On the Number of Variables in Special Classes of Random Lambda-Terms

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Abstract

We investigate the number of variables in two special subclasses of lambda-terms that are restricted by a bound of the number of abstractions between a variable and its binding lambda, and by a bound of the nesting levels of abstractions, respectively. These restrictions are on the one hand very natural from a practical point of view, and on the other hand they simplify the counting problem compared to that of unrestricted lambda-terms in such a way that the common methods of analytic combinatorics are applicable.

We will show that the total number of variables is asymptotically normally distributed for both subclasses of lambda-terms with mean and variance asymptotically equal to $C_1 n$ and $C_2 n$, respectively, where the constants C_1 and C_2 depend on the bound that has been imposed. So far we just derived closed formulas for the constants in case of the class of lambda-terms with a bounded number of abstractions between each variable and its binding lambda. However, for the other class of lambda-terms that we consider, namely lambda-terms with a bounded number of nesting levels of abstractions, we investigate the number of variables in the different abstraction levels and thereby exhibit very interesting results concerning the distribution of the variables within those lambda-terms.

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1 Introduction

The lambda calculus was invented by Church and Kleene in the 30ies as a tool for the investigation of decision problems. Today it still plays an important role in computability theory and for automatic proof systems. Furthermore, it represents the basis for some programming languages, such as LISP. For a thorough introduction to the lambda calculus we refer to [1]. This paper does not require any preliminary knowledge of lambda calculus in order to follow the proofs. Instead we will study the basic objects of lambda calculus, namely lambda-terms, by considering them as combinatorial objects, or more precisely as a special class of directed acyclic graphs (DAGs).

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▶ **Definition 1** (lambda-terms, [9, Definition 3]). Let $\mathcal{V}$ be a countable set of variables. The set Λ of lambda-terms is defined by the following grammar:

1. every variable in $\mathcal{V}$ is a lambda-term,
2. if T and S are lambda-terms then TS is a lambda-term, (application)
3. if T is a lambda-term and x is a variable then $\lambda x.T$ is a lambda-term. (abstraction)

The name application arises, since lambda-terms of the form TS can be regarded as functions $T(S)$, where the function T is applied to S, which in turn can be a function itself. An abstraction can be considered as a quantifier that binds the respective variable in the sub-lambda-term within its scope. Both application and repeated abstraction are not commutative, *i.e.*, in general the lambda-terms TS and ST, as well as $\lambda x.\lambda y.M$ and $\lambda y.\lambda x.M$, are different (with the exceptions of $T = S$ and none of the variables x or y occurring in M, respectively). Each λ binds exactly one variable (which may occur several times in the terms), and since we will just focus on a special subclass of closed lambda-terms, each variable is bound by exactly one λ.

We will consider lambda-terms modulo α-equivalence, which means that we identify two lambda-terms if they only differ by the names of their bound variables. For example $\lambda x.(\lambda y.(xy)) \equiv \lambda y.(\lambda z.(yz))$. There is a combinatorial interpretation of lambda-terms that considers them as DAGs and thereby naturally identifies two α-equivalent terms to be equal. Combinatorially, lambda-terms can be seen as rooted unary-binary trees containing special additional directed edges. Note that in general the resulting structures are not trees in the sense of graph theory, but due to their close relation to trees (see Definition 2) some authors call them lambda-trees or enriched trees. We will call them lambda-DAGs in order to emphasise that these structures are in fact DAGs, if we consider the undirected edges of the underlying tree to be directed away from its root.

▶ **Definition 2** (lambda-DAG, [9, Definition 5]). With every lambda-term T, the corresponding lambda-DAG $G(T)$ can be constructed in the following way:

1. If x is a variable then $G(x)$ is a single node labeled with x. Note that x is unbound.
2. $G(PQ)$ is a lambda-DAG with a binary node as root, having the two lambda-DAGs $G(P)$ (to the left) and $G(Q)$ (to the right) as subgraphs.
3. The DAG G($\lambda x.P$) is obtained from $G(P)$ in four steps:
   a. Add a unary node as new root.
   b. Connect the new root by an undirected edge with the root of G(P).
   c. Connect all leaves of $G(P)$ labelled with x by directed edges with the new root, where the root is start vertex of these edges.
   d. Remove all labels x from $G(P)$. Note that now x is bound.

Obviously, applications correspond to binary nodes and abstractions correspond to unary nodes of the underlying Motzkin-tree that is obtained by removing all directed edges. Of course in the lambda-DAG some of the vertices that were former unary nodes might have gained out-going edges, so they are no unary nodes in the lambda-DAG anymore. However, when we speak of unary nodes in the following, we mean the unary nodes of the underlying unary-binary tree, that forms the skeleton of the lambda-DAG.

Since the skeleton of a lambda-DAG is a tree, we sometimes call the variables leaves (*i.e.*, the nodes with out-degree zero), and the path connecting the root with a leaf (consisting of undirected edges) is called a branch. There are different approaches as to how one can define the size of a lambda-term ([4], [11]), but within this paper the size will be defined as the total number of nodes in the corresponding lambda-DAG.

Figure 1 The lambda-DAGs representing the terms $\lambda x.((\lambda y.(xy))x)$ and $(\lambda x.(x(\lambda y.y))(\lambda x.(\lambda y.xy))$.

Recently rising interest in the number and structural properties of lambda-terms can be observed, due to the direct relationship between these random structures acting as computer programs and mathematical proofs ([7]). At first sight lambda-terms appear to be very simple structures, in the sense that their construction can easily be described, but so far no one has yet accomplished to derive their asymptotic number. However, the asymptotic equivalent of the logarithm of this number can be determined up to the second-order term (see [5]). The difficulty of counting unrestricted lambda-terms arises due to the fact that their number increases superexponentially with increasing size. Thus, if we translate the counting problem into generating functions, then the resulting generating function has a radius of convergence equal to zero, which makes the common methods of analytic combinatorics inapplicable. This fast growth of the number of lambda-terms can be explained by the numerous possible bindings of leaves by lambdas, *i.e.*, by unary nodes. Consequently, lately some simpler subclasses of lambda-terms, which reduce these multiple binding possibilities, have been studied, e.g. lambda-terms with prescribed number of unary nodes ([4]), or lambda-terms in which every lambda binds a prescribed ([5],[2],[9]) or a bounded ([6],[2],[9]) number of leaves. In this paper we will investigate structural properties of lambda-terms with a bounded number of abstractions between every variable and its binding lambda and lambda-terms with a bounded number of nesting levels of abstractions, which both have been introduced in [3] and [4]. From a practical point of view these restrictions appear to be very natural, since the number of abstractions in lambda-terms which are used for computer programming is in general assumed to be very low compared to their size.

Particular interest lies in the number and distribution of the variables within these special subclasses of lambda-terms. We will show within this paper that the total number of leaves in lambda-DAGs with bounded number of abstractions between the leaves and their respective binding lambdas as well as in lambda-terms with bounded number of nested abstractions is asymptotically normally distributed with mean and variance asymptotically Cn and $\tilde{C}n$, respectively, where the constants C and $\tilde{C}$ depend on the bound that has been imposed. For the latter class of lambda-terms we will also investigate the number of leaves on the different abstraction levels (so called unary levels, *cf.* Definition 11), which shows a very interesting behaviour. We will see that on the lower unary levels, *i.e.*, near the root of the lambda-DAG, there are very few leaves, while the majority of the leaves is located at the upper unary levels and these two domains will turn out to be strictly separated.

For lambda-terms that are locally restricted by a bounded number of abstractions located between the leaves and their binding lambdas the number of unary levels is not bounded and will tend to infinity for increasing size. The expected number of unary levels is unknown, which implies that the correct scaling cannot be determined. Thus, we have not been able to establish results concerning the leaves in the different unary levels for this class of lambda-terms so far. Nevertheless, further studies on this subject seem to be very interesting already for the simpler combinatorial class of Motzkin-trees.

Figure 2 The lambda-DAG of the term $\lambda x.((\lambda y.xy)(\lambda z.(z(\lambda t.tx))z))$, where left the unary length of all bindings, and right the unary height of the leaves is depicted at the respective leaves.

2 Main results

In this section we will introduce the basic definitions and summarize the main results that will be presented in this paper.

First, we will investigate the total number of leaves in lambda-DAGs with bounded unary length of their bindings, *i.e.*, with a bounded number of abstractions between each leaf and its binding lambda.

▶ **Definition 3** (unary length of a binding, [4, Definition 1]). Consider a lambda-term T and its associated lambda-DAG $G(T)$. The unary length of the binding of a leaf e by some abstraction v in T (directed edge from v to e in $G(T)$) is defined as the number of unary nodes on the path connecting v and e in the underlying Motzkin tree (*cf.* Figure 2, left).

Our first main result is the asymptotic distribution of the number of variables in random closed lambda-terms with bounded unary length of their bindings.

▶ **Theorem 4.** *Let X_n be the total number of leaves in lambda-DAGs of size n where the unary length of each binding is at most k. Then X_n is asymptotically normally distributed with*

$$\mathbb{E}X_n \sim \frac{k}{\sqrt{k}+2k}n, \qquad \text{and} \qquad \mathbb{V}X_n \sim \frac{k^2}{2\sqrt{k}(\sqrt{k}+2k)^2}n, \qquad \text{as } n \longrightarrow \infty.$$

▶ Remark 5. Note that the number of leaves equals the number of binary nodes plus one. For $k=1$ this implies that expections of the number of unary, the number of binary nodes, and the number of leaves are all asymptotically equal. Since the subtree attached to a unary node cannot contain further unary nodes, asymptotically almost all such trees are only a single leaf. So, almost all unary nodes are on the fringe of the tree.

On the other hand, as $k \to \infty$, we have $\mathbb{E}X_n \longrightarrow \frac{n}{2}$, and $\mathbb{V}X_n \longrightarrow 0$ for $k \longrightarrow \infty$. So, we can expect that a general lambda-term has $o(n)$ unary nodes and looks therefore like a slightly perturbed binary tree. So far, nothing isknown on the distribution of the locations of the unary nodes.

Next we turn to lambda-terms of bounded unary height.

▶ **Definition 6** (unary height, [4, Definition 1]). Consider a lambda-term T and its associated lambda-DAG $G(T)$. The unary height $h_u(v)$ of a vertex v of $G(T)$ is defined as the number of unary nodes on the path from the root to v in the underlying Motzkin tree.

The unary height of the lambda-term T is defined as the maximum number of unary nodes occurring in the separate branches of the underlying Motzkin tree (*cf.* Figure 2, right).

Table 1 The coefficients occurring in the variance and the mean $k = 1, \dots, 12$ and some larger values close to 135, the next value of some special sequence (*cf.* Definition 10), which is indicated by the lines in bold. The second column tells the number of (nested) radicands which must be considered for the determination of the dominant singularity.

bound k	$j+1$	$B''(1) + B'(1) - B'(1)^2$	$B'(1)$
1	**2**	**0**	**0**
2	2	0.0385234386	0.4381229337
3	2	0.0210625856	0.4414407371
4	2	0.0167136805	0.4463973717
5	2	0.0148700270	0.4504258849
6	2	0.0138224393	0.4536185043
7	2	0.0131157948	0.4561987871
8	**3**	**0.048**	**0.4**
9	3	0.0582322465	0.4566104777
10	3	0.0470481360	0.4560418340
11	3	0.0396601986	0.4560810348
12	3	0.0345090124	0.4564489368
⋮	⋮	⋮	⋮
133	3	0.0077469541	0.4821900098
134	3	0.0077234960	0.4822482745
135	**4**	**0.0108490182**	**0.4782608696**

▶ **Theorem 7.** *Let $\rho_k(u)$ be the root of smallest modulus of the function $z \mapsto R_{j+1,k}(z,u)$, where*

$$R_{j+1,k}(z,u) = 1-4(k-j)z^2u-2z+2z\sqrt{1-4(k-j+1)z^2u-2z+\sqrt{\dots+2z\sqrt{1-4kz^2u}}},$$

and let us define $B(u) = \rho_k(u)/\rho_k(1)$.

If $B''(1) + B'(1) - B'(1)^2 \neq 0$, then the total number of leaves in lambda-DAGs with bounded unary height at most k is asymptotically normally distributed with asymptotic mean μn and asymptotic variance $\sigma^2 n$, where $\mu = B'(1)$ and $\sigma^2 = B''(1) + B'(1) - B'(1)^2$.

▶ Remark 8. The requirement $B''(1) + B'(1) - B'(1)^2 \neq 0$ obviously results from the fact that otherwise the variance would be equal to zero. However, this inequality seems to be very difficult to verify, since $B(u) = \frac{\rho_k(1)}{\rho_k(u)}$ and we do not know anything about the function $\rho_k(u)$, except for some crude bounds and its analyticity. In Table 1 we give *inter alia* the coefficients $B''(1) + B'(1) - B'(1)^2$ and $B'(1)$ for the variances and the mean values, respectively, for the first few values for k.

▶ Remark 9. No clear conclusion can be inferred from the numerical values given in Table 1. The mean seems to be slightly increasing, except for the special values belonging to the sequence given in Definition 10. But $k = 10$ is another exception in the interval $k = 9, \dots, 134$ (not listed completely). The variance seems decreasing in any interval between two special values. If k belongs to the special sequence given in Definition 10 then we observe irregularities.

Lambda-terms of bounded unary height have been studied in [4], where a very unusual behaviour has been discovered. The asymptotic behaviour of the number of lambda-terms belonging to this subclass differs depending on whether the bound for the unary height is an element of a certain sequence $(N_i)_{i\geq 0}$, which will be given in Definition 10, or not (in Table 1

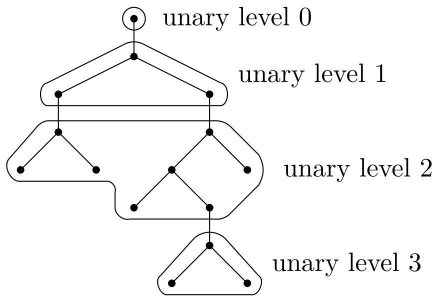

Figure 3 Underlying Motzkin tree of e.g. the lambda term $\lambda x.((\lambda y.yx)(\lambda z.(z(\lambda t.tx))z))$, where the different unary levels are encircled.

the rows belonging to elements of this sequence are therefore in bold). Though the behaviour of the counting sequences differs for these two cases, the result in Theorem 7 concerning lambda-terms of bounded unary height is the same after all. However, the method of proof is different in the two cases. For our subsequent results the distinction of cases will have an impact on the asymptotic behaviour of the investigated structures. Thus, we will have to distinguish between these two cases.

▶ **Definition 10** (auxiliary sequences $(u_i)_{i\geq 0}$ and $(N_i)_{i\geq 0}$, [4, Definition 6])**.** Let $(u_i)_{i\geq 0}$ and $(N_i)_{i\geq 0}$ be the integer sequence defined by $u_0 = 0$, $u_{i+1} = u_i^2 + i + 1$ for $i \geq 0$, and $N_i = u_i^2 - u_i + i$, for $i \geq 0$.

Finally, in the last section we investigate the number of leaves in lambda-DAGs with bounded unary height that are located in the different unary levels throughout the tree.

▶ **Definition 11** (unary level)**.** A node is said to be in the i-th unary level, if there are exactly i unary nodes on the branch from the root to that node (the node itself is not counted). Thus, the i-th unary level contains all nodes with unary height i (*cf.* Figure 3).

The following theorem includes the results that we will present in Section 5, where we show that the number of leaves near the root of the lambda-DAG, *i.e.*, in the lower unary levels, is very low, while there are many leaves in the upper unary levels. Furthermore these two domains are strictly separated and the "separating level", *i.e.*, the first level with many leaves, depends on the bound of the unary height. We will show a very interesting behaviour, namely that, with growing bound of the unary height, the number of leaves within the unary level that is directly below the critical separating level increases, until the bound reaches a certain number, which makes this adjacent leaf-filled level become the new separating level.

▶ **Theorem 12.** *Let $\tilde{\rho}_{k,l}(u)$ be the root of smallest modulus of the function $z \mapsto \tilde{R}_{j+1,k}(z,u)$, where*

$$\tilde{R}_{i,k}(z,u) = 1-4(k-j)z^2-2z+\sqrt{\ldots+2z\sqrt{1-4(k-l)z^2u-2z+2z\sqrt{\ldots+2z\sqrt{1-4kz^2}}}},$$

i.e., the u is inserted only in the $(l+1)$-th radicand, and let us define $\tilde{B}_l(u) = \frac{\tilde{\rho}_{k,l}(u)}{\tilde{\rho}_{k,l}(1)}$.

1. *If $k \in (N_j, N_{j+1})$, then the average number of leaves in the first $k-j$ unary levels is $O(1)$, as $n \longrightarrow \infty$, while it is $\theta(n)$ for the last $j+1$ unary levels.*
   In particular, if $\tilde{B}_l''(1) + \tilde{B}_l'(1) - \tilde{B}_l'(1)^2 \neq 0$, the number of leaves in each of the last $j+1$ unary levels (i.e., $l = 0, \ldots, j$) is asymptotically normally distributed.

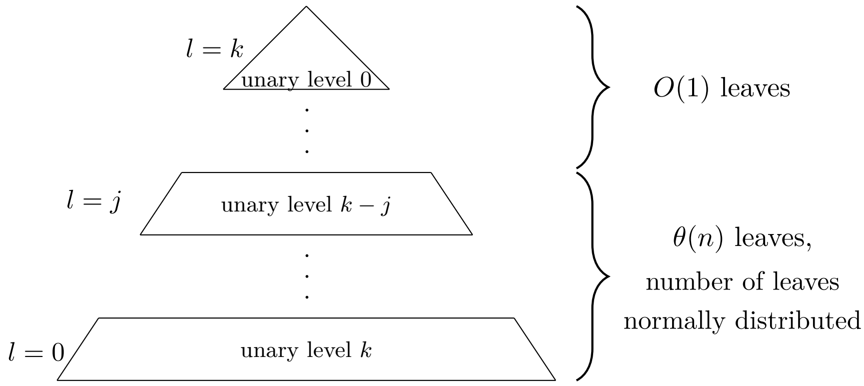

Figure 4 Summary of the mean values of the number of leaves in the different unary levels for the case $k \in (N_j, N_{j+1})$ in lambda-terms of unary height at most k.

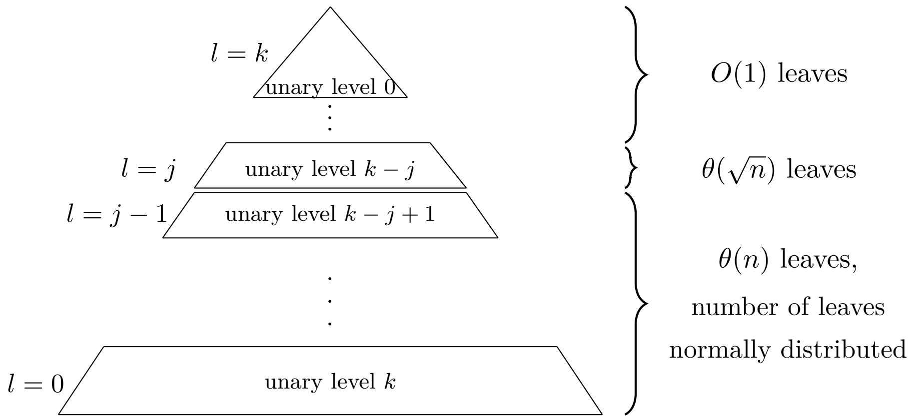

Figure 5 Summary of the mean values of the number of leaves in the different unary levels for the case $k = N_j$ in lambda-terms with unary height at most k.

2. *If $k = N_j$, then the average number of leaves in the first $k - j$ unary levels is $O(1)$, as $n \longrightarrow \infty$, while the average number of leaves in the j-th unary level is $\theta(\sqrt{n})$. The last j unary levels have asymptotically $\theta(n)$ leaves.*
   In particular, if $\tilde{B}_l''(1) + \tilde{B}_l'(1) - \tilde{B}_l'(1)^2 \neq 0$, the number of leaves in each of the last j unary levels (i.e., $l = 0, \dots, j-1$) is asymptotically normally distributed.

3 Total number of leaves in lambda-terms with bounded unary length of bindings

In this section we investigate the asymptotic number of all leaves in lambda-terms with bounded unary length of their bindings (*cf.* Definition 3). In order to get some quantitative results on this restricted class of lambda-terms we will use the well-known symbolic method (see [8]) and therefore we introduce certain combinatorial classes as it has been done in [4]: $\mathcal{Z}$ denotes the class of atoms, $\mathcal{A}$ the class of application nodes (*i.e.*, binary nodes), $\mathcal{U}$ the class of abstraction nodes (*i.e.*, unary nodes), and $\hat{\mathcal{P}}^{(i,k)}$ the class of unary-binary trees such that every leaf e can be labelled in $\min\{h_u(e) + i, k\}$ ways. The classes $\hat{\mathcal{P}}^{(i,k)}$ can be specified by

$$\hat{\mathcal{P}}^{(k,k)} = k\mathcal{Z} + (\mathcal{A} \times \hat{\mathcal{P}}^{(k,k)} \times \hat{\mathcal{P}}^{(k,k)}) + (\mathcal{U} \times \hat{\mathcal{P}}^{(k,k)}),$$

and

$$\hat{\mathcal{P}}^{(i,k)} = i\mathcal{Z} + (\mathcal{A} \times \hat{\mathcal{P}}^{(i,k)} \times \hat{\mathcal{P}}^{(i,k)}) + (\mathcal{U} \times \hat{\mathcal{P}}^{(i+1,k)}) \qquad \text{for } \ i < k.$$

Translating into generating functions with z marking the size and u marking the number of leaves, and solving for $\hat{P}^{(i,k)}(z,u)$ yields

$$\hat{P}^{(i,k)}(z,u) = \frac{1 - \mathbf{1}_{[i=k]}z - \sqrt{\hat{R}_{k-i+1}(z,u)}}{2z},$$

with $\hat{R}_{1,k}(z,u) = (1-z)^2 - 4kuz^2$, $\hat{R}_{2,k}(z,u) = 1 - 4(k-1)z^2u - 2z + 2z^2 + 2z\sqrt{\hat{R}_{1,k}(z,u)}$, and $\hat{R}_{i,k}(z,u) = 1 - 4(k-i+1)z^2u - 2z + 2z\sqrt{\hat{R}_{i-1,k}(z,u)}$,for $3 \le i \le k+1$.

Since the class $\hat{\mathcal{P}}^{(0,k)}$ is isomorphic to the class $\mathcal{G}_k$ of lambda-terms where all bindings have unary lengths not larger than k, we get for the corresponding bivariate generating function

$$G_k(z,u) = \hat{P}^{(0,k)}(z,u) = \frac{1 - \sqrt{\hat{R}_{k+1,k}(z,u)}}{2z}.$$

From [4] we know that the dominant singularity of $G_k(z,1)$ comes from the innermost radical and is of type $\frac{1}{2}$. Due to continuity arguments this implies that in a sufficiently small neighbourhood of $u = 1$ the dominant singularity $\hat{\rho}_k(u)$ of $G_k(z,u)$ comes also from the innermost radical and is also of type $\frac{1}{2}$. By calculating the smallest positive root of $\hat{R}_{1,k}(z,u)$ we get $\hat{\rho}_k(u) = \frac{1}{1+2\sqrt{ku}}$. Now we will determine the expansions of the radicands in a neighbourhood of the dominant singularity $\hat{\rho}_k(u)$.

▶ **Proposition 13.** *Let $\hat{\rho}_k(u)$ be the root of the innermost radicand $\hat{R}_{1,k}(z,u)$, i.e., $\hat{\rho}_k(u) = \frac{1}{1+2\sqrt{ku}}$. Then*

$$\hat{R}_{1,k}(\hat{\rho}_k(u)(1-\epsilon), u) = \left(2\hat{\rho}_k(u) - 2\hat{\rho}_k^2(u) + 8ku\hat{\rho}_k^2(u)\right)\epsilon + \mathcal{O}(\epsilon^2),$$

$$\hat{R}_{j,k}(\hat{\rho}_k(u)(1-\epsilon), u) = c_j\hat{\rho}_k^2(u) + \frac{4\hat{\rho}_k^2(u)((ku)^{\frac{1}{4}} + \sqrt{2ku})}{\prod_{l=2}^{j}\sqrt{c_l}}\sqrt{\epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}}),$$

for $2 \le j \le k+1$, where $c_1(u) = 1$ and $c_j(u) = 4(j-1)u - 1 + 2\sqrt{c_{j-1}(u)}$ for $2 \le j \le k+1$.

▶ **Theorem 14.** *Let for any fixed k, $G_k(z,u)$ denote the bivariate generating function of lambda-terms where all bindings have unary lengths not larger than k. Then*

$$[z^n]G_k(z,u) = \sqrt{\frac{\sqrt{ku} + 2ku}{4\pi\prod_{l=2}^{k+1}c_l(u)}}(1+2\sqrt{ku})^n n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad \textit{for } n \longrightarrow \infty,$$

where $c_1(u) = 1$ and $c_j(u) = 4(j-1)u - 1 + 2\sqrt{c_{j-1}(u)}$, for $2 \le j \le k+1$.

From [4, Theorem 1] we know the following result.

$$[z^n]G_k(z,1) = \sqrt{\frac{\sqrt{k} + 2k}{4\pi\prod_{l=2}^{k+1}c_l(1)}}(1+2\sqrt{k})^n n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \longrightarrow \infty, \qquad (1)$$

with c_l defined as in Proposition 13.

Now we want to apply the well-known Quasi-Power Theorem.

▶ **Theorem 15** (Quasi-Power Theorem, [10]). *Let X_n be a sequence of random variables with the property that*

$$\mathbb{E}u^{X_n} = A(u)B(u)^{\lambda_n}\left(1+\mathcal{O}\left(\frac{1}{\phi_n}\right)\right)$$

holds uniformly in a complex neighbourhood of $u=1$, $\lambda_n \longrightarrow \infty$ and $\phi_n \longrightarrow \infty$, and $A(u)$ and $B(u)$ are analytic functions in a neighbourhood of $u=1$ with $A(1)=B(1)=1$. Set $\mu = B'(1)$ and $\sigma^2 = B''(1)+B'(1)-B'(1)^2$. If $\sigma^2 \neq 0$, then

$$\frac{X_n - \mathbb{E}X_n}{\sqrt{\mathbb{V}X_n}} \longrightarrow \mathcal{N}(0,1),$$

with $\mathbb{E}X_n = \mu\lambda_n + A'(1) + \mathcal{O}(1/\phi_n))$ and $\mathbb{V}X_n = \sigma^2\lambda_n + A''(1)+A'(1)-A'(1)^2+\mathcal{O}(1/\phi_n))$.

Using Theorem 14 and (1), we get

$$\mathbb{E}u^{X_n} = \frac{[z^n]G_k(z,u)}{[z^n]G_k(z,1)} = \left(\frac{1+2\sqrt{ku}}{1+2\sqrt{k}}\right)^n \sqrt{\frac{\sqrt{ku}+2ku}{2k+\sqrt{k}}\prod_{j=2}^{k+1}\frac{c_j(1)}{c_j(u)}}\left(1+O\left(\frac{1}{n}\right)\right),$$

where $c_1(u)=1$ and $c_j(u) = 4ju-4u-1+2\sqrt{c_{j-1}(u)}$.

Thus, all assumptions for the Quasi-Power Theorem are fulfilled, and we get that the number of leaves in lambda-DAGs with bounded unary length of their bindings is asymptotically normally distributed with

$$\mathbb{E}X_n \sim \frac{k}{\sqrt{k}+2k}n, \quad \text{and} \quad \mathbb{V}X_n \sim \frac{k^2}{2\sqrt{k}(\sqrt{k}+2k)^2}n, \quad \text{as } n \longrightarrow \infty,$$

and therefore Theorem 4 is shown.

4 Total number of leaves in lambda-terms with bounded unary height

This section is devoted to the enumeration of leaves in lambda-terms of bounded unary height (*cf.* Definition 6). As in [4] let us denote by $\mathcal{P}^{(i,k)}$ the class of unary-binary trees such that the unary height $h_u(e)$ of each leaf e is at most $k-i$ and every leaf can be colored with one out of $i+h_u(e)$ colors. These classes can be specified by

$$\mathcal{P}^{(k,k)} = k\mathcal{Z} + (\mathcal{A}\times\mathcal{P}^{(k,k)}\times\mathcal{P}^{(k,k)}),$$

and

$$\mathcal{P}^{(i,k)} = i\mathcal{Z} + (\mathcal{A}\times\mathcal{P}^{(i,k)}\times\mathcal{P}^{(i,k)}) + (\mathcal{U}\times\mathcal{P}^{(i+1,k)}) \qquad \text{for} \quad i<k.$$

Their bivariate generating functions can be derived analogously as the univariate ones in [4] and read as

$$P^{(i,k)}(z,u) = \frac{1-\sqrt{R_{k-i+1,k}(z,u)}}{2z},$$

where $R_{1,k}(z,u) = 1-4kz^2u$, and $R_{i,k}(z,u) = 1-4(k-i+1)z^2u-2z+2z\sqrt{R_{i-1,k}(z,u)}$, for $2\le i\le k+1$.

For the bivariate generating function of lambda-terms with bounded unary height this implies

$$H_k(z,u) = P^{(0,k)}(z,u) = \frac{1-\sqrt{R_{k+1,k}(z,u)}}{2z}. \qquad (2)$$

Thus, the generating function consists again of $k+1$ nested radicals, but as stated in Section 2, the counting sequence of lambda-terms with bounded unary height has a very unusual behaviour, namely the location and the type of the dominant singularity changes with the bound k. More precisely, the following result has been shown in [4].

▶ **Theorem 16** ([4, Theorem 3]). *Let $(u_i)_{i\geq 0}$ and $(N_i)_{i\geq 0}$ be the integer sequences defined in Definition 10.*

(i) *If there exists $j \geq 0$ such that $N_j < k < N_{j+1}$, then there exists a constant h_k such that*

$$[z^n]H_k(z) \sim h_k n^{-3/2} \rho_k(1)^{-n}, \text{ as } n \longrightarrow \infty.$$

(ii) *If there exists j such that $k = N_j$, then the following asymptotic relation holds:*

$$[z^n]H_k(z) \sim h_k n^{-5/4} \rho_k(1)^{-n} = h_k n^{-5/4} (2u_j)^n \text{ as } n \longrightarrow \infty.$$

Thus, in order to investigate structural properties of this class of lambda-terms we perform a distinction of cases whether the bound k is an element of the sequence $(N_i)_{i\geq 0}$ or not.

4.1 The case $N_j < k < N_{j+1}$

From [4] we know that in this case the dominant singularity of the generating function $H_k(z,1)$ comes from the $(j+1)$-th radicand $R_{j+1,k}$ and is of type $\frac{1}{2}$. As in the previous section we can again use continuity arguments to guarantee that sufficiently close to $u=1$ the dominant singularity $\rho_k(u)$ of $H_k(z,u)$ comes from the $(j+1)$-th radicand $R_{j+1,k}(z,u)$ and is of type $\frac{1}{2}$. Now we will determine the expansions of the radicands in a neighbourhood of the dominant singularity.

▶ **Proposition 17.** *Let $\rho_k(u)$ be the dominant singularity of $H_k(z,u)$. Then*

(i) $\forall i < j+1$ *(inner radicands)* : $R_{i,k}(\rho_k(u)(1-\epsilon),u) = R_{i,k}(\rho_k(u),u) + \mathcal{O}(\epsilon)$

(ii) $R_{j+1,k}(\rho_k(u)(1-\epsilon),u) = \rho_k(u)\gamma_{j+1}(u)\epsilon + \mathcal{O}(\epsilon^2)$, *with* $\gamma_{j+1}(u) = -\frac{\partial}{\partial z}R_{j+1,k}(\rho_k(u),u)$

(iii) $\forall i > j+1$ *(outer radicands)* : $R_{i,k}(\rho_k(u)(1-\epsilon),u) = a_i(u) + b_i(u)\sqrt{\epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}})$, *with* $a_{i+1}(u) = 1-4(k-i)\rho_k^2(u)u - 2\rho_k(u) + 2\rho_k(u)\sqrt{a_i(u)}$, *and* $b_{i+1}(u) = \frac{b_i(u)\rho_k(u)}{\sqrt{a_i(u)}}$ *for* $j+2 \leq i \leq k$, *with* $a_{j+2}(u) = 1-4(k-j)\rho_k^2(u)u - 2\rho_k(u)$ *and* $b_{j+2}(u) = 2\rho_k(u)\sqrt{\rho_k(u)\gamma_{j+1}(u)}$.

We know that for sufficiently large i the sequence u_i is given by $u_i = \lfloor \chi^{2^i} \rfloor$, with $\chi \approx 1.36660956\ldots$ (see [4, Lemma 18]). Therefore we have $N_j \sim u_j^2 \sim \chi^{2^j 2}$ and $N_j < k < N_{j+1} = O(N_j^2)$, which gives $j \asymp \log\log k$. This implies that $j+1 < k+1$, *i.e.*, that the dominant singularity $\rho_{j+1,k}(u)$ cannot come from the outermost radical.

▶ Remark 18. Obviously the same is true for the case $k = N_j$. Thus, the dominant singularity never comes from the outermost radical.

Using Proposition 17 and (2) we can prove

$$[z^n]H_k(z,u) = h_k(u)\rho_{j+1,k}(u)^{-n}\frac{n^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}\left(1+O\left(\frac{1}{n}\right)\right), \quad \text{as } n \longrightarrow \infty,$$

with $h_k(u) = -\frac{b_{k+1,k}(u)}{4\rho_{j+1,k}(u)\sqrt{a_{k+1}(u)}}$.

Taking a look at the recursive definitions of $a_i(u)$ and $b_i(u)$ (see Proposition 17), it can easily be seen that these functions are not equal to zero in a neighbourhood of $u = 1$, which implies that $h_k(u) \neq 0$ and thus we can apply the Quasi-Power Theorem. What is still left to show is, that $\sigma^2 = B''(1) + B'(1) - B'(1)^2 \neq 0$ with $B(u) = \frac{\rho_k(1)}{\rho_k(u)}$. Unfortunately, as stated in Section 2 this task appears to be quite difficult, since there is only very little known about the function $\rho_k(u)$. However, it seems very likely that this condition will be fulfilled for arbitrary $k \in (N_j, N_{j+1})$, so that the Quasi-Power Theorem can be applied and we get that the number of leaves in lambda-terms of bounded unary height is asymptotically normally distributed with asymptotic mean and variance μn and $\sigma^2 n$, respectively, where $\mu = B'(1)$ and $\sigma^2 = B''(1) + B'(1) - B'(1)^2$, with $B(u) = \frac{\rho_k(1)}{\rho_k(u)}$.

4.2 The case $k = N_j$

We know from [4] that in the case $k = N_j$ both radicands $R_{j,k}(z, 1)$ and $R_{j+1,k}(z, 1)$ vanish simultaneously and the dominant singularity is therefore of type $\frac{1}{4}$.

Now we will investigate how the radicands behave in a neighbourhood of the dominant singularity $\rho_k(u)$ for $u \neq 1$.

▶ **Lemma 19.** *Let $z = \rho_k(u)$ be the dominant singularity of the bivariate generating function $H_k(z, u)$. Then*

(i) $R_{j,k}\left(\rho_k(u)\left(1 + \frac{t}{n}\right), 1 + \frac{s}{n}\right) = \frac{1}{n}\left(c_{j,1} \cdot t + c_{j,2} \cdot s\right) + O\left(\frac{|t|^2+|s|^2}{n^2},\right)$ *with* $c_{j,1} = 4\rho_k(1)^2 - 2\rho_k(1) - 8(k - j + 1)\rho_k(1)^2$, *and* $c_{j,2} = 4\rho_k(1)\rho_k'(1) - 4(k - j + 1)\rho_k(1)^2 - 8(k - j + 1)\rho_k(1)\rho_k'(1) - 2\rho_k'(1)$.

(ii) $R_{j+1,k}\left(\rho_k(u)\left(1 + \frac{t}{n}\right), 1 + \frac{s}{n}\right) = \frac{1}{n}\left(c_{j+1,1} \cdot t + c_{j+1,2} \cdot s\right) + 2\rho_k(1)\sqrt{R_{j,k}} + O\left(n^{-3/2}\right)$, *with* $c_{j+1,1} = -8(k - j)\rho_k(1)^2 - 2\rho_k(1)$, *and* $c_{j+1,2} = -2\rho_k'(1) - 4(k - j)\rho_k(1)^2 - 8\rho_k(1)\rho_k'(1)(k - j)$.

(iii) $R_{j+p,k}\left(\rho_k(u)\left(1 + \frac{t}{n}\right), 1 + \frac{s}{n}\right) = \hat{C}_{j+p} + \hat{D}_{j+p}\sqrt[4]{R_{j,k}} + O\left(\frac{1}{n}p_{j+p}(t, s)\right)$, *for* $2 \leq p \leq k - j + 1$, *where* $p_{j+p}(t, s)$ *is a polynomial that is linear in* t *and* s, *and* $\hat{C}_{j+p}$ *and* $\hat{D}_{j+p}$ *are constants.*

▶ **Proposition 20.** *Let $H_k(z, u)$ be the bivariate generating function of the class of lambda-terms with unary height at most k. Then the n-th coefficient of $H_k(z, u)$ is given by*

$$[z^n]H_k(z, u) = \tilde{h}_k(u)\rho_k(u)^{-n}n^{-\frac{5}{4}}\left(1 + O\left(n^{-\frac{3}{4}}\right)\right), \quad as \longrightarrow \infty,$$

with a constant $\tilde{h}_k(u) \neq 0$.

Thus, we apply the Quasi-Power Theorem and like in the previous case (where $k \in (N_j, N_{j+1})$) what is left to show is that the variance $\sigma^2 = B''(1) + B'(1) - B'(1)^2$ with $B(u) = \frac{\rho_k(1)}{\rho_k(u)}$ is positive. Assuming this requirement is valid we get that the total number of leaves in a lambda-term of bounded unary-height is asymptotically normally distributed for arbitrary bounds k.

5 Number of leaves in the unary levels in lambda-terms with bounded unary height

The aim of this section is the investigation of the distribution of the number of leaves in the different unary levels in lambda-terms with bounded unary height (*cf.* Definition 11). In order to do so, let us consider that each unary level in such a lambda-term corresponds to one or more binary trees that contain different types of leaves, where the number of types depends on the respective level (*cf.* Figure 3). Let $\mathcal{C}$ be the class of binary trees. Using the notation from the previous sections we can specify this class by $\mathcal{C} = \mathcal{Z} + (\mathcal{A} \times \mathcal{C} \times \mathcal{C})$. Translating into generating functions and solving for $C(z,u)$, with z marking the size (*i.e.*, the total number of nodes) and u marking the number of leaves, yields $C(z,u) = \frac{1-\sqrt{1-4uz^2}}{2z}$.

Let ${}_{k-l}H_k(z,u)$ be the generating function of lambda-terms with unary height at most k, where z marks the size and u marks the number of leaves on the $(k-l)$-th unary level $(0 \le l \le k)$. Then

$${}_{k-l}H_k(z,u) = C(z, C(z, 1 + \ldots + C(z, (k-l)\cdot u + \ldots + C(z, (k-1) + C(z,k)))\ldots)\ldots)),$$

which can be written as

$${}_{k-l}H_k(z,u) = \frac{1-\sqrt{\tilde{R}_{k+1}(z,u)}}{2z},$$

with $\tilde{R}_1(z,u) = 1-4z^2k$, $\tilde{R}_i(z,u) = 1-4z^2(k-i+1)-2z+2z\sqrt{\tilde{R}_{i-1}(z,u)}$, for $2 \le i \le k+1$, $i \ne l+1$, and $\tilde{R}_{l+1}(z,u) = 1-4z^2u(k-l)-2z+2z\sqrt{\tilde{R}_{l-1}(z,u)}$.

▶ Remark 21. Note that the radicands $\tilde{R}_i$ that are introduced above are very similar to the radicands $R_{i,k}$ that were used in the previous section. The only difference is that now we have a u only in the $(l+1)$-th radicand, while in the previous case u was occurring in all radicands. Thus, we will have further distinction of cases now depending on the relative position (w.r.t. l) of the radicand(s) where the dominant sigularity comes from.

We obtain the following result for the asymptotic mean values of the number of leaves in the different unary levels.

▶ **Proposition 22.** *Let X_n denote the number of leaves in the $(k-l)$-th unary level in a random lambda-term of unary height at most k with size n.*

1. *If $k \in (N_j, N_{j+1})$, then we get for the asymptotic mean*
   - *in the case $l > j$:*

$$\mathbb{E}X_n = \frac{[z^n]\left(\frac{\partial}{\partial u}\ {}_{k-l}H_k(z,u)\right)|_{u=1}}{[z^n]_{k-l}H_k(z,1)} = C_{k,l}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

   - *and in the case $l \le j$:*

$$\mathbb{E}X_n = \frac{[z^n]\left(\frac{\partial}{\partial u}\ {}_{k-l}H_k(z,u)\right)|_{u=1}}{[z^n]_{k-l}H_k(z,1)} = \tilde{C}_{k,l}\cdot n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

   with constants $C_{k,l}$ and $\tilde{C}_{k,l}$ depending on l and k.
2. *If $k = N_j$, then the asymptotic mean reads as*
   - *in the case $l > j$:*

$$\mathbb{E}X_n = \frac{[z^n]\left(\frac{\partial}{\partial u}\ {}_{k-l}H_k(z,u)\right)|_{u=1}}{[z^n]_{k-l}H_k(z,1)} = D_{k,l}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

   - *in the case $l = j$:*

$$\mathbb{E}X_n = \frac{[z^n]\left(\frac{\partial}{\partial u}\ {}_{k-l}H_k(z,u)\right)|_{u=1}}{[z^n]_{k-l}H_k(z,1)} = \hat{D}_{k,l}\cdot\sqrt{n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

- *and in the case $l < j$:*

$$\mathbb{E}X_n = \frac{[z^n]\left(\frac{\partial}{\partial u}\ {}_{k-l}H_k(z,u)\right)|_{u=1}}{[z^n]_{k-l}H_k(z,1)} = \tilde{D}_{k,l} \cdot n\left(1+\mathcal{O}\left(n^{-\frac{1}{4}}\right)\right),$$

with constants $D_{k,l}$, $\hat{D}_{k,l}$ and $\tilde{D}_{k,l}$ depending on l and k.

Now that we derived the mean values for the number of leaves in the different unary levels, we are interested in their distribution. Therefore we perform the same distinction of cases as we did for the mean values. However, so far we only know the distribution of the leaves in the all those levels, which contain many leaves.

▶ **Proposition 23.** *Let $z = \tilde{\rho}_{k,l}(u)$ denote the dominant singularity of ${}_{k-l}H_k(z,u)$.*

1. *If $k \in (N_j, N_{j+1})$, then we get for $l \leq j$*

$$\frac{[z^n]\ {}_{k-l}H_k(z,u)}{[z^n]\ {}_{k-l}H_k(z,1)} = \frac{\tilde{h}_k(u)}{h_k}\left(\frac{\tilde{\rho}_{k,l}(1)}{\tilde{\rho}_{k,l}(u)}\right)^n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

with constants $\tilde{h}_k(u)$ and h_k that are not equal to zero.

2. *If $k = N_j$, then it holds for $l < j$*

$$\frac{[z^n]\ {}_{k-l}H_k(z,u)}{[z^n]\ {}_{k-l}H_k(z,1)} = \frac{\hat{h}_k(u)}{h_k}\left(\frac{\tilde{\rho}_{k,l}(1)}{\tilde{\rho}_{k,l}(u)}\right)^n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

with constants $\hat{h}_k(u)$ and h_k that are not equal to zero.

Finally, by using the Quasi-Power Theorem the proof of Theorem 12 is finished. Therefore we have to assume again that the variance is not equal to zero.

As stated before the generating function ${}_{k-j}H_k(z,u)$ consists of $k+1$ nested radicals, where a u is inserted in the $(l+1)$-th radicand counted from the innermost. In the case $k \in (N_j, N_{j+1})$ we know that the dominant singularity $\tilde{\rho}_k(u)$ comes from the $(j+1)$-th radicand. Thus, if $l > j$ then $\tilde{\rho}_k(u)$ is independent of u and we will not get a quasi-power. The same holds for the case $k = N_j$ and $l < j$, since we showed that in this case the dominant singularity comes from the j-th radicand. The $(j+1)$-th unary level for $k = N_j$ is a special case, because we do not know whether the dominant singularity comes from the j-th or the $(j+1)$-th radicand. However, it seems very unlikely that the number of leaves in this level will be asymptotically normally distributed, but further studies on this subject might be very interesting.

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