S_{22}) + \exp(S_{22})\right).$$

This is immediate by considering all possible situation for the rooted edge. Observe that, despite the negative terms, $\frac{\partial B}{\partial y}$ is in fact a positive function of the generating functions $\{S_{ij}\}$. Hence, $\frac{\partial B}{\partial y}$ has also a squareroot singularity:

$$\frac{\partial B}{\partial y} = b_0(y, y_0, y_1, y_2) + b_1(y, y_0, y_1, y_2) \left(1 - \frac{x}{R(y, y_0, y_1, y_2)}\right)^{1/2} + \cdots,$$

where $b_1(y, y_0, y_1, y_2) < 0$ for positive y, y_0, y_1, y_2. Next, by applying the proof method of [4, Lemma 2.28], we can integrate $\frac{\partial B}{\partial y}$ with respect to y and then take the derivative with respect to y_0 and obtain the same kind of squareroot singularity for $\frac{\partial B}{\partial y_0}$

$$\frac{\partial B}{\partial y_0} = b_{1,0}(y, y_0, y_1, y_2) + b_{1,1}(y, y_0, y_1, y_2) \left(1 - \frac{x}{R(y, y_0, y_1, y_2)}\right)^{1/2} + \cdots,$$

and consequently the following representation of $\frac{\partial^2 B}{\partial y_0^2}$:

$$\frac{\partial^2 B}{\partial y_0^2} = b_{2,-1}(y, y_0, y_1, y_2) \left(1 - \frac{x}{R(y, y_0, y_1, y_2)}\right)^{-1/2} + b_{2,1}(y, y_0, y_1, y_2) + \cdots,$$

which implies that (5) holds for $i = 0$. This completes the proof for maximal independent sets in series-parallel graphs.

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A Maximal matchings

A.1 Maximal matchings in block-stable classes of graphs

In this subsection we deal with the case of maximal matchings. Most of the definitions and concepts are the natural analogues of the ones developed in the case of maximal independent sets. Hence, we will skip unnecessary repetitions.

A *matched block* is a triple (I, M, b) with a block $b \in \mathcal{B}$, a matching M in b, and an independent set I of b, and where no element of I is incident to an edge in M. In other words, we split the set of vertices of b in three disjoint subsets: matched vertices, vertices in I, and the rest. A *pointed* matched block is a triple (I, M, b°), where $b^\circ \in \mathcal{B}^\circ$ and M and I are respectively a matching and an independent set of b, and where again no element of I is incident to any edge in M. Let $\overline{B}(x, z_0, z_1, z_2)$ be the generating function counting matched blocks, where the variable x marks vertices, z_0 marks vertices in I, z_1 marks vertices matched by M, and z_2 the remaining ones. For $i \in \{0, 1, 2\}$, let $\overline{B}_i = \overline{B}_i(x, z_0, z_1, z_2)$ be the generating function counting pointed matched blocks where the pointed vertex is either in I, is incident with M or none of the previous cases. In particular,

$$\overline{B}_i = \frac{1}{x} \cdot \frac{\partial \overline{B}}{\partial z_i}, \text{ for } i \in \{0, 1, 2\}.$$

A *matched graph* is a triple (I, M, g) consisting of a connected graph g in $\mathcal{C} \subseteq \mathcal{G}$, a matching M of g, and an independent set $I \subset V(g) \setminus V(M)$. Similarly, a *pointed* matched graph is a triple (M, I, g°) where now g° is a pointed graph. Let $\overline{C}(x, z_0, z_1, z_2)$ be the generating function counting matched graphs, where x, z_0, z_1 and z_2 respectively mark vertices, vertices incident with I, vertices incident with M, and the rest of the vertices. Notice that when $z_2 = 0$, $\overline{C} := \overline{C}(x, z_0, z_1) = \overline{C}(x, z_0, z_1, 0)$ encodes matched graphs where M is maximal. For each $i \in \{0, 1, 2\}$, let us define the following generating function

$$\overline{C}_i = \overline{C}_i(x, z_0, z_1) = \frac{1}{x} \cdot \frac{\partial \overline{C}}{\partial z_i}(x, z_0, z_1, 0).$$

Observe then that $\overline{C}_0$ counts pointed matched graphs, where the matching is maximal and the pointed vertex belongs to the independent set, $\overline{C}_1$ counts pointed matched graphs, where the matching is maximal and the pointed vertex belongs to the matching, whereas $\overline{C}_2$ counts pointed matched graphs, where the matching is not necessarily maximal and the pointed vertex does not belong to either the independent set or the matching. In the latter case, the matching is maximal except for possibly the pointed vertex, which might be unmatched and adjacent to other unmatched vertices. In particular, this implies that the generating function of pairs of connected graphs and maximal matchings is given by

$$\frac{\partial \overline{C}}{\partial x} = z_0\overline{C}_0 + z_1\overline{C}_1. \tag{8}$$

The main idea behind this encoding of the problem is that vertices in the independent set I play the role of vertices that will not be matched in the block decomposition. In particular, we exploit independence in order to ensure that the matching cannot be extended. On the other hand, the set of vertices that are unmatched and not in I will be matched by an attached block of the decomposition.

The following lemma relates all the previous generating functions. Note that the generating functions $\overline{C}(x, z_0, z_1, 0)$ and $\overline{G}(x, z_0, z_1) = \exp(\overline{C}(x, z_0, z_1))$ directly follow from the solution of the next system.

▶ **Lemma 6.** *The following equalities hold:*

$$\begin{aligned} \overline{C}_0 &= \exp(\overline{B}_0(x, z_0\overline{C}_0, z_1\overline{C}_2, z_1\overline{C}_1)), \\ \overline{C}_1 &= \overline{C}_2\,\overline{B}_1(x, z_0\overline{C}_0, z_1\overline{C}_2, z_1\overline{C}_1), \\ \overline{C}_2 &= \exp(\overline{B}_2(x, z_0\overline{C}_0, z_1\overline{C}_2, z_1\overline{C}_1)). \end{aligned} \tag{9}$$

Proof. Let (M, I, g°) be a pointed matched graph, with pointed vertex v. Suppose first that $v \in I$, i.e. the case counted by $\overline{C}_0$. It therefore is the pointed vertex of a (possibly empty) set of adjacent pointed blocks (I_j, M_j, b_j°), in which $v \in I_j$, and is not adjacent to any other pointed block. This means that all the pointed blocks adjacent to v are counted by $\overline{B}_0$. Suppose next that $v \in V(M)$, i.e. the case counted by $\overline{C}_1$. Then the edge of M incident with v must belong to a single pointed block whose pointed vertex (v) is incident to an edge of the respective matching. Hence, attached to v are this one block together with any number (possibly null) of pointed blocks counted by $\overline{B}_2$, since v is already incident to an edge of a matching. Suppose finally that we are in the case counted by $\overline{C}_2$. Then v is neither in I nor in $V(M)$. Therefore, any block attached to it must not have its pointed vertex in an independent set or incident to and edge of a matching. This means that v belongs to a (possibly empty) set of blocks counted by $\overline{B}_2$.

Let now $\{(I_i, M_i, b_i^\circ) : i = 1, \dots k\}$ be the pointed blocks in the decomposition of (M, I, g°) and fix a $j \in \{1, \dots k\}$. Then using the same arguments as just above, we see that to a vertex in I_j must be attached a pointed matched graph counted by $\overline{C}_0$, to a vertex in $V(M_j)$ one counted by $\overline{C}_2$ and to any other vertex must be attached a pointed matched graph counted by $\overline{C}_1$, as we need to extend the matching to maximality. ◀

A.2 Maximal matchings in trees

Observe that in this case $\overline{B}(x, z_0, z_1, z_2) = \frac{x^2}{2}\left(2z_0z_2 + z_1^2 + z_2^2\right)$, which gives

$$\overline{B}_0 = xz_2, \qquad \overline{B}_1 = xz_1, \qquad \overline{B}_2 = x(z_0 + z_2).$$

Hence, we are in a similar situation as above and Lemma 5 applies. This completes the proof for maximal matchings in trees.

A.3 Maximal matchings in series-parallel graphs

We proceed similarly to Subsection 4.2. Let G be a series-parallel graph with a matching M and an independent set I such that $I \cap V(M) = \emptyset$. A vertex v of G is said to be of type 0 when $v \in I$, of type 1 when $v \in V(M)$ and of type 2 otherwise.

Series-parallel networks

Let $D_{ij}(x, y, z_0, z_1, z_2)$ be the exponential generating function counting matchings in series-parallel networks whose poles are of type i and j. As before, observe that $D_{ij} = D_{ji}$ and for $ij \in \{00, 01, 02, 11, 12, 22\}$, define S_{ij} and P_{ij} to be the generating functions counting matchings in networks that are respectively series and parallel.

The following system of 18 equations and 18 unknowns holds:

$$\begin{aligned}
D_{ij} &= e_{ij} + S_{ij} + Pij,\\
S_{ij} &= (D_{i0} - S_{i0})xz_0D_{0j} + (D_{i1} - S_{i1})xz_1D_{2j} + (D_{i2} - S_{i2})x(z_1D_{1j} + z_2D_{2j}),\\
P_{00} &= \exp_{\geq 2}(S_{00}),\\
P_{01} &= S_{01}(y\exp(S_{02}) + \exp_{\geq 1}(S_{02})),\\
P_{02} &= y\exp_{\geq 1}(S_{02}) + \exp_{\geq 2}(S_{02}),\\
P_{11} &= (yS_{11} + (1+y)S_{12}^2)\exp(S_{22}) + (y + S_{11})\exp_{\geq 1}(S_{22}),\\
P_{12} &= S_{12}(y\exp(S_{22}) + \exp_{\geq 1}(S_{22})),\\
P_{22} &= y\exp_{\geq 1}(S_{22}) + \exp_{\geq 2}(S_{22}),
\end{aligned}$$

where this time $e_{02} = e_{11} = e_{22} = y$ and $e_{ij} = 0$.

2-connected series-parallel graphs

It remains to check the relevent analytic properties of
$B(x, y, z_0, z_1, z_2)$ in order to assure that Lemma 5 can be applied. Eliminating D_{ij} from the above system, we again get a positive and strongly connected system of equations for the set of generating functions $\{S_{ij}, P_{ij}\}$, where the right hand-side consists of entire functions. In particular, the functions S_{ij} and P_{ij} all have a common singular behaviour that is of squareroot type.

And we have that

$$\begin{aligned}
\frac{\partial B}{\partial y} &= x^2z_0z_1S_{01}\exp(S_{02}) + x^2z_0z_2\exp(S_{02}) + x^2z_1z_2S_{12}\exp(S_{22})\\
&\quad + \frac{x^2}{2}z_2^2\exp(S_{22}) + \frac{x^2}{2}z_1^2\left(S_{11} + S_{12}^2 + 1\right)\exp(S_{22}).
\end{aligned}$$

Finally, using the very same arguments as in the case of maximal independent sets, we show that (5) is satisfied in the context of maximal matchings in series-parallel graphs. Thus completing the proof.