Distribution of the Number of Corners in Tree-like and Permutation Tableaux

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Abstract

In this abstract, we study tree-like tableaux and some of their probabilistic properties. Tree-like tableaux are in bijection with other combinatorial structures, including permutation tableaux, and have a connection to the partially asymmetric simple exclusion process (PASEP), an important model of interacting particles system. In particular, in the context of tree-like tableaux, a corner corresponds to a node occupied by a particle that could jump to the right while inner corners indicate a particle with an empty node to its left. Thus, the total number of corners represents the number of nodes at which PASEP can move, i.e., the total current activity of the system. As the number of inner corners and regular corners is connected, we limit our discussion to just regular corners and show that, asymptotically, the number of corners in a tableaux of length n is normally distributed. Furthermore, since the number of corners in tree-like tableaux are closely related to the number of corners in permutation tableaux, we will discuss the corners in the context of the latter tableaux.

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1 Introduction

In this report, we study tree-like tableaux, a combinatorial object introduced in [1]. They are in bijection with permutation tableaux and alternative tableaux but are interesting in their own right as they exhibit a natural tree structure. Aside from being in bijection with permutations and permutation tableaux, they can be used to study the partially asymmetric simple exclusion process (PASEP). The PASEP (see e. g. [6, 9] and references therein) is a model in which n nodes on a 1-dimensional lattice each either contains a particle or not. At each time interval, a particle can either move left or right to an empty adjacent node with fixed probabilities and the probability of a move left is q times the probability of jumping to the right. New particle may also enter from the left with probability α (if the first node

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is unoccupied) and a particle on the nth node may leave the lattice with probability β. A state of the PASEP is a configuration of occupied and unoccupied nodes and it naturally corresponds to border edges of tree-like tableaux. In this association, corners in tree-like tableaux, correspond to sites at which a particle can move (we will give more details below, see also [13] for an explanation). In physics literature this is known as (total) current activity [7, 8] and was studied for the TASEP (a special case of the PASEP with $q = 0$) in [15].

It was conjectured (see [13, Conjecture 4.1]) that the expected number of corners in a randomly chosen tree-like tableaux of size n is $(n+4)/6$. This conjecture (and its companion for symmetric tree-like tableaux) was proved in [12, Theorem 4] and subsequently also in [10, Theorem 4.1]). However, not much beyond that has been known (even the asymptotic value of the variance). In the present paper we take the next step in the analysis of tree-like tableaux. First, since permutation tableaux are in bijection with tree-like tableaux and the number of corners in both are related, we shift our discussion to be solely concerned with permutation tableaux and derive our results in that context. In particular, we obtain the variance of the number of corners. Furthermore, we also show that the number of corners in random permutation, and therefore tree-like tableau of size n is asymptotically normal as n goes to infinity.

The rest of the paper is organized as follows. In the next section we introduce the necessary definitions and notation. We also explain the relation between the tree-like tableaux and the PASEP. In Section 3 we present a recursive relation for the generating function involving the corners in a similar combinatorial object, namely permutation tableaux. This recursion will be used in Section 4 to obtain a recursion for the moment generation function of the number of corners in permutation tableaux and in Section 5 to establish its asymptotic normality. As mentioned earlier, this will imply the same result for the number of corners in tree-like tableaux.

2 Preliminaries

2.1 Tree-like Tableaux and Permutation Tableaux

We endeavor to introduce the background for studying tree-like tableaux. We start by recalling the necessary notions and properties.

▶ **Definition 1.** A Ferrers diagram is an up and left justified arrangement of cells with weakly decreasing number of cells in rows. Depending on the situation, some rows may or may not be empty. The length of a Ferrers diagram is the number of columns plus the number of rows.

Let us recall the following definition introduced in [1].

▶ **Definition 2.** A tree-like tableau of size n is a Ferrers diagram of length $n + 1$ with no empty rows and with some cells (called pointed cells) filled with a point according to the following rules:

1. The cell in the first column and first row is always pointed (this point is known as the root point).
2. Every row and every column contains at least one pointed cell.
3. For every non-root pointed cell, either all the cells above are empty or all the cells to the left are empty (but not both).

We denote the set of all tree-like tableaux of size n by $\mathcal{T}_n$.

We will also need a notion of permutation tableaux originally introduced in [14].

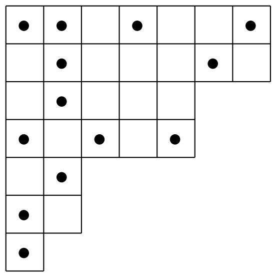

Figure 1 A tree-like tableau of size 13.

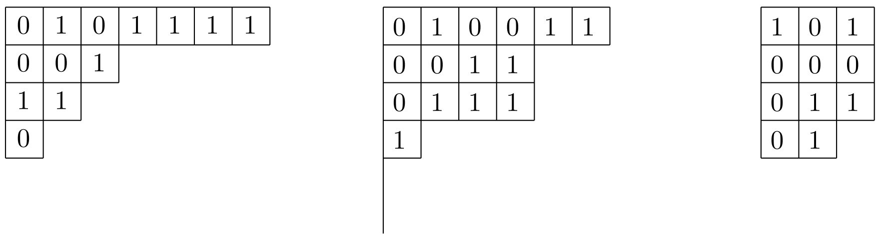

Figure 2 Examples of permutation tableaux. The tableau in the middle has two empty rows.

▶ **Definition 3.** A permutation tableau of size n is a Ferrers diagram of length n whose non-empty rows are filled with 0's and 1's according to the following rules:

1. Each column has at least one 1.
2. Any 0 cannot have a 1 both above it and to the left of it simultaneously.

We denote the set of all permutation tableaux of size n by $\mathcal{P}_n$.

In a tree-like or a permutation tableau, the edges outlining the southeast border are often called border edges. We also refer to those edges as steps. Each step is either a south step or a west step if we move along border edges from northeast to southwest or a north step or an east step if we move in the opposite direction.

▶ **Definition 4.** A corner in a tableau is a south step followed immediately by a west step as we traverse the border edges starting from the northeast and going to the southwest end. We denote by $c(T)$ the number of corners of the tableau T. If $\mathcal{T}$ is a set of tableaux we let

$$c(\mathcal{T}) = \sum_{T \in \mathcal{T}} c(T)$$

denote the total number of corners of tableaux in $\mathcal{T}$.

Tree-like tableaux correspond to the states of the PASEP as follows: traverse the border edges of a tree-like tableau beginning at the southwest end. Ignoring the first and the last step, a north step corresponds to an unoccupied node and an east step corresponds to an occupied node. Thus, for example, the tree-like tableau depicted in Figure 1 corresponds to the following state of the PASEP on 12 nodes: (In this state of the PASEP a particle could enter from the left, the particle in the second node could jump in either direction, the particle in the fifth or the tenth node could jump to the left and a particle in the seventh or the eleventh node could jump to the right.)

Figure 3 The state of the PASEP corresponding to the tree-like tableau in Figure 1.

With this association, the corners in tree-like tableau correspond to occupied sites, in which the particle could jump to the right (or enter from the left, or leave to the right) and any inner corner (north step followed by the east step) corresponds to an occupied node with a particle that can jump to the left. Since the number of inner corners is one less than the number of corners, the total number of possible moves for the PASEP in a state corresponding to $T \in \mathcal{T}_n$ is $2c(T) - 1$. For example the tableau in Figure 1 has four corners and thus the PASEP in the state depicted in Figure 3 has seven possible moves as described above. As we mentioned earlier, in physics literature the number of nodes at which a particle can move is called the current activity of the system, see e. g. [7, 8, 15].

It is known (see [1, Proposition 3.1]) that tree-like tableaux of length $n+1$ are in bijection with permutation tableaux of length n (and both are in bijection with permutations of $[n]$, see e. g. [3, 5, 14, 1]). The corners need not be preserved, but a difference between their number in a tableau and its image under that bijection is at most one (see [12, Section 3]). Therefore, in order to study corners in tree-like tableaux it will be enough to study corners in permutation tableaux, and this is what we are going to do. We need a few more notions associated with permutation tableaux.

▶ **Definition 5.** We say a zero in a permutation tableau is restricted if it has a one above it. Otherwise, the zero is unrestricted. We say a row is restricted if it contains a restricted zero, otherwise it is unrestricted. We denote by $u(T)$ the number of unrestricted rows of T.

In the first example in Figure 2, the top and the third row are unrestricted, but the other two rows are restricted. Note that the top row of a permutation tableau is necessarily unrestricted.

An important feature of permutation tableaux is that they can be constructed recursively. Given a permutation tableau, we can increase its length incrementally and fill in the new columns as they come.

▶ **Definition 6.** We say a tableau $T' \in \mathcal{P}_{n+1}$ is an extension of a tableau $T \in \mathcal{P}_n$ if T' is obtained either by adding a south step to the southwest corner of T or by adding a west step and filling the new column according to the rules.

Notice that there is only one way to extend a tableau by adding a south step, but multiple ways by adding a west step. When a west step is added, a new column is formed which must be filled. In a cell that is part of a restricted row, it must have a zero. The cells that are part of the unrestricted rows leave us options. It is not difficult to count the number of extensions (see e. g. [4, 11]) and we have:

▶ **Proposition 7.** *The number of extensions of $T \in \mathcal{P}_n$ into $T' \in \mathcal{P}_{n+1}$ is $2^{u(T)}$.*

This, however, tells us nothing of the number of unrestricted rows of the extended tableau, which is often of relevance. But the evolution of the number of unrestricted rows can be traced down (see [4] or [11]) and is given by:

▶ **Proposition 8.** *Let $T \in \mathcal{P}_n$ be a permutation tableau of length n, and let $u(T)$ be the number of unrestricted rows of T. The number of ways to extend T so that the extension has exactly k unrestricted rows, $1 \le k \le u(T)$, is:*

$$\sum_{j=1}^{k} \binom{u(T)-j}{k-j} = \binom{u(T)}{k-1}.$$

In the following sections we prefer to use probabilistic language and thus, instead of talking about the number of corners in tableaux, we let $\mathbb{P}_n$ be the uniform probability measure on $\mathcal{X}_n$ (where $\mathcal{X}_n$ is either $\mathcal{T}_n$ or $\mathcal{P}_n$) and consider a random variable C_n on the probability space $(\mathcal{X}_n, \mathbb{P}_n)$ defined by $C_n(T) = c(T)$, the number of corners of $T \in \mathcal{X}_n$. A tableau chosen from $\mathcal{X}_n$ according to the probability measure $\mathbb{P}_n$ is usually referred to as a random tableau of size n and C_n is referred to as the number of corners in a random tableau of size n. We let $\mathbb{E}_n$ denote the expected value with respect to the measure $\mathbb{P}_n$. Then, of course, we have:

$$\mathbb{E}_n C_n = \frac{c(\mathcal{X}_n)}{|\mathcal{X}_n|}.$$

As we will see below, the variance of the number of corners, $\mathbb{V}\mathrm{ar}(C_n)$, grows to infinity as $n \to \infty$ (in fact, $\mathbb{V}\mathrm{ar}(C_n) \sim 11n/180$). Furthermore if $\phi_n : \mathcal{T}_n \to \mathcal{P}_n$ is the bijection described in [1, 12] then for $T \in \mathcal{T}_n$, $c(T) = c(\phi_n(T)) + I$, where I is 0 or 1 depending on the shape of T. Therefore, for every $x \in \mathbb{R}$

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : \frac{C_n(T) - \mathbb{E}C_n}{\sqrt{\mathbb{V}\mathrm{ar}(C_n)}} \le x\right) = \mathbb{P}_n\left(T \in \mathcal{P}_n : \frac{C_n(T) - \mathbb{E}C_n + O(1)}{\sqrt{\mathbb{V}\mathrm{ar}(C_n)}} \le x\right).$$

Thus, the limiting distribution of the number of corners in a random tree-like tableau is the same as that of the number of corners in a random permutation tableau, so we will focus on the latter.

3 Generating Function and the First Two Moments

We wish to construct a generating function for the number of corners in permutation tableaux of length n. We can do it recursively by using the extension procedure for permutation tableaux mentioned earlier. In order to do this we need to keep track of the number of unrestricted rows, and we use it as a 'catalytic' variable. Proposition 8 allows us to follow the evolution of the number of unrestricted rows under the extension and with its help we can derive a recurrence for the bivariate generating function. Because of the space limitation, the presentation of our proof is deferred to the full version of the paper.

▶ **Proposition 9.** *Let for $n \ge 0$*

$$C_n(x,z) = \sum_{T \in \mathcal{P}_n} x^{c(T)} z^{u(T)}$$

be the bivariate generation function of permutation tableaux of length n, where x marks the number of corners and z marks the number of unrestricted rows. Then we have the following recurrence for $C_n(x,z)$:

$$C_n(x,z) = zC_{n-1}(x,z+1) + (x-1)\Big(z(z+1)C_{n-2}(x,z+1) - z^2 C_{n-2}(x,z)\Big) \tag{1}$$

with

$$C_0(x,z) = 1, \quad C_1(x,z) = z.$$

3.1 Expectation

The above proposition allows us to recover the expected value of the number of corners, a result conjectured in [13], first proved in [12], and then also in [10]. To do this, note that it is clear from (1) that

$$C_n(1,z) = zC_{n-1}(1,z+1) = \cdots = z^{\overline{n}},$$

where

$$z^{\overline{n}} = z(z+1)\cdot\cdots\cdot(z+n-1),$$

is the rising factorial. We can treat

$$\frac{C_n(x,z)}{C_n(1,z)} = \frac{C_n(x,z)}{z^{\overline{n}}}$$

as the probability generating function of a random variable that depends on a parameter z and, in fact, is defined on a probability space that depends on z. Ultimately, we will be interested in $z = 1$ but it is convenient to proceed with more generality.

When we write $C_n(x,z)$ in the form

$$C_n(x,z) = \sum_{m=0}^{\lfloor n/2 \rfloor} c_{n,m}(z)(x-1)^m,$$

then the expected value of such random variable is $c_{n,1}(z)/z^{\overline{n}}$. Note that (1) yields

$$c_{n,m}(z) = zc_{n-1,m}(z+1) + z(z+1)c_{n-2,m-1}(z+1) - z^2 c_{n-2,m-1}(z),$$

with the initial conditions $c_{n,0} = z^{\overline{n}}$, $n \geq 0$. Iteration gives

$$\begin{aligned} c_{n,m}(z) &= z(z+1)c_{n-2,m}(z+2) \\ &\quad + z(z+1)\Big((z+2)c_{n-3,m-1}(z+2) - (z+1)c_{n-3,m-1}(z+1)\Big) \\ &\quad + z\Big((z+1)c_{n-2,m-1}(z+1) - zc_{n-2,m-1}(z)\Big) \\ &= z^{\overline{k}}c_{n-k,m}(z+k) \\ &\quad + \sum_{j=1}^{k} z^{\overline{j}}\Big((z+j)c_{n-j-1,m-1}(z+j) - (z+j-1)c_{n-j-1,m-1}(z+j-1)\Big) \\ &= z^{\overline{n-2m}}c_{2m,m}(z+n-2m) \qquad (2) \\ &\quad + \sum_{j=1}^{n-2m} z^{\overline{j}}\Big((z+j)c_{n-j-1,m-1}(z+j) - (z+j-1)c_{n-j-1,m-1}(z+j-1)\Big). \end{aligned}$$

When $m = 1$ this becomes

$$\begin{aligned} c_{n,1}(z) &= z^{\overline{n-2}} c_{2,1}(z+n-2) + \\ &\quad \sum_{j=1}^{n-2} z^{\overline{j}} \Big((z+j) c_{n-j-1,0}(z+j) - (z+j-1) c_{n-j-1,0}(z+j-1) \Big) \\ &= z^{\overline{n-2}}(z+n-2) + \\ &\quad \sum_{j=1}^{n-2} z^{\overline{j}} \Big((z+j)(z+j)^{\overline{n-j-1}} - (z+j-1)(z+j-1)^{\overline{n-j-1}} \Big) \\ &= z^{\overline{n-1}} + z^{\overline{n-2}} \sum_{j=1}^{n-2} \Big((z+j)(z+n-2) - (z+j-1)^2 \Big) \\ &\quad = z^{\overline{n-1}} + z^{\overline{n-2}}(n-2)(z+n-2) + \\ &\quad z^{\overline{n-2}} \sum_{j=1}^{n-2} \Big((z+j-1)(z+n-2) - (z+j-1)^2 \Big) \\ &\quad = z^{\overline{n-2}} \Big((n-1)(z+n-2) + \sum_{j=1}^{n-2} (z+j-1)(n-j-1) \Big) \\ &\quad = z^{\overline{n-2}}(n-1) \Big((z+n-2) + \frac{(n-2)(n+3z-3)}{6} \Big) \\ &\quad = z^{\overline{n-2}}(n-1) \frac{n^2+3zn+n-6}{6}. \end{aligned}$$

Therefore,

$$\frac{c_{n,1}(z)}{z^{\overline{n}}} = \frac{(n-1)(n^2+3zn+n-6)}{6(z+n-1)_2}$$

where $(w)_k = w(w-1)\dots(w-(k-1))$ is the falling factorial. When $z = 1$ the above formula gives

$$\mathbb{E}C_n = \frac{n^2+4n-6}{6n} = \frac{n+4}{6} - \frac{1}{n}$$

which agrees with [12, Theorem 2].

3.2 Variance

Calculation of the expected value can be pushed further and we can obtain the variance of the number of corners, which has not been known before.

▶ **Proposition 10.** *For $n \geq 4$ we have*

$$\mathbb{V}ar(C_n) = \frac{11n^4 - 191n^2 + 360n + 180}{180n^2(n-1)} \sim \frac{11}{180} n$$

as $n \to \infty$. In addition

$$\mathbb{V}ar(C_1) = 0, \quad \mathbb{V}ar(C_2) = \frac{1}{4}, \quad \mathbb{V}ar(C_3) = \frac{5}{36}.$$

Proof. Consider $n \geq 4$ (the other three cases can be calculated directly). Our first goal is to extract $c_{n,2}(z)$. From (2) used with $m = 2$ we have

$$\begin{aligned} c_{n,2}(z) &= z^{\overline{n-4}} c_{4,2}(z+n-4) \\ &\quad + \sum_{j=1}^{n-4} z^{\overline{j}} \Big((z+j) c_{n-j-1,1}(z+j) - (z+j-1) c_{n-j-1,1}(z+j-1) \Big). \end{aligned}$$

Since

$$c_{4,2}(z) = z(z+1)c_{2,1}(z+1) - z^2 c_{2,1}(z) = z(z+1)^2 - z^3 = z(2z+1),$$

we see that

$$z^{\overline{n-4}} c_{4,2}(z+n-4) = z^{\overline{n-3}}(2(z+n)-7). \tag{3}$$

Furthermore,

$$\begin{aligned} & z^{\overline{j}}(z+j)c_{n-j-1,1}(z+j) \\ &= z^{\overline{n-3}}(z+j)(n-j-2)\frac{(n-j-1)^2 + 3(z+j)(n-j-1) + n-j-7}{6} \end{aligned}$$

and similarly,

$$\begin{aligned} & z^{\overline{j}}(z+j-1)c_{n-j-1,1}(z+j-1) \\ &= z^{\overline{n-4}}(z+j-1)^2(n-j-2)\frac{(n-j-1)^2 + 3(z+j-1)(n-j-1) + n-j-7}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{j=1}^{n-4} z^{\overline{j}} \Big((z+j)c_{n-j-1,1}(z+j) - (z+j-1)c_{n-j-1,1}(z+j-1) \Big) \\ &= \frac{z^{\overline{n-4}}}{6} \sum_{j=1}^{n-4} (n-j-2) \Big\{ (z+j)(z+n-4)\Big((n-j-1)^2 + 3(z+j)(n-j-1) + n-j-7 \Big) \\ &\quad -(z+j-1)^2 \Big((n-j-1)^2 + 3(z+j-1)(n-j-1) + n-j-7 \Big) \Big\}. \end{aligned}$$

When $z = 1$ this equals

$$\frac{(n-4)!}{360}(n-3)(n-4)(5n^4 + 26n^3 - 38n^2 - 83n - 150).$$

Combining with (3) we get

$$\begin{aligned} c_{n,2}(1) &= (n-3)!(2n-5) + \frac{(n-4)!}{360}(n-3)(n-4)(5n^4 + 26n^3 - 38n^2 - 83n - 150) \\ &= \frac{(n-2)!}{360}(5n^4 + 16n^3 - 110n^2 - 151n + 600). \end{aligned}$$

The second factorial moment for the number of corners is thus given by:

$$\mathbb{E}(C_n)_2 = \mathbb{E}C_n(C_n - 1) = \frac{2!}{n!}c_{n,2}(1) = \frac{5n^4 + 16n^3 - 110n^2 - 151n + 600}{180n(n-1)}$$

and therefore,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(C_n) &= \mathbb{E}(C_n)_2 - (\mathbb{E}C_n)^2 + \mathbb{E}C_n \\ &= \frac{5n^4+16n^3-110n^2-151n+600}{180n(n-1)} - \left(\frac{n+4}{6}-\frac{1}{n}\right)^2 + \left(\frac{n+4}{6}-\frac{1}{n}\right) \\ &= \frac{11n^4-191n^2+360n+180}{180n^2(n-1)}\end{aligned}$$

as claimed. ◀

It is, however, increasingly difficult to find $c_{n,m}$ for higher m. Instead, we will use (1) to derive a recurrence for the moment generating function and rely on method of moments (see e. g. [2, Theorem 30.2]) to establish the asymptotic normality of suitably normalized (C_n).

4 Moment Generating Function

Consider

$$P_n(t,z) := e^{-\mu_n(z)t}\frac{C_n(e^t,z)}{z^{\overline{n}}}, \quad P_0(t,z) = P_1(t,z) = 1$$

where

$$\mu_0(z) = \mu_1(z) = 0; \quad \mu_n(z) = \frac{(n-1)(n^2+3zn+n-6)}{6(z+n-1)_2}, \quad n \geq 2.$$

(Notice that $\mu_n(1)$ is the expected value of C_n, the number of corners in permutation tableaux of size n.) Then, recurrence (1) translates into

$$\begin{aligned}P_n(t,z) &= e^{\alpha_n(z)t}P_{n-1}(t,z+1) \\ &+ \frac{e^t-1}{(z+n-1)_2}\Big((z+1)(z+n-2)e^{\beta_n(z)t}P_{n-2}(t,z+1) - z^2 e^{\delta_n(z)t}P_{n-2}(t,z)\Big),\end{aligned}$$

where

$$\begin{aligned}\alpha_n(z) &= \mu_{n-1}(z+1) - \mu_n(z) = -\frac{n+zn-z-2}{(z+n-1)_2}, \\ \beta_n(z) &= \mu_{n-2}(z+1) - \mu_n(z), \\ \delta_n(z) &= \mu_{n-2}(z) - \mu_n(z).\end{aligned}$$

This gives a linear recurrence of the first order for $P_n^{(m)}(0,z)$; first

$$\begin{aligned}P_n^{(m)}(t,z) &= e^{\alpha_n(z)t}P_{n-1}^{(m)}(t,z+1) + \sum_{k=0}^{m-1}\binom{m}{k}\alpha_n^{m-k}(z)e^{\alpha_n(z)t}P_{n-1}^{(k)}(t,z+1) \\ &+ \frac{e^t}{(z+n-1)_2}\sum_{k=0}^{m-1}\binom{m}{k}\Big[(z+1)(z+n-2)\sum_{i=0}^{k}\binom{k}{i}\beta_n^{k-i}(z)e^{\beta_n(z)t}P_{n-2}^{(i)}(t,z+1) \\ &\qquad - z^2\sum_{i=0}^{k}\binom{k}{i}\delta_n^{k-i}(z)e^{\delta_n(z)t}P_{n-2}^{(i)}(t,z)\Big] \\ &+ \frac{e^t-1}{(z+n-1)_2}\Big((z+1)(z+n-2)e^{\beta_n(z)t}P_{n-2}^{(i)}(t,z+1) - z^2 e^{\delta_n(z)t}P_{n-2}^{(i)}(t,z)\Big)^{(m)}.\end{aligned}$$

At $t = 0$ the last term vanishes and letting $P_n^{(m)}(z) := P_n^{(m)}(0, z)$ we get

$$\begin{aligned} P_n^{(m)}(z) &= P_{n-1}^{(m)}(z+1) + \sum_{k=0}^{m-1} \binom{m}{k} \alpha_n^{m-k}(z) P_{n-1}^{(k)}(z+1) + \\ &+ \frac{1}{(z+n-1)_2} \sum_{k=0}^{m-1} \Bigg\{ (z+1)(z+n-2) P_{n-2}^{(k)}(z+1) \sum_{i=k}^{m-1} \binom{m}{i}\binom{i}{k} \beta_n^{i-k}(z) \\ &\qquad - z^2 P_{n-2}^{(k)}(z) \sum_{i=k}^{m-1} \binom{m}{i}\binom{i}{k} \delta_n^{i-k}(z) \Bigg\}. \end{aligned} \tag{4}$$

This recurrence is the starting point for establishing asymptotic normality for the number of corners in permutation tableaux. We outline the argument in the forthcoming section.

5 Main Result

We can now state our main result.

▶ **Theorem 11.** *Let $\{C_n\}$ be a sequence of random variables where C_n is the number of corners in a random permutation tableau of length n. Let:*

$$\mu_n = \frac{n+4}{6} - \frac{1}{n} \sim \frac{n}{6}$$

and

$$\sigma_n^2 = \mathbb{V}ar(C_n) \sim \frac{11}{180} n.$$

Then

$$\frac{C_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{or} \quad \frac{C_n - \frac{n}{6}}{\sqrt{\frac{11}{180} n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\xrightarrow{d}$ is convergence in distribution and $\mathcal{N}(0,1)$ is the standard normal random variable.

Proof. *(Sketch)* The proof relies on the method of moments (see e. g. [2, Theorem 30.2] and on the analysis of recurrence (4) for the moments which will allow us to establish that:

$$\frac{P_n^{(m)}(1)}{(\frac{11}{180} n)^{\frac{m}{2}}} \to \begin{cases} 0, & m \text{ odd} \\ \frac{m!}{2^{\frac{m}{2}} \cdot (m/2)!}, & m \text{ even.} \end{cases} \tag{5}$$

The complete proof will be presented in the full version of the paper; here we indicate the main steps in the argument. First, we retain only the two highest degree terms in recurrence (4) (the remaining terms are of lower order and thus do not contribute significantly). Then (4) simplifies to

$$\begin{aligned} P_n^{(m)}(z) &= P_{n-1}^{(m)}(z+1) + \binom{m}{m-1} \alpha_n(z) P_{n-1}^{(m-1)}(z+1) + \binom{m}{m-2} \alpha_n^2(z) P_{n-1}^{(m-2)}(z+1) \\ &+ \frac{1}{(z+n-1)_2} \Bigg\{ (z+1)(z+n-2) P_{n-2}^{(m-1)}(z+1) \binom{m}{m-1} - z^2 P_{n-2}^{(m-1)}(z) \binom{m}{m-1} \\ &\quad + (z+1)(z+n-2) P_{n-2}^{(m-2)}(z+1) \left[\binom{m}{m-2} + \binom{m}{m-1}\binom{m-1}{m-2} \beta_n(z) \right] \\ &\qquad - z^2 P_{n-2}^{(m-2)}(0,z) \left[\binom{m}{m-2} + \binom{m}{m-1}\binom{m-1}{m-2} \delta_n(z) \right] \Bigg\}. \end{aligned}$$

The next simplification is based on the observation (which, again, will be justified in the full version of the paper) that $P_{n-2}^{(k)}(z+1) \sim P_{n-1}^{(k)}(z+1) \sim P_{n-2}^{(k)}(z)$. Thus, replacing all the k^{th} derivatives in the curly brackets by $P_{n-1}^{(k)}(z+1)$, $k = m-1, m-2$ the above is

$$P_n^{(m)}(z) = P_{n-1}^{(m)}(z+1) + mP_{n-1}^{(m-1)}(z+1)\left[\alpha_n(z) + \frac{(z+1)(z+n-2) - z^2}{(z+n-1)_2}\right]$$
$$+\binom{m}{2}P_{n-1}^{(m-2)}(z+1)\left\{\alpha_n^2(z) + \frac{(z+1)(z+n-2)(1+2\beta_n(z)) - z^2(1+2\delta_n(z))}{(z+n-1)_2}\right\}.$$

By the definition of $\alpha_n(z)$, the term in the square brackets is zero. Denote the term in the curly brackets by $T_{n-1}(z+1)$. Then, our recurrence becomes

$$P_n^{(m)}(z) = P_{n-1}^{(m)}(z+1) + \binom{m}{2}T_{n-1}(z+1)P_{n-1}^{(m-2)}(z+1).$$

Iterating and using $P_1^{(m)}(z) = 0$ for all z and $m \geq 1$, we get

$$P_n^{(m)}(z) = \binom{m}{2}\sum_{j=1}^{n-1} T_{n-j}(z+j)P_{n-j}^{(m-2)}(z+j). \tag{6}$$

Let now $m = 2r$ be even. Iterating (6) yields

$$\begin{aligned}
P_n^{(2r)}(z) &= \binom{2r}{2}\sum_{j=1}^{n-1} T_{n-j}(z+j)P_{n-j}^{(2r-2)}(z+j) \\
&= \binom{2r}{2}\binom{2r-2}{2}\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{n-j_1-1} T_{n-j_1}(z+j_1)T_{n-j_1-j_2}(z+j_1+j_2)P_{n-j_1-j_2}^{(2r-4)}(z+j_1+j_2) \\
&= \binom{2r}{2}\binom{2r-2}{2}\cdots\binom{4}{2}\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{n-j_1-1}\cdots\sum_{j_r=1}^{n-1-\sum_{i=1}^{r-1} j_i}\prod_{i=1}^{r} T_{n-\sum_{l=1}^{i} j_l}(z+\sum_{l=1}^{i} j_l) \\
&= \frac{(2r)!}{2^r}\sum_{1\le k_1<k_2<\cdots<k_r<n}\prod_{i=1}^{r} T_{n-k_i}(z+k_i) \\
&= \frac{(2r)!}{2^r}\frac{1}{r!}\sum_{\substack{1\le k_1,\ldots,k_r<n \\ \text{distinct}}}\prod_{i=1}^{r} T_{n-k_i}(z+k_i) \\
&= \frac{(2r)!}{2^r}\frac{1}{r!}\Bigg(\sum_{\substack{1\le k_1,\ldots,k_r<n \\ \text{all}}}\prod_{i=1}^{r} T_{n-k_i}(z+k_i) - \sum_{\substack{1\le k_1,\ldots,k_r<n \\ \text{not all distinct}}}\prod_{i=1}^{r} T_{n-k_i}(z+k_i)\Bigg).
\end{aligned}$$

Set $z = 1$. The first sum is

$$\left(\sum_{k=1}^{n-1} T_{n-k}(k+1)\right)^r = \left(P_n^{(2)}(1)\right)^r \sim \left(\frac{11}{180}n\right)^r$$

where the first equality follows from (6) used with $m = 2$ and $P_n^{(0)}(z) = 1$. (This can also be verified by evaluating the sum of the $T_{n-k}(1+k)$ directly; for the purpose of asymptotic evaluation it suffices to use the highest order term approximating $T_n(z)$, i.e.,

$$T_n(z) \sim \frac{nz(2n^4 + 6n^3z + n^2z^2 + 3z^4)}{3(n+z)^6}$$

which is obtained by using the highest order term approximations, for example, $\mu_n(z) \sim n^2(n+3z)/(6(z+n-1)_2)$ and similarly for $\alpha_n(z)$, $\beta_n(z)$, and $\delta_n(z)$, but all of these approximations require justifications.)

For the second summation, observe that $|T_{n-j}(1+j)| \le C$ for all $1 \le j < n$ and a universal constant C. Thus:

$$\left| \sum_{\substack{1 \le k_1,\dots,k_r < n \\ \text{not all distinct}}} \prod_{i=1}^{r} T_{n-k_i}(1+k_i) \right| \le \sum_{\substack{1 \le k_1,\dots,k_r < n \\ \text{not all distinct}}} \prod_{i=1}^{r} |T_{n-k_i}(1+k_i)| \le C^r \cdot O(n^{r-1}),$$

which is of lower order than the first sum. This proves (5) for m even.

Let now $m = 2r+1$ be odd. We wish to show that $P_n^{(2r+1)}(1) = O(n^r)$ as $n \to \infty$. In fact, we proceed to prove by induction that for every $r \ge 0$, $P_{n-k}^{(2r+1)}(k+1) = O(n^r)$ uniformly in $1 \le k < n$ as $n \to \infty$. Since $P_n^{(1)}(z) = 0$ (and approximation errors are bounded) this is true for $r = 0$. Now assume $P_{n-k}^{(2r-1)}(k+1) = O(n^{r-1})$ uniformly in $1 \le k < n$. Then from (6) we have that:

$$\begin{aligned} |P_{n-k}^{(2r+1)}(k+1)| &\le \binom{2r+1}{2} \max_{1 \le k \le n-1} |P_{n-k}^{(2r-1)}(k+1)| \sum_{j=1}^{n-1} |T_{n-j}(z+j)| \\ &\le \binom{2r+1}{2} O(n^{r-1}) \cdot Cn = O(n^r) \end{aligned}$$

as desired. ◀

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