Analysis of Summatory Functions of Regular Sequences: Transducer and Pascal's Rhombus

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Abstract

The summatory function of a q-regular sequence in the sense of Allouche and Shallit is analysed asymptotically. The result is a sum of periodic fluctuations multiplied by a scaling factor. Each summand corresponds to an eigenvalues of absolute value larger than the joint spectral radius of the matrices of a linear representation of the sequence. The Fourier coefficients of the fluctuations are expressed in terms of residues of the corresponding Dirichlet generating function. A known pseudo Tauberian argument is extended in order to overcome convergence problems in Mellin–Perron summation.

Two examples are discussed in more detail: The case of sequences defined as the sum of outputs written by a transducer when reading a q-ary expansion of the input and the number of odd entries in the rows of Pascal's rhombus.

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1 Introduction

In this paper, we study the asymptotic behaviour of the summatory function of q-regular sequences.[1] Regular sequences have been introduced by Allouche and Shallit [2] (see also [3, Chapter 16]); these are sequences which are intimately related to the q-ary expansion of their arguments. Many special cases have been investigated in the literature; this is also due to their relation to divide-and-conquer algorithms. Our goal is to provide a single result decomposing the summatory function into periodic fluctuations multiplied by some scaling functions and to provide the Fourier coefficients of these periodic fluctuations.

Note that it is well-known that the summatory function of a q-regular sequence is itself q-regular. (This is an immediate consequence of [2, Theorem 3.1].) Similarly, the sequence of differences of a q-regular sequence is q-regular. Therefore, we might also start to analyse a regular sequence by considering it to be the summatory function of its sequence of differences. However, when modelling a quantity by a regular sequences, its asymptotic behaviour is often not smooth, but the asymptotic behaviour of its summatory functions is. Moreover, we will see throughout this work that from a technical perspective, considering partial sums is appropriate. Therefore, we adopt this point of view of summatory functions of q-regular sequences in this paper. This also enlightens us about the expectation of a random element of the sequence (with respect to uniform distribution on the non-negative integers smaller than a certain N).

In the remaining paper, we first recall the definition of q-regular sequences in Section 1.1, then formulate a somewhat simplified version of our main result in Section 1.2. In Section 1.3, we give a heuristic non-rigorous argument to explain why the result is expected. We outline the relation to previous work in Section 1.4. We give two examples in Sections 2 and 3. In principle, these examples are straight-forward applications of the results, but still, we have to reformulate the relevant questions in terms of a q-regular sequence and will then provide shortcuts for the computation of the Fourier series. The first example is generic and deals with sequences defined as the sum of outputs of transducer automata; the second example—which motivated us to conduct this study at this point—is a concrete problem counting the number of odd entries in Pascal's rhombus.

The full formulation of our results is given in the appendix; their proofs are given in the appendix of the arXiv version [17] of this extended abstract.

1.1 q-Regular Sequences

We start by giving a definition of q-regular sequences, see Allouche and Shallit [2]. Let $q \geq 2$ be a fixed integer and $(x(n))_{n\geq 0}$ be a sequence.

Then $(x(n))_{n\geq 0}$ is said to be $(\mathbb{C}, q)$-*regular* (briefly: q-*regular* or simply *regular*) if the $\mathbb{C}$-vector space generated by its q-*kernel*

$$\bigl\{\bigl(x(q^j n + r)\bigr)_{n\geq 0} \colon \text{integers } j \geq 0,\, 0 \leq r < q^j\bigr\}$$

has finite dimension. In other words, $(x(n))_{n\geq 0}$ is q-regular if there is an integer D and sequences $(x_1(n))_{n\geq 0}$, …, $(x_D(n))_{n\geq 0}$ such that for every $j \geq 0$ and $0 \leq r < q^j$ there exist integers c_1, …, c_D such that

$$x(q^j n + r) = c_1 x_1(n) + \cdots + c_D x_D(n) \qquad \text{for all } n \geq 0.$$

[1] In the standard literature [2, 3] these sequences are called k-regular sequences (instead of q-regular sequences).

By Allouche and Shallit [2, Theorem 2.2], $(x(n))_{n\geq 0}$ is q-regular if and only if there exists a vector valued sequence $(v(n))_{n\geq 0}$ whose first component coincides with $(x(n))_{n\geq 0}$ and there exist square matrices $A_0, \dots, A_{q-1} \in \mathbb{C}^{d\times d}$ such that

$$v(qn+r) = A_r v(n) \qquad \text{for } 0 \le r < q,\ n \ge 0. \tag{1.1}$$

This is called a *q-linear representation* of $x(n)$.

The best-known example for a 2-regular function is the binary sum-of-digits function.

▶ **Example 1.** For $n \geq 0$, let $x(n) = s(n)$ be the binary sum-of-digits function. We clearly have

$$\begin{aligned} x(2n) &= x(n), \\ x(2n+1) &= x(n) + 1 \end{aligned} \tag{1.2}$$

for $n \geq 0$.

Indeed, we have

$$x(2^j n + r) = x(n) + x(r) \cdot 1$$

for integers $j \geq 0$, $0 \leq r < 2^j$ and $n \geq 0$; i.e., the complex vector space generated by the 2-kernel is generated by $(x(n))_{n\geq 0}$ and the constant sequence $(1)_{n\geq 0}$.

Alternatively, we set $v(n) = (x(n), 1)^\top$ and have

$$\begin{aligned} v(2n) &= \begin{pmatrix} x(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n), \\ v(2n+1) &= \begin{pmatrix} x(n)+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v(n) \end{aligned}$$

for $n \geq 0$. Thus (1.1) holds with

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We defer the discussion of other examples, both generic such as sequences defined by transducer automata as well as a specific example involving the number of odd entries in Pascal's rhombus to Sections 2 and 3.

At this point, we note that a linear representation (1.1) immediately leads to an explicit expression for $x(n)$ by induction.

▶ Remark. Let $r_{\ell-1}\dots r_0$ be the q-ary digit expansion[2] of n. Then

$$x(n) = e_1 A_{r_0} \cdots A_{r_{\ell-1}} v(0)$$

where $e_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$.

[2] Whenever we write that $r_{\ell-1}\dots r_0$ is the q-ary digit expansion of n, we mean that $r_j \in \{0,\dots,q-1\}$ for $0 \leq j < \ell$, $r_{\ell-1} \neq 0$ and $n = \sum_{j=0}^{\ell-1} r_j q^j$. In particular, the q-ary expansion of zero is the empty word.

1.2 Main Result

We are interested in the asymptotic behaviour of the summatory function $X(N) = \sum_{0\le n<N} x(n)$.

At this point, we give a simplified version of our results. We choose any vector norm $\|\cdot\|$ on $\mathbb{C}^d$ and its induced matrix norm. We set $C := \sum_{r=0}^{q-1} A_r$. We choose $R > 0$ such that $\|A_{r_1}\cdots A_{r_\ell}\| = O(R^\ell)$ holds for all $\ell \ge 0$ and $0 \le r_1, \ldots, r_\ell < q$. In other words, R is an upper bound for the joint spectral radius of $A_1, \ldots, A_{q-1}$. The spectrum of C, i.e., the set of eigenvalues of C, is denoted by $\sigma(C)$. For $\lambda \in \mathbb{C}$, let $m(\lambda)$ denote the size of the largest Jordan block of C associated with λ; in particular, $m(\lambda) = 0$ if $\lambda \notin \sigma(C)$. Finally, we consider the Dirichlet series[3]

$$\mathcal{X}(s) = \sum_{n\ge1} n^{-s}x(n), \qquad \mathcal{V}(s) = \sum_{n\ge1} n^{-s}v(n)$$

where $v(n)$ is the vector valued sequence defined in (1.1). Of course, $\mathcal{X}(s)$ is the first component of $\mathcal{V}(s)$. The principal value of the complex logarithm is denoted by log. The fractional part of a real number z is denoted by $\{z\} := z - \lfloor z \rfloor$.

▶ **Theorem 2.** *With the notations above, we have*

$$X(N) = \sum_{\substack{\lambda\in\sigma(C)\\|\lambda|>R}} N^{\log_q \lambda} \sum_{0\le k<m(\lambda)} (\log_q N)^k \Phi_{\lambda k}(\{\log_q N\}) + O\bigl(N^{\log_q R}(\log N)^{\max\{m(\lambda)\colon |\lambda|=R\}}\bigr) \tag{1.3}$$

for suitable 1*-periodic continuous functions* $\Phi_{\lambda k}$*. If there are no eigenvalues* $\lambda \in \sigma(C)$ *with* $|\lambda| \le R$*, the* O*-term can be omitted.*

For $|\lambda| > R$ *and* $0 \le k < m(\lambda)$*, the function* $\Phi_{\lambda k}$ *is Hölder continuous with any exponent smaller than* $\log_q(|\lambda|/R)$*.*

The Dirichlet series $\mathcal{V}(s)$ *converges absolutely and uniformly on compact subsets of the half plane* $\Re s > \log_q R + 1$ *and can be continued to a meromorphic function on the half plane* $\Re s > \log_q R$*. It satisfies the functional equation*

$$(I - q^{-s}C)\mathcal{V}(s) = \sum_{n=1}^{q-1} n^{-s}v(n) + q^{-s}\sum_{r=0}^{q-1} A_r \sum_{k\ge1} \binom{-s}{k}\Bigl(\frac{r}{q}\Bigr)^k \mathcal{V}(s+k) \tag{1.4}$$

for $\Re s > \log_q R$*. The right side converges absolutely and uniformly on compact subsets of* $\Re s > \log_q R$*. In particular,* $\mathcal{V}(s)$ *can only have poles where* $q^s \in \sigma(C)$*.*

For $\lambda \in \sigma(C)$ *with* $|\lambda| > \max\{R, 1/q\}$*, the Fourier series*

$$\Phi_{\lambda k}(u) = \sum_{\ell\in\mathbb{Z}} \varphi_{\lambda k \ell} \exp(2\ell\pi i u)$$

converges pointwise for $u \in \mathbb{R}$ *where*

$$\varphi_{\lambda k\ell} = \frac{(\log q)^k}{k!} \operatorname{Res}\Biggl(\frac{\bigl(x(0) + \mathcal{X}(s)\bigr)\bigl(s - \log_q \lambda - \frac{2\ell\pi i}{\log q}\bigr)^k}{s}, s = \log_q \lambda + \frac{2\ell\pi i}{\log q}\Biggr) \tag{1.5}$$

for $\ell \in \mathbb{Z}$*,* $0 \le k < m(\lambda)$*.*

[3] Note that the summatory function $X(N)$ contains the summand $x(0)$ but the Dirichlet series cannot. This is because the choice of including $x(0)$ into $X(N)$ will lead to more consistent results.

This theorem is proved in the arXiv version [17, Appendix G] of this extended abstract. Note that we write $\Phi_{\lambda k}(\{\log_q N\})$ to optically emphasise the 1-periodicity; technically, we have $\Phi_{\lambda k}(\{\log_q N\}) = \Phi_{\lambda k}(\log_q N)$. Note that the arguments in the proof could be used to meromophically continue the Dirichlet series to the complex plane, but we do not need this result for our purposes. See [1] for the corresponding argument for automatic sequences.

We come back to the binary sum of digits.

▶ **Example 3** (Continuation of Example 1). We have $C = A_0 + A_1 = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. As A_0 is the identity matrix, any product $A_{r_1} \cdots A_{r_\ell}$ has the shape $A_1^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ where k is the number of factors A_1 in the product. This implies that R with $\|A_{r_1} \cdots A_{r_\ell}\| = O(R^\ell)$ may be chosen to be any number greater than 1. As C is a Jordan block itself, we simply read off that the only eigenvalue of C is $\lambda = 2$ with $m(2) = 2$.

Thus Theorem 2 yields

$$X(N) = N(\log_2 N)\,\Phi_{21}(\{\log_2 N\}) + N\,\Phi_{20}(\{\log_2 N\})$$

for suitable 1-periodic continuous functions Φ_{21} and Φ_{20}.

In principle, we can now use the functional equation (1.4). Due to the fact that one component of v is the constant sequence where everything is known, it is more efficient to use an ad-hoc calculation for $\mathcal{X}$ by splitting the sum according to the parity of the index and using the recurrence relation (1.2) for $x(n)$. We obtain

$$\begin{aligned}
\mathcal{X}(s) &= \sum_{n\ge1} \frac{x(2n)}{(2n)^s} + \sum_{n\ge0} \frac{x(2n+1)}{(2n+1)^s} \\
&= 2^{-s} \sum_{n\ge1} \frac{x(n)}{n^s} + \sum_{n\ge0} \frac{x(n)}{(2n+1)^s} + \sum_{n\ge0} \frac{1}{(2n+1)^s} \\
&= 2^{-s}\mathcal{X}(s) + \frac{x(0)}{1^s} + \sum_{n\ge1} \frac{x(n)}{(2n)^s} + \sum_{n\ge1} x(n)\Big(\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}\Big) \\
&\qquad + 2^{-s} \sum_{n\ge0} \frac{1}{\big(n+\frac12\big)^s} \\
&= 2^{1-s}\mathcal{X}(s) + 2^{-s}\,\zeta\big(s, \tfrac12\big) + \sum_{n\ge1} x(n)\Big(\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}\Big)
\end{aligned}$$

where the Hurwitz zeta function $\zeta(s,\alpha) \coloneqq \sum_{n+\alpha>0}(n+\alpha)^{-s}$ has been used. We get

$$(1 - 2^{1-s})\mathcal{X}(s) = 2^{-s}\,\zeta\big(s, \tfrac12\big) + \sum_{n\ge1} x(n)\Big(\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}\Big). \tag{1.6}$$

As the sum of digits is bounded by the length of the expansion, we have $x(n) = O(\log n)$. By combining this estimate with

$$(2n+1)^{-s} - (2n)^{-s} = \int_{2n}^{2n+1} \Big(\frac{\mathrm{d}}{\mathrm{d}t} t^{-s}\Big)\,\mathrm{d}t = \int_{2n}^{2n+1} (-s)t^{-s-1}\,\mathrm{d}t = O(|s|n^{-\Re s-1}),$$

we see that the sum in (1.6) converges absolutely for $\Re s > 0$ and is therefore analytic for $\Re s > 0$.

Therefore, the right side of (1.6) is a meromorphic function for $\Re s > 0$ whose only pole is simple and at $s = 1$ which originates from $\zeta\big(s, \frac12\big)$. Therefore, $\mathcal{X}(s)$ is a meromorphic function for $\Re s > 0$ with a double pole at $s = 1$ and simple poles at $1 + \frac{2\ell\pi i}{\log 2}$ for $\ell \in \mathbb{Z} \setminus \{0\}$.

Thus

$$\begin{aligned}\Phi_{21}(u) = \varphi_{210} &= (\log 2)\operatorname{Res}\Bigl(\frac{\mathcal{X}(s)(s-1)}{s}, s=1\Bigr)\\ &= (\log 2)\operatorname{Res}\Bigl(\frac{2^{-s}(s-1)}{1-2^{1-s}}\,\zeta\bigl(s,\tfrac{1}{2}\bigr), s=1\Bigr) = \frac{1}{2}\end{aligned} \tag{1.7}$$

by (1.5) and (1.6).

We conclude that

$$X(N) = \frac{1}{2}N\log_2 N + N\,\Phi_{20}(\{\log_2 N\}).$$

We refrain from computing the Fourier coefficients of $\Phi_{20}(u)$ explicitly at this point; numerically, they could be computed from (1.6). However, an explicit expression can be obtained by rewriting the residues of $\mathcal{X}(s)$ in terms of shifted residues of $\sum_{n\ge 1}\bigl(x(n)-x(n-1)\bigr)n^{-s}$ and computing the latter explicitly; see [18, Proof of Corollary 2.5]. This yields the well-known result by Delange [6].

It will also turn out that (1.7) being a constant function is an immediate consequence of the fact that $\begin{pmatrix}0 & 1\end{pmatrix}$ is a left eigenvector of both A_0 and A_1 associated with the eigenvalue 1.

1.3 Heuristic Approach: Mellin–Perron Summation

The purpose of this section is to explain why the formula (1.5) for the Fourier coefficients is expected. The approach here is heuristic and non-rigorous because we do not have the required growth estimates. See also [7].

By the Mellin–Perron summation formula of order 0 (see, for example, [12, Theorem 2.1]), we have

$$\sum_{1\le n<N} x(n) + \frac{x(N)}{2} = \frac{1}{2\pi i}\int_{\max\{\log_q R+2,1\}-i\infty}^{\max\{\log_q R+2,1\}+i\infty} \mathcal{X}(s)\frac{N^s\,\mathrm{d}s}{s}.$$

By Remark 1.1 and the definition of R, we have $x(N) = O(R^{\log_q N}) = O(N^{\log_q R})$. Adding the summand $x(0)$ to match our definition of $X(N)$ amounts to adding $O(1)$. Shifting the line of integration to the left—we have *no analytic justification* that this is allowed—and using the location of the poles of $\mathcal{X}(s)$ claimed in Theorem 2 yield

$$\begin{aligned}X(N) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}}\sum_{\ell\in\mathbb{Z}}\operatorname{Res}\Bigl(\frac{\mathcal{X}(s)N^s}{s}, s=\log_q\lambda+\frac{2\ell\pi i}{\log q}\Bigr)\\ + \frac{1}{2\pi i}\int_{\log_q R+\varepsilon-i\infty}^{\log_q R+\varepsilon+i\infty}\mathcal{X}(s)\frac{N^s\,\mathrm{d}s}{s} + O(N^{\log_q R}+1)\end{aligned}$$

for some $\varepsilon>0$. Expanding N^s as

$$N^s = \sum_{k\ge 0}\frac{(\log N)^k}{k!}N^{\log_q\lambda+\frac{2\ell\pi i}{\log q}}\Bigl(s-\log_q\lambda-\frac{2\ell\pi i}{\log q}\Bigr)^k$$

and assuming that the remainder integral converges absolutely yields

$$\begin{aligned}X(N) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q\lambda}\sum_{0\le k<m_{\lambda\ell}}(\log_q N)^k\sum_{\ell\in\mathbb{Z}}\varphi_{\lambda k\ell}\exp\bigl(2\ell\pi i\log_q N\bigr)\\ + O(N^{\log_q R+\varepsilon}+1)\end{aligned}$$

where $m_{\lambda\ell}$ denotes the order of the pole of $\mathcal{X}(s)/s$ at $\log_q \lambda + \frac{2\ell\pi i}{\log q}$ and $\varphi_{\lambda k\ell}$ is as in (1.5).

Summarising, this heuristic approach explains most of the formulæ in Theorem 2. Some details (exact error term and order of the poles) are not explained by this approach. A result "repairing" the zeroth order Mellin–Perron formula is known as Landau's theorem, see [4, § 9]. It is not applicable to our situation due to multiple poles along vertical lines which then yield the periodic fluctuations. Instead, we prove a theorem which provides the required justification (not by estimating the relevant quantities, but by reducing the problem to higher order Mellin–Perron summation). The essential assumption is that the summatory function can be decomposed into fluctuations multiplied by some growth factors such as in (1.3).

1.4 Relation to Previous Work

Sequences defined as the output sum of transducer automata in the sense of [18] are a special case of regular sequences; these are a generalisation of many previously studied concepts. In that case, much more is known (variance, limiting distribution, higher dimensional input). See [18] for references and results. A more detailed comparison can be found in Section 2. Divide and Conquer recurrences (see [19] and [8]) can also be seen as special cases of regular sequences.

The asymptotics of the summatory function of specific examples of regular sequences has been studied in [14], [15], [11].

Dumas [9, 10] finally proved the first part of Theorem 2. We re-prove it here in a self-contained way because we need more explicit results than obtained by Dumas (e.g., we need explicit expressions for the fluctuations) for proving Hölder continuity and to explicitly get the precise structure depending on the eigenspaces. Notice that Dumas' paper introduces linear representations as we do in (1.1), but then the order of factors is reversed in his equivalent of Remark 1.1, which means that some transpositions have to be silently introduced.

The first version of our pseudo-Tauberian argument was provided in [12]: there, no logarithmic factors were allowed and the growth conditions on the Dirichlet series were stronger.

2 Sequences Defined by Transducer Automata

Let $q \geq 2$ be a positive integer. We consider a complete deterministic subsequential transducer $\mathcal{T}$ with input alphabet $\{0, \dots, q-1\}$ and output alphabet $\mathbb{C}$, see [5, Chapter 1] and [18]. Recall that a transducer is said to be *deterministic* and *complete* if for every state and every digit of the input alphabet, there is exactly one transition starting in this state with this input label. A *subsequential* transducer has a final output label for every state.

For a non-negative integer n, let $\mathcal{T}(n)$ be the sum of the output labels (including the final output label) encountered when the transducer reads the q-ary expansion of n. This concept has been thoroughly studied in [18]: there, $\mathcal{T}(n)$ is considered as a random variable defined on the probability space $\{0, \dots, N-1\}$ equipped with uniform distribution. The expectation in this model corresponds (up to a factor of N) to our summatory function $\sum_{0\leq n<N} \mathcal{T}(n)$. We remark that in [18], the variance and limiting distribution of the random variable $\mathcal{T}(n)$ have also been investigated. Most of the results there are also valid for higher dimensional input.

The purpose of this section is to show that $\mathcal{T}(n)$ is a q-regular sequence and to see that our results here coincide with the corresponding results in [18]. We note that the binary sum of digits considered in Example 1 is the special case of $q = 2$ and the transducer consisting of a single state which implements the identity map. For additional special cases of this concept, see [18]. Note that our result here for the summatory function contains (fluctuating) terms for all eigenvalues λ of the adjacency matrix of the underlying digraph with $1 < |\lambda|$ whereas in [18] only contributions of those eigenvalues λ with $|\lambda| = q$ are available, all other contributions are absorbed by the error term there.

By a *component* of a digraph we always mean a strongly connected component. We call a component *final* if there are no arcs leaving the component. The *period* of a component is the greatest common divisor of its cycle lengths. The *final period* of a digraph is the least common multiple of the periods of its final components.

We consider the states of $\mathcal{T}$ to be numbered by $\{1, \dots, d\}$ for some positive integer $d \geq 1$ such that the initial state is state 1. We set $\mathcal{T}_j(n)$ to be the sum of the output labels (including the final output label) encountered when the transducer reads the q-ary expansion of n when starting in state j. By construction, we have $\mathcal{T}(n) = \mathcal{T}_1(n)$ and $\mathcal{T}_j(0)$ is the final output label of state j. We set $y(n) = (\mathcal{T}_1(n), \dots, \mathcal{T}_d(n))$. For $0 \leq r < q$, we define the $(d \times d)$-$\{0,1\}$-matrix P_r in such a way that there is a one in row j, column k if and only if there is a transition from state j to state k with input label r. The vector o_r is defined by setting its jth coordinate to be the output label of the transition from state j with input label r.

For $n_0 \geq 1$, we set

$$\mathcal{X}(s) = \sum_{n\geq 1} n^{-s}\mathcal{T}(n), \qquad \mathcal{Y}_{n_0}(s) = \sum_{n\geq n_0} n^{-s} y(n), \qquad \zeta_{n_0}(s,\alpha) = \sum_{n \geq n_0} (n+\alpha)^{-s}.$$

The last Dirichlet series is a truncated version of the Hurwitz zeta function.

▶ **Corollary 4.** *Let $\mathcal{T}$ be a transducer as described at the beginning of this section. Let M and p be the adjacency matrix and the final period of the underlying digraph, respectively. For $\lambda \in \mathbb{C}$ let $m(\lambda)$ be the size of the largest Jordan block associated with the eigenvalue λ of M.*

Then $(\mathcal{T}(n))_{n\geq 0}$ is a q-regular sequence and

$$\begin{aligned}\sum_{0\leq n<N} \mathcal{T}(n) = e_{\mathcal{T}} N \log_q N + N\Phi(\log_q N) \\ + \sum_{\substack{\lambda\in\sigma(M)\\ 1<|\lambda|<q}} N^{\log_q \lambda} \sum_{0\leq k<m(\lambda)} (\log_q N)^k \Phi_{\lambda k}(\log_q N) \\ + O\bigl((\log N)^{\max\{m(\lambda) :\, |\lambda|=1\}}\bigr)\end{aligned} \tag{2.1}$$

for some continuous p-periodic function Φ, some continuous 1-periodic functions $\Phi_{\lambda k}$ for $\lambda \in \sigma(M)$ with $1 < |\lambda| < q$ and $0 \leq k < m(\lambda)$ and some constant $e_{\mathcal{T}}$.

Furthermore,

$$\Phi(u) = \sum_{\ell\in\mathbb{Z}} \varphi_\ell \exp\Bigl(\frac{2\ell\pi i}{p} u\Bigr)$$

with

$$\varphi_\ell = \operatorname{Res}\Bigl(\frac{\mathcal{X}(s)}{s}, s = 1 + \frac{2\ell\pi i}{p\log q}\Bigr)$$

for $\ell \in \mathbb{Z}$. The Fourier series expansion of $\Phi_{\lambda k}$ for $\lambda \in \sigma(M)$ with $1 < |\lambda| < q$ is given in Theorem 2.

The Dirichlet series $\mathcal{Y}_{n_0}(s)$ satisfies the functional equation

$$(I - q^{-s}M)\mathcal{Y}_{n_0}(s) = \sum_{n_0 \le n < qn_0} n^{-s} y(n) + q^{-s} \sum_{0 \le r < q} \zeta_{n_0}\bigl(s, \tfrac{r}{q}\bigr) o_r + q^{-s} \sum_{0 \le r < q} P_r \sum_{k \ge 1} \binom{-s}{k} \Bigl(\frac{r}{q}\Bigr)^k \mathcal{Y}_{n_0}(s+k).$$

Sketch of the Proof. The proof is split into several steps.

Recursive Description. We set $v(n) = \bigl(\mathcal{T}_1(n), \ldots, \mathcal{T}_d(n), 1\bigr)^\top$. For $1 \le j \le d$ and $0 \le r < q$, we define $t(j,r)$ and $o(j,r)$ to be the target state and output label of the unique transition from state j with input label r, respectively. Therefore,

$$\mathcal{T}_j(qn + r) = \mathcal{T}_{t(j,r)}(n) + o(j,r) \tag{2.2}$$

for $1 \le j \le d$, $n \ge 0$, $0 \le r < q$ with $qn + r > 0$.

For $0 \le r < q$, define $A_r = (a_{rjk})_{1 \le j,\, k \le d+1}$ by

$$a_{rjk} = \begin{cases} [t(j,r) = k] & \text{if } j,\, k \le d, \\ o(j,r) & \text{if } j \le d,\ k = d+1, \\ [k = d+1] & \text{if } j = d+1 \end{cases}$$

where we use Iverson's convention $[\mathit{expr}] = 1$ if *expr* is true and 0 otherwise; see Graham, Knuth, and Patashnik [16]. Then (2.2) is equivalent to

$$v(qn + r) = A_r v(n)$$

for $n \ge 0$, $0 \le r < q$ with $qn + r > 0$.

q-Regular Sequence. If we insist on a proper formulation as a regular sequence, we rewrite (2.2) to

$$\mathcal{T}_j(qn + r) = \mathcal{T}_{t(j,r)}(n) + o(j,r) + [r = 0][n = 0]\bigl(\mathcal{T}_j(0) - \mathcal{T}_{t(j,0)}(0) - o(j,0)\bigr) \tag{2.3}$$

for $1 \le j \le d$, $n \ge 0$, $0 \le r < q$. Setting $\widetilde{v}(n) = (\mathcal{T}_1(n), \ldots, \mathcal{T}_d(n), 1, [n = 0])$ and $\widetilde{A}_r = (\widetilde{a}_{rjk})_{1 \le j,\, k \le d+2}$ with

$$\widetilde{a}_{rjk} = \begin{cases} [t(j,r) = k] & \text{if } j,\, k \le d, \\ o(j,r) & \text{if } j \le d,\ k = d+1, \\ [r = 0]\bigl(\mathcal{T}_j(0) - \mathcal{T}_{t(j,0)}(0) - o(j,0)\bigr) & \text{if } j \le d,\ k = d+2, \\ [k = d+1] & \text{if } j = d+1, \\ [k = d+2][r = 0] & \text{if } j = d+2, \end{cases}$$

the system (2.3) is equivalent to

$$\widetilde{v}(qn + r) = \widetilde{A}_r \widetilde{v}(n)$$

for $n \ge 0$, $0 \le r < q$.

The rest of the proof (relating the eigenvalues of M with those of C) can be found in the arXiv version [17, Appendix H] of this extended abstract. ◀

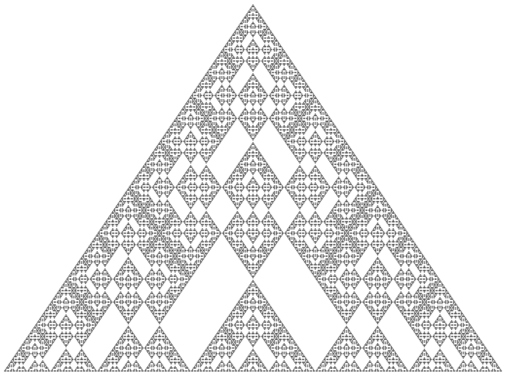

Figure 3.1 Pascal's rhombus modulo 2.

3 Pascal's Rhombus

We consider Pascal's rhombus $\mathfrak{R}$ which is, for integers $i \geq 0$ and j, the array with entries $r_{i,j}$, where

- $r_{0,j} = 0$ all j,
- $r_{1,0} = 1$ and $r_{1,j} = 0$ for all $j \neq 0$,
- and
$$r_{i,j} = r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j}$$
for $i \geq 1$.

Let $\mathfrak{X}$ be equal to $\mathfrak{R}$ but with entries takes modulo 2; see also Figure 3.1. We partition $\mathfrak{X}$ into the four sub-arrays

- $\mathfrak{E}$ consisting only of the rows and columns of $\mathfrak{X}$ with even indices, i.e., the entries $r_{2i,2j}$,
- $\mathfrak{Y}$ consisting only of the rows with odd indices and columns with even indices, i.e., the entries $r_{2i-1,2j}$,
- $\mathfrak{Z}$ consisting only of the rows with even indices and columns with odd indices, i.e., the entries $r_{2i,2j-1}$, and
- $\mathfrak{N}$ consisting only of the rows and columns with odd indices, i.e., the entries $r_{2i-1,2j-1}$.

Note that $\mathfrak{E} = \mathfrak{X}$ and $\mathfrak{N} = 0$; see [13].

3.1 Recurrence Relations and 2-Regular Sequences

Let $X(N)$, $Y(N)$ and $Z(N)$ be the number of ones in the first N rows (starting with row index 1) of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ respectively.

Using results by Goldwasser, Klostermeyer, Mays and Trapp [13] leads to recurrence relations for the backward differences $x(n) = X(n) - X(n-1)$, $y(n) = Y(n) - Y(n-1)$ and $z(n) = Z(n) - Z(n-1)$, namely

$$x(2n) = x(n) + z(n), \qquad x(2n+1) = y(n+1), \tag{3.1a}$$
$$y(2n) = x(n-1) + z(n), \qquad y(2n+1) = x(n+1) + z(n), \tag{3.1b}$$
$$z(2n) = 2x(n), \qquad z(2n+1) = 2y(n+1) \tag{3.1c}$$

for $n \geq 1$, and $x(0) = y(0) = z(0) = 0$, $x(1) = 1$, $y(1) = 1$ and $z(1) = 2$. (See the arXiv version [17, Appendix I.1] of this extended abstract for details.) These $x(n)$, $y(n)$ and $z(n)$ are the number of ones in the nth row of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ respectively.

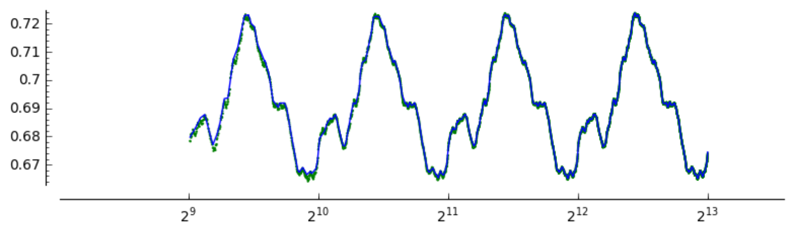

Figure 3.2 Fluctuation in the main term of the asymptotic expansion of $X(N)$. The figure shows $\Phi(\log_2 N)$ (blue) approximated by its trigonometric polynomial of degree 99 as well as $X(N)/N^\kappa$ (green).

Let use write our coefficients as the vector

$$v(n) = \big(x(n), x(n+1), y(n+1), z(n), z(n+1)\big)^\top. \tag{3.2}$$

It turns out that the components included into $v(n)$ are sufficient for a self-contained linear representation of $v(n)$. In particular, it is not necessary to include $y(n)$. By using the recurrences (3.1), we find that

$$v(2n) = A_0 v(n) \qquad \text{and} \qquad v(2n+1) = A_1 v(n)$$

for all[4] $n \geq 0$ with the matrices

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

and with $v(0) = (0, 1, 1, 0, 2)^\top$. Therefore, the sequences $x(n)$, $y(n)$ and $z(n)$ are 2-regular.

3.2 Asymptotics

▶ **Corollary 5.** *We have*

$$X(N) = \sum_{1 \leq n \leq N} x(n) = N^\kappa\, \Phi(\{\log_2 N\}) + O(N \log_2 N) \tag{3.3}$$

with $\kappa = \log_2\big(3 + \sqrt{17}\big) - 1 = 1.83250638358045\ldots$ *and a* 1*-periodic function* Φ *which is Hölder continuous with any exponent smaller than* $\kappa - 1$.

Moreover, we can effectively compute the Fourier coefficients of Φ.

We get analogous results for the sequences $Y(N)$ and $Z(N)$ (each with its own periodic function Φ, but the same exponent κ). The fluctuation Φ of $X(N)$ is visualized in Figure 3.2 and its first few Fourier coefficients are shown in Table 3.1.

At this point, we only prove (3.3) of Corollary 5. We deal with the Fourier coefficients in the arXiv version [17, Appendix I.2] of this extended abstract. As in the introductory example of the binary sum-of-digits functions (Example 1), we could get Fourier coefficients by

[4] Note that $v(0) = A_0 v(0)$ and $v(1) = A_1 v(0)$ are indeed true.

Table 3.1 Fourier coefficients of Φ (Corollary 5). All stated digits are correct.

ℓ	α_ℓ
0	$0.6911615112341912755021246$
1	$-0.01079216311240407872950510 - 0.0023421761940286789685827i$
2	$0.0027937863735049517211671 2 - 0.00066736128659728911347756i$
3	$-0.00020078258323645842522640 - 0.0031973663977645462669373i$
4	$0.00024944678921746747281338 - 0.00059129954670760614976 50i$
5	$-0.0003886698612765803447578 + 0.0000672386631993014856843 1i$
6	$-0.0006223575988893574655258 + 0.0004321722061493985978154 2i$
7	$0.00023034317364181383130476 - 0.00058663168772856091427688i$
8	$0.0005339060804798716172593 - 0.0002119380802590974909465i$
9	$0.0000678898389770175928529 - 0.0003830782328548623528018 5i$
10	$-0.00019981745997355255061991 - 0.00031394569060142799808175i$

Theorem 2 and the 2-linear representation of Section 3.1 directly. However, the information in the vector $v(n)$ (see (3.2)) is redundant with respect to the asymptotic main term as it contains $x(n)$ and $z(n)$ as well as $x(n+1)$ and $z(n+1)$; both pairs are asymptotically equal in the sense of (3.3). Therefore, we head for an only 3-dimensional functional system of equations for our Dirichlet series of $x(n)$, $y(n)$ and $z(n)$ (instead of a 5-dimensional system).

Proof of (3.3)**.** We use Theorem 2.

Joint Spectral Radius. First we compute the joint spectral radius ρ of A_0 and A_1. Both matrices have a maximum absolute row sum equal to 2, thus $\rho \leq 2$, and both matrices have 2 as an eigenvalue. Therefore we obtain $\rho = 2$. Moreover, the finiteness property of the linear representation is satisfied by considering only products with exactly one matrix factor A_0 or A_1.

Thus, we have $R = \rho = 2$.

Eigenvalues. Next, we compute the spectrum $\sigma(C)$ of $C = A_0 + A_1$. The matrix C has the eigenvalues $\lambda_1 = \bigl(3 + \sqrt{17}\bigr)/2 = 3.5615528128088\ldots$, $\lambda_2 = 2$, $\lambda_3 = -2$, $\lambda_4 = -1$ and $\lambda_5 = \bigl(3 - \sqrt{17}\bigr)/2 = -0.5615528128088\ldots$ (each with multiplicity one). (Note that λ_1 and λ_5 are the zeros of the polynomial $U^2 - 3U - U$.)

Asymptotic Formula. By using Theorem 2, we obtain an asymptotic formula for $X(N-1)$. Shifting from $N-1$ to N does not change this asymptotic formula, as this shift is absorbed by the error term $O(N \log_2 N)$. ◀

3.3 Dirichlet Series and Meromorphic Continuation

Let $n_0 \geq 2$ be an integer and define

$$\mathcal{X}_{n_0}(s) = \sum_{n \geq n_0} \frac{x(n)}{n^s}, \qquad \mathcal{Y}_{n_0}(s) = \sum_{n \geq n_0} \frac{y(n)}{n^s}, \qquad \mathcal{Z}_{n_0}(s) = \sum_{n \geq n_0} \frac{z(n)}{n^s}.$$

▶ **Lemma 6.** *Set*

$$C = I - \begin{pmatrix} 2^{-s} & 2^{-s} & 2^{-s} \\ 2^{1-s} & 0 & 2^{1-s} \\ 2^{1-s} & 2^{1-s} & 0 \end{pmatrix}.$$

Then

$$C \begin{pmatrix} \mathcal{X}_{n_0}(s) \\ \mathcal{Y}_{n_0}(s) \\ \mathcal{Z}_{n_0}(s) \end{pmatrix} = \begin{pmatrix} \mathcal{J}_{n_0}(s) \\ \mathcal{K}_{n_0}(s) \\ \mathcal{L}_{n_0}(s) \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned}
\mathcal{J}_{n_0}(s) &= 2^{-s}\,\Sigma(s, -\tfrac{1}{2}, \mathcal{Y}_{n_0}) + \mathcal{I}_{\mathcal{J}_{n_0}}(s),\\
&\qquad \mathcal{I}_{\mathcal{J}_{n_0}}(s) = -\frac{y(n_0)}{(2n_0-1)^s} + \sum_{n_0 \le n < 2n_0} \frac{x(n)}{n^s},\\
\mathcal{K}_{n_0}(s) &= 2^{-s}\,\Sigma(s, 1, \mathcal{X}_{n_0}) + 2^{-s}\,\Sigma(s, -\tfrac{1}{2}, \mathcal{X}_{n_0}) + 2^{-s}\,\Sigma(s, \tfrac{1}{2}, \mathcal{Z}_{n_0}) + \mathcal{I}_{\mathcal{K}_{n_0}}(s),\\
&\qquad \mathcal{I}_{\mathcal{K}_{n_0}}(s) = \frac{x(n_0-1)}{(2n_0)^s} - \frac{x(n_0)}{(2n_0-1)^s} + \sum_{n_0 \le n < 2n_0} \frac{y(n)}{n^s},\\
\mathcal{L}_{n_0}(s) &= 2^{1-s}\,\Sigma(s, -\tfrac{1}{2}, \mathcal{Y}_{n_0}) + \mathcal{I}_{\mathcal{L}_{n_0}}(s),\\
&\qquad \mathcal{I}_{\mathcal{L}_{n_0}}(s) = -\frac{2y(n_0)}{(2n_0-1)^s} + \sum_{n_0 \le n < 2n_0} \frac{z(n)}{n^s},
\end{aligned}$$

with

$$\Sigma(s, \beta, \mathcal{D}) = \sum_{k \ge 1} \binom{-s}{k} \beta^k \mathcal{D}(s+k)$$

provides meromorphic continuations of the Dirichlet series $\mathcal{X}_{n_0}(s)$, $\mathcal{Y}_{n_0}(s)$, and $\mathcal{Z}_{n_0}(s)$ for $\Re s > \kappa_0 = 1$ with the only possible poles at $\kappa + \chi_\ell$ for $\ell \in \mathbb{Z}$, all of which are simple poles.

The proof of Lemma 6 can be found in the arXiv version [17, Appendix I] of this extended abstract. The idea is to rewrite the Dirichlet series corresponding to (3.1a), (3.1b) and (3.1c) to obtain the functional equation. The poles in the meromorphic continuation come from

$$\Delta(s) = \det C = 2^{-3s}(2^{2s} - 3 \cdot 2^s - 2)(2^s + 2).$$

The Fourier coefficients (rest of Corollary 5) can then be computed by applying Theorem 2.

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A Results

We formulate the full version of our results here in Appendix A. Formulating them will need quite a number of definitions provided in Appendix A.2. The proofs are given in the appendix of the arXiv version [17] of this extended abstract.

As announced in the introduction, we study matrix products instead of regular sequences.

A.1 Problem Statement

Let $q \geq 2$, $d \geq 1$ be fixed integers and $A_0, \ldots, A_{q-1} \in \mathbb{C}^{d\times d}$. We investigate the sequence $(f(n))_{n\geq 0}$ of $d \times d$ matrices such that

$$f(qn + r) = A_r f(n) \qquad \text{for } 0 \leq r < q,\ 0 \leq n \text{ with } qn + r \neq 0 \tag{A.1}$$

and $f(0) = I$.

Let n be an integer with q-ary expansion $r_{\ell-1} \ldots r_0$. Then it is easily seen that (A.1) implies that

$$f(n) = A_{r_0} \ldots A_{r_{\ell-1}}. \tag{A.2}$$

We are interested in the asymptotic behaviour of $F(N) := \sum_{0\leq n<N} f(n)$.

A.2 Definitions and Notations

In this section, we give all definitions and notations which are required in order to state the results.

The following notations are essential:

- Let $\|\cdot\|$ denote a fixed norm on $\mathbb{C}^d$ and its induced matrix norm on $\mathbb{C}^{d\times d}$.
- We set $B_r := \sum_{0\leq r'<r} A_{r'}$ for $0 \leq r < q$ and $C := \sum_{0\leq r<q} A_r$.
- The joint spectral radius of $A_0, \ldots, A_{q-1}$ is denoted by

  $$\rho := \inf_\ell \sup\bigl\{\|A_{r_1} \ldots A_{r_\ell}\|^{1/\ell} \colon r_1, \ldots, r_\ell \in \{0, \ldots, q-1\}\bigr\}.$$

  If the set of matrices $A_0, \ldots, A_{q-1}$ has the finiteness property, i.e., there is an $\ell > 0$ such that

  $$\rho = \sup\bigl\{\|A_{r_1} \ldots A_{r_\ell}\|^{1/\ell} \colon r_1, \ldots, r_\ell \in \{0, \ldots, q-1\}\bigr\},$$

  then we set $R = \rho$. Otherwise, we choose $R > \rho$ in such a way that there is no eigenvalue λ of C with $\rho < |\lambda| \leq R$.
- The spectrum of C, i.e., the set of eigenvalues of C, is denoted by $\sigma(C)$.
- For a positive integer n_0, set

  $$\mathcal{F}_{n_0}(s) := \sum_{n\geq n_0} n^{-s} f(n)$$

  for a complex variable s.
- Set $\chi_k := \frac{2\pi ik}{\log q}$ for $k \in \mathbb{Z}$.

In the formulation of Theorem 7 and Corollary 8, the following constants are needed additionally:

- Choose a regular matrix T such that $TCT^{-1} =: J$ is in Jordan form.
- Let D be the diagonal matrix whose jth diagonal element is 1 if the jth diagonal element of J is not equal to 1; otherwise the jth diagonal element of D is 0.
- Set $C' := T^{-1}DJT$.
- Set $K := T^{-1}DT(I - C')^{-1}(I - A_0)$.
- For a $\lambda \in \mathbb{C}$, let $m(\lambda)$ be the size of the largest Jordan block associated with λ. In particular, $m(\lambda) = 0$ if $\lambda \notin \sigma(C)$.

- For $m \geq 0$, set
$$\vartheta_m := \frac{1}{m!} T^{-1}(I-D)T(C-I)^{m-1}(I-A_0);$$
here, ϑ_0 remains undefined if $1 \in \sigma(C)$.[5]
- Define $\vartheta := \vartheta_{m(1)}$.

All implicit O-constants depend on q, d, the matrices A_0, …, A_{q-1} (and therefore on ρ) as well as on R.

A.3 Decomposition into Periodic Fluctuations

Instead of considering $F(N)$, it is certainly enough to consider $wF(N)$ for all generalised left eigenvectors w of C, e.g., the rows of T. The result for $F(N)$ then follows by taking appropriate linear combinations.

▶ **Theorem 7.** *Let w be a generalised left eigenvector of rank m of C corresponding to the eigenvalue λ.*

1. *If $|\lambda| < R$, then*
$$wF(N) = wK + (\log_q N)^m w\vartheta_m + O(N^{\log_q R}).$$
2. *If $|\lambda| = R$, then*
$$wF(N) = wK + (\log_q N)^m w\vartheta_m + O(N^{\log_q R}(\log N)^m).$$
3. *If $|\lambda| > R$, then there are 1-periodic continuous functions $\Phi_k \colon \mathbb{R} \to \mathbb{C}^d$, $0 \leq k < m$, such that*
$$wF(N) = wK + (\log_q N)^m w\vartheta_m + N^{\log_q \lambda} \sum_{0 \leq k < m} (\log_q N)^k \Phi_k(\{\log_q N\})$$
for $N \geq q^{m-1}$. The function Φ_k is Hölder-continuous with any exponent smaller than $\log_q|\lambda|/R$.
If, additionally, the left eigenvector $w(C-\lambda I)^{m-1}$ of C happens to be a left eigenvector to each matrix A_0, …, A_{q-1} associated with the eigenvalue 1, then
$$\Phi_{m-1}(u) = \frac{1}{q^{m-1}(m-1)!} w(C-qI)^{m-1}$$
is constant.

Here, $wK = 0$ for $\lambda = 1$ and $w\vartheta_m = 0$ for $\lambda \neq 1$.

Note that in general, the three summands in the theorem have different growths: a constant, a logarithmic term and a term whose growth depends essentially on the joint spectral radius and the eigenvalues larger than the joint spectral radius, respectively. The vector w is not directly visible in front of the third summand; instead, the vectors of its Jordan chain are part of the function Φ_k.

Expressing the identity matrix as linear combinations of generalised left eigenvalues and summing up the contributions of Theorem 7 essentially yields the following corollary.

---

[5] If $1 \in \sigma(C)$, then the matrix $C - I$ is singular. In that case, ϑ_0 will never be used.

▶ **Corollary 8.** *With the notations above, we have*

$$F(N) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>\rho}} N^{\log_q \lambda} \sum_{0\le k<m(\lambda)} (\log_q N)^k \Phi_{\lambda k}(\{\log_q N\}) + (\log_q N)^{m(1)}\vartheta + K + O\bigl(N^{\log_q R}(\log N)^{\max\{m(\lambda)\colon |\lambda|=R\}}\bigr)$$

for suitable 1*-periodic continuous functions* $\Phi_{\lambda k}$*. If* 1 *is not an eigenvalue of* C*, then* $\vartheta = 0$*. If there are no eigenvalues* $\lambda \in \sigma(C)$ *with* $|\lambda| \le \rho$*, then the* O*-term can be omitted.*

For $|\lambda| > R$*, the function* $\Phi_{\lambda k}$ *is Hölder continuous with any exponent smaller than* $\log_q(|\lambda|/R)$*.*

A.4 Dirichlet Series

This section gives the required result on the Dirichlet series $\mathcal{F}_{n_0}$. For theoretical purposes, it is enough to study $\mathcal{F} := \mathcal{F}_1$; for numerical purposes, however, convergence improves for larger values of n_0.

▶ **Theorem 9.** *Let* n_0 *be a positive integer. Then the Dirichlet series* $\mathcal{F}_{n_0}(s)$ *converges absolutely and uniformly on compact subsets of the half plane* $\Re s > \log_q \rho + 1$*, thus is analytic there.*

We have

$$(I - q^{-s}C)\mathcal{F}_{n_0}(s) = \mathcal{G}_{n_0}(s) \tag{A.3}$$

for $\Re s > \log_q \rho + 1$ *with*

$$\mathcal{G}_{n_0}(s) = \sum_{n=n_0}^{qn_0-1} n^{-s} f(n) + q^{-s} \sum_{r=0}^{q-1} A_r \sum_{k\ge 1} \binom{-s}{k} \Bigl(\frac{r}{q}\Bigr)^k \mathcal{F}_{n_0}(s+k). \tag{A.4}$$

The series in (A.4) *converge absolutely and uniformly on compact sets for* $\Re s > \log_q \rho$*. Thus* (A.3) *gives a meromorphic continuation of* $\mathcal{F}_{n_0}$ *to the half plane* $\Re s > \log_q \rho$ *with possible poles at* $s = \log_q \lambda + \chi_\ell$ *for each* $\lambda \in \sigma(C)$ *with* $|\lambda| > \rho$ *and* $\ell \in \mathbb{Z}$ *whose pole order is at most* $m(\lambda)$*.*

Let $\delta > 0$*. For real* z*, we set*

$$\mu_\delta(z) = \max\{1 - (z - \log_q \rho - \delta), 0\},$$

i.e., the linear function on the interval $[\log_q \rho + \delta, \log_q \rho + \delta + 1]$ *with* $\mu_\delta(\log_q \rho + \delta) = 1$ *and* $\mu_\delta(\log_q \rho + \delta + 1) = 0$*. Then*

$$\mathcal{F}_{n_0}(s) = O\bigl(|\Im s|^{\mu_\delta(\Re s)}\bigr) \tag{A.5}$$

holds uniformly for $\log_q \rho + \delta \le \Re s$ *and* $|q^s - \lambda| \ge \delta$ *for all eigenvalues* $\lambda \in \sigma(C)$*. Here, the implicit* O*-constant also depends on* δ*.*

▶ Remark. By the identity theorem for analytic functions, the meromorphic continuation of $\mathcal{F}_{n_0}$ is unique on the domain given in the theorem. Therefore, the bound (A.5) does not depend on the particular expression for the meromorphic continuation given in (A.3) and (A.4).

A.5 Fourier Coefficients

As discussed in Section 1.3, we would like to apply the zeroth order Mellin–Perron summation formula but need analytic justification. In the following theorem we prove that whenever it is known that the result is a periodic fluctuation, the use of zeroth order Mellin–Perron summation can be justified. In contrast to the remaining paper, this theorem does *not* assume that $f(n)$ is a matrix product.

▶ **Theorem 10.** *Let $f(n)$ be a sequence, let $\kappa_0 \in \mathbb{R} \setminus \{0\}$ and $\kappa \in \mathbb{C}$ with $\Re\kappa > \kappa_0 > -1$, $\delta > 0$, $q > 1$ be real numbers with $\delta \le \pi/(\log q)$ and $\delta < \Re\kappa - \kappa_0$, and let m be a positive integer. Moreover, let Φ_k be Hölder-continuous (with exponent α with $\Re\kappa - \kappa_0 < \alpha \le 1$) 1-periodic functions for $0 \le k < m$ such that*

$$F(N) \coloneqq \sum_{1\le n<N} f(n) = \sum_{0\le k<m} N^{\kappa}(\log_q N)^k \Phi_k(\{\log_q N\}) + O(N^{\kappa_0}) \tag{A.6}$$

for integers $N \to \infty$.

For the Dirichlet series $\mathcal{F}(s) \coloneqq \sum_{n\ge1} n^{-s} f(n)$ assume that

- *there is some real number $\sigma_a \ge \Re\kappa$ such that $\mathcal{F}(s)$ converges absolutely for $\Re s > \sigma_a$;*
- *the Dirichlet series $\mathcal{F}(s)$ can be continued to a meromorphic function for $\Re s > \kappa_0 - \delta$ such that poles can only occur at $\kappa + \chi_\ell$ for $\ell \in \mathbb{Z}$ and such that these poles have order at most m;*
- *there is some real number $\eta > 0$ such that for $\kappa_0 \le \Re s \le \sigma_a$ and $|s - \kappa - \chi_\ell| \ge \delta$ for all $\ell \in \mathbb{Z}$, we have*

$$\mathcal{F}(s) = O\bigl(|\Im s|^{\eta}\bigr) \tag{A.7}$$

for $|\Im s| \to \infty$.

All implicit O-constants may depend on f, q, m, κ, κ_0, α, δ, σ_a and η.

Then

$$\Phi_k(u) = \sum_{\ell\in\mathbb{Z}} \varphi_{k\ell} \exp(2\ell\pi i u)$$

for $u \in \mathbb{R}$ where

$$\varphi_{k\ell} = \frac{(\log q)^k}{k!} \operatorname{Res}\Bigl(\frac{\mathcal{F}(s)(s-\kappa-\chi_\ell)^k}{s}, s = \kappa + \chi_\ell\Bigr) \tag{A.8}$$

for $\ell \in \mathbb{Z}$ and $0 \le k < m$.

If $-1 < \kappa_0 < 0$ and $\kappa \notin \frac{2\pi i}{\log q}\mathbb{Z}$, then $\mathcal{F}(0) = 0$.

The theorem is more general than necessary for q-regular sequences because Theorem 9 shows that we could use some $0 < \eta < 1$. However, it might be applicable in other cases, so we prefer to state it in this more general form.