Asymptotic Expansions for Sub-Critical Lagrangean Forms

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Abstract

Asymptotic expansions for the Taylor coefficients of the Lagrangean form $\phi(z) = zf(\phi(z))$ are examined with a focus on the calculations of the asymptotic coefficients. The expansions are simple and useful, and we discuss their use in some enumerating sequences in trees, lattice paths and planar maps.

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1 Introduction

Singularity analysis and saddle-point method represent the two major standard approaches used in analytic combinatorics to compute the asymptotics of, say the Taylor coefficients $[z^n]\phi(z)$ for large n; see [9, Chap. VI & VII]. The choice of which method to use depends crucially on the growth order of the functions in question near the dominant singularity or the saddle-point. The general principle is to apply the saddle-point method when the growth order of f near the saddle-point is large (e.g., $\log\phi(z) \gg (\log|1-z|)^{1+\varepsilon}$) and to apply the singularity analysis otherwise. In most cases, only one of the two works if one is interested in more precise asymptotic approximations. The Lagrangean form (frequently encountered in diverse areas; see [9, §A.6])

$$\phi(z) = zf(\phi(z)) \tag{1}$$

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with both ϕ and f analytic functions, is one of the few situations in which both methods apply, and the key tool bridging the two different approaches is the Lagrange Inversion Formula [9, §A.6]

$$[z^n]\phi(z) = n^{-1}[t^{n-1}]f(t)^n \qquad (n \geqslant 1). \tag{2}$$

This form of large powers shows generally that saddle-point method is a good candidate for deriving the corresponding asymptotics, while the functional form (1) favors the use of singularity analysis (coupling with the implicit function theorem).

For the purpose of more precise asymptotics, we assume the following conditions.

C1 (nonnegativity and aperiodicity) $a_j := [t^j]f(t) \geqslant 0$ for every $j \geqslant 0$ and $\gcd\{j : a_j > 0\} = 1$;
C2 (analyticity) f is analytic in $|z| < R$ for $0 < R \leqslant \infty$;
C3 (sub-criticality) there exists an $r \in (0, R)$ such that $rf'(r) = f(r)$.

Note that the conditions C1 and C3 imply that

$$a_0 = f(0) = \sum_{j \geqslant 2}(j-1)a_j r^j > 0.$$

Note further that the condition C3 fails when f is linear, namely, $f(z) = a_0 + a_1 z$, which gives rise to

$$\phi(z) = \frac{a_0 z}{1 - a_1 z} \Longrightarrow [z^n]\phi(z) = a_0 a_1^{n-1}.$$

Under the conditions C1–C3, it is known, by singularity analysis or saddle-point method, that (see [9, §IV.7] or [13, 14])

$$[z^n]\phi(z) \sim C\, n^{-\frac{3}{2}}\, \rho^{-n}, \quad \text{where } \rho := \frac{r}{f(r)} \text{ and } C := \sqrt{\frac{f(r)}{2\pi f''(r)}}.$$

The aim of this extended abstract is to examine the asymptotic expansions of the Lagrangean form (1). In particular, we will prove the following theorem, which can be regarded as an alternative version of Theorem VI.6 in [9, §VI.7] with the coefficients not just "computable" but by a more precise formula. Also we prefer the use of binomial coefficients to negative powers of n.

▶ **Theorem 1.** *Assume that ϕ and f satisfy* (1). *Then, under the conditions C1–C3, we have*

$$[z^n]\phi(z) \sim \rho^{-n} \sum_{k \geqslant 0} c_{2k+1} \binom{n-k-\frac{3}{2}}{n}, \tag{3}$$

where the coefficients c_j's are expressible in yet another Lagrangean form

$$c_k = k^{-1}[v^{k-1}]F(v)^k, \quad \text{with} \quad F(v) := -\left(\frac{1 - \frac{(r+v)f(r)}{rf(r+v)}}{v^2}\right)^{-\frac{1}{2}} \qquad (k \geqslant 1). \tag{4}$$

This succinct expression for c_k shows that the Lagrangean form (1) is not only useful for computing the Taylor expansion of ϕ at $z = 0$ (as is most commonly used), but also at the dominant singularity in subcritical situations (the latter is little known). The singular Lagrangean form (4) can further be used to derive the asymptotic behavior of c_k (although in most cases the sub-criticality condition C3 fails), which in turn will be helpful in determining the number of terms used in order to reduce the numerical errors; see Section 4–6 for the discussion of some examples.

Let $\tau := \sqrt{\frac{2f(r)}{f''(r)}}$. Then we have (with $f_j := f^{(j)}(r)$)

$$\frac{c_1}{\tau} = -1 \quad \text{and} \quad \frac{c_3}{\tau^3} = -\frac{1}{8r^2} - \frac{f_2}{4f_0} + \frac{f_3}{4rf_2} - \frac{5f_3^2}{72f_2^2} + \frac{f_4}{24f_2}.$$

While the expressions of c_k are becoming messy as k increases, the neat expression (4) is not commonly available in most asymptotic expansions and reflects certain intrinsic properties of the Lagrangean form (1).

The asymptotic expansion (3) is to be compared with the usual one (see [9, Theorem VI.6]):

$$[z^n]\phi(z) \sim \rho^{-n} \sum_{k\geqslant 0} d_k n^{-k-\frac{3}{2}}, \tag{5}$$

where the coefficients d_k can be computed recursively but no simple expression such as (4) is available; see for example the next section for the usual constructive procedures to compute d_k. Alternatively, we can convert (3) to (5) by the following argument. Recall first Euler's reflection formula

$$\binom{n-k-\frac{3}{2}}{n} = \frac{\Gamma\left(n-k-\frac{1}{2}\right)}{n!\Gamma\left(-k-\frac{1}{2}\right)} = \frac{(-1)^{k+1}\Gamma\left(k+\frac{3}{2}\right)}{\pi} \cdot \frac{\Gamma\left(n-k-\frac{1}{2}\right)}{\Gamma(n+1)}. \tag{6}$$

Then we need the following asymptotic expansion.

▶ **Lemma 2** ([17]). *For $\alpha \in \mathbb{C}$*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim \sum_{j\geqslant 0} \lambda_j(\alpha) z^{\alpha-j} \qquad (|z| \to \infty),$$

uniformly for $|\arg(z)| \leqslant \pi - \varepsilon$, $\varepsilon > 0$. Here $\lambda_0(\alpha) = 1$ and

$$\lambda_j(\alpha) = \frac{1}{j} \sum_{0\leqslant l<j} \binom{\alpha - l}{j+1-l} \lambda_l(\alpha) \qquad (j \geqslant 1).$$

This expression of $\lambda_j(\alpha)$ is simpler than that given in [8, Proposition 1]. Applying Lemma 2 to (6), we obtain

$$\binom{n-k-\frac{3}{2}}{n} \sim \frac{(-1)^{k+1}\Gamma\left(k+\frac{3}{2}\right)}{\pi} \sum_{j\geqslant 0} \lambda_j\left(-k-\tfrac{1}{2}\right) n^{-k-\frac{3}{2}-j},$$

from which we deduce the relation between d_k and c_k, which in turn results in the effective version (5) of [9, Theorem VI.6].

▶ **Theorem 3.** *Assume that ϕ and f satisfy* (1)*. Then, under the conditions C1–C3, the expansion* (5) *holds with*

$$d_k = \frac{1}{\pi} \sum_{0\leqslant j\leqslant k} (-1)^j c_{2j+1} \Gamma\left(j+\tfrac{3}{2}\right) \lambda_{k-j}\left(-j-\tfrac{1}{2}\right) \qquad (k \geqslant 0). \tag{7}$$

In view of the computational complexity of the coefficients, the expansion (3) is preferable and recommended for most numerical purposes because the binomial coefficients can be easily computed in most softwares.

On the other hand, the expansion (3) can be extended to a more general context of the form (or Lagrange-Bürmann formula)

$$[z^n]G(\phi(z)) = n^{-1}[t^{n-1}]G'(t)f(t)^n \qquad (n \geqslant 1).$$

▶ **Theorem 4.** *Let G be an analytic function in $|z| \leqslant r$. Under the conditions C1–C3, we have*

$$[z^n]G(\phi(z)) \sim \rho^{-n} \sum_{k\geqslant 0} e_{2k+1} \binom{n-k-\frac{3}{2}}{n}, \tag{8}$$

where $e_k = k^{-1}[v^{k-1}]G'(r+v)F(v)^k$ for $k \geqslant 1$.

Since $e_1 = -G'(r)\tau$, we see that $e_1 = 0$ when $G'(r) = 0$ (very common in the context of planar maps [1]), and we then get the $n^{-\frac{5}{2}}$-asymptotics

$$[z^n]G(\phi(z)) \sim \rho^{-n} \sum_{k\geqslant 0} e_{2k+3} \binom{n-k-\frac{5}{2}}{n},$$

where in particular (with $G_j = G^{(j)}(r)$) $\frac{e_3}{\tau^3} = -\frac{G_3}{6} - \frac{G_2}{2r} + \frac{f_3 G_2}{6 f_2}$. The usefulness of the two expansions (3) and (8) will be demonstrated through a few examples of trees and planar maps.

In the next section, we give a procedure to compute the coefficients d_k in (5). Then we prove (3) and (8) in Section 3. Some applications are discussed in the remaining sections.

2 An asymptotic expansion by saddle-point method

For comparison and for more methodological interest, we derive (5) in this section by a direct saddle-point method. Since the analysis is standard (see [9, 14]), we focus on the computation of the asymptotic coefficients d_k as follows.

1. Compute first the expansion $f(re^v) = \sum_{k\geqslant 0} d_k^{[1]} v^k$, where ($S(k,j)$ being Stirling numbers of the second kind)

$$d_k^{[1]} = \frac{1}{k!} \sum_{0\leqslant j\leqslant k} S(k,j) f^{(j)}(r) r^j \qquad (k \geqslant 0).$$

2. Expand $\log f(re^v) = \sum_{k\geqslant 0} d_k^{[2]} v^k$, where $d_0^{[2]} = \log f(r)$ and

$$d_k^{[2]} = \frac{d_k^{[1]}}{f(r)} - \frac{1}{kf(r)} \sum_{1\leqslant j<k} j d_j^{[2]} d_{k-j}^{[1]} \qquad (k \geqslant 2).$$

By $rf'(r) = r$, we see that $d_1^{[2]} = 1$ and $d_2^{[2]} = \frac{r^2 f''(r)}{2f(r)}$.

3. Now expand

$$\exp\Big(vx + \frac{1}{2d_2^{[2]}} \sum_{j\geqslant 1} d_{j+2}^{[2]} v^{j+3} x^j\Big) = \sum_{k\geqslant 0} d_k^{[3]}(v) x^k,$$

where $d_0^{[3]} = 1$ and

$$d_k^{[3]}(v) = \frac{v}{k} d_{k-1}^{[3]}(v) + \frac{1}{2d_2^{[2]} k} \sum_{1\leqslant j\leqslant k} j v^{j+2} d_{j+2}^{[2]} d_{k-j}^{[3]}(v) \qquad (k \geqslant 1).$$

Note that $d_k^{[3]}(v)$ contains only powers of v with the same parity as k of degree $3k$.

4. Then, with $\sigma := r\sqrt{\frac{f''(r)}{f(r)}}$,

$$[z^{n-1}]f(z)^n \sim \frac{r^{1-n}f(r)^n}{\sqrt{2\pi n}\,\sigma}\left(1+\sum_{k\geqslant 1}\frac{d_k^{[4]}}{\sigma^{2k}n^k}\right),$$

where

$$d_k^{[4]} := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac12 t^2} d_{2k}^{[3]}(it)\,\mathrm{d}t = \sum_{0\leqslant j\leqslant 3k}(-1)^j\frac{(2j)!}{j!2^j}[v^{2j}]d_{2k}^{[3]}(v).$$

Thus, by comparing with (5), we have (with $\tau := \sqrt{\frac{2f(r)}{f''(r)}}$)

$$d_k = \frac{r}{\sqrt{2\pi}\,\sigma^{2k+1}}\,d_k^{[4]} = \frac{\tau^{2k+1}}{2^{k+\frac12}r^{2k}}\,d_k^{[4]}. \tag{9}$$

A non-recursive procedure is possible via Bell polynomials but not simpler; see [7] and the references therein. In particular (with $f_j := f^{(j)}(r)$), $d_1 = \frac18 - \frac{3r^2 f_2}{8f_0} + \frac{rf_3}{4f_2} - \frac{r^2f_3^2}{24f_2^2} + \frac{r^2 f_4}{8f_2}$. For larger k the expressions of d_n become very messy.

3 An asymptotic expansion by singularity analysis

We prove (3) and (8) in this section by singularity analysis. As in the previous section, we focus on the computations of c_k, the analytic justification being done as in [9, Theorem VI.6]. Following the exposition there, the idea starts from the equation (writing $w = \phi(z)$)

$$\rho - z = \frac{r}{f(r)} - \frac{w}{f(w)}. \tag{10}$$

Then invert (10) by expanding w in terms of $\rho - z$. For convenience, we find that it is simpler to use the expansion

$$1 - \frac{wf(r)}{rf(w)} = \sum_{j\geqslant 2} b_j(w-r)^j. \tag{11}$$

In particular, we have (with $f_j = f^{(j)}(r)$)

$$b_2 = \frac{f_2}{2f_0},\quad b_3 = -\frac{b_2}{r}+\frac{f_3}{6f_0},\quad b_4 = -\frac{b_3}{r} - \frac{f_2^2}{4f_0^2}+\frac{f_4}{24f_0}.$$

Now write $w = r+t$, and rearrange the expansion (11) as

$$f(r+t) - \frac{f(r)}{r}(r+t) = f(r+t)\sum_{j\geqslant 2} b_j t^j,$$

which then leads to the recurrence

$$b_m = \frac{f_m}{m!f_0} - \sum_{2\leqslant j\leqslant m-2} b_{m-j}\frac{f_j}{f_0 j!} - \frac{b_{m-1}}{r} \qquad (m \geqslant 3).$$

These coefficients can be computed in linear time (in m) once the derivatives of f at r are available.

Let $\Delta := \sqrt{1-z}$. We now examine the local behavior of Δ for $z \sim 1$ by first inverting the relation

$$\Delta^2 = 1 - \frac{(r+t)f(r)}{rf(r+t)} = \sum_{j\geqslant 2} b_j t^j,$$

or

$$\Delta^2 = t^2 \sum_{j\geqslant 0} b_{j+2} t^j \quad \Longrightarrow \quad t = \Delta F(t), \tag{12}$$

where

$$F(t) := -\Big(\sum_{j\geqslant 0} b_{j+2} t^j\Big)^{-\frac{1}{2}} = -\left(\frac{1-\frac{(r+t)f(r)}{rf(r+t)}}{t^2}\right)^{-\frac{1}{2}}.$$

Interestingly, this is again of a Lagrangean form, and we see that

$$t = \sum_{k\geqslant 1} c_k \Delta^k,$$

where c_k is given in (4). Then we are led to the singular expansion

$$t = \phi(\rho z) - r = \sum_{k\geqslant 1} c_k (1-z)^{\frac{k}{2}},$$

which is convergent in a neighborhood of unity excluding the branch-cut $[1,\infty)$ (the exact range depending on the zeros or singularities of F). Then, by singularity analysis, we obtain (3).

The proof for (8) is similar, because

$$G(\phi(\rho z)) = G(r+t) = G(r) + \sum_{k\geqslant 1} e_k \Delta^k.$$

4 Applications I: $[z^n]\phi(z)$ and the $n^{-\frac{3}{2}}$-asymptotics

We discuss in this section the use of our asymptotic expansions in some popular counting sequences in combinatorics.

The following simple observation is useful for justifying sub-criticality of the Lagrangean form (1); see also [9, Proposition IV.5] for a slightly more general version.

▶ **Lemma 5** (Sub-criticality). *Let the radius of convergence of the series $f(z) = \sum_{j\geqslant 0} a_j z^j$ be $R > 0$ with $a_0 > 0$ and $a_j \geqslant 0$ for $j \geqslant 1$. If f is not linear and $\lim_{z\to R} f(z) \to \infty$, then the condition C3 is satisfied, namely, there exists an $r \in (0,R)$ such that $rf'(r) = f(r)$.*

Proof. Consider the function $g(z) := \frac{z}{f(z)}$, which is well-defined at least in $[0,R)$. Since $g(0) = 0$ (because $a_0 > 0$) and $\lim_{z\to R} g(z) \to 0$, by Rolle's Theorem, there exists an $r \in (0,R)$ such that $g'(r) = 0$. But $g'(r) = 0$ is equivalent to $rf'(r) - f(r) = 0$. ◀

In particular, if f is a rational function of z with nonnegative Taylor coefficients, then the Lagrangean form (1) is always sub-critical. For example, $f(z) = \frac{1}{1-z}$ gives the Catalan numbers $\phi(z) = \frac{1-\sqrt{1-4z}}{2}$ (A000108 in Sloane's OEIS). We see that no further treatment is needed because the singular expansion contains only one term.

Motzkin numbers (A001006).

Consider now the Motzkin numbers with $f(z) = 1 + z + z^2$ and

$$\phi(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}. \tag{13}$$

Note that $zf'(z) - f(z) = z^2 - 1$, implying that $r = 1$ and $\rho = \frac{1}{3}$. Thus

$$[z^n]\phi(z) = \sum_{0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{j!(j+1)!(n-1-2j)!} \sim 3^n \sum_{k \geqslant 0} c_{2k+1} \binom{n-k-\frac{3}{2}}{n}. \tag{14}$$

For finite k, the coefficients c_k can be readily computed by (4) with $F(v) = -\sqrt{3 + 3v + v^2}$. We observe that while the asymptotics of the left-hand side of (14) remains less visible even for the exponential order, that of the right-hand side is transparent if we regard the binomial factor as decreasing powers in n. Furthermore, the right-hand side is a direct consequence of Theorem 1, and thus even without any explicit formula for $[z^n]\phi(z)$, which is often the case, we can still apply the expansion (3) and obtain very effective approximations.

We now look at the large k-asymptotics of c_k. Note that $vF'(v) - F(v) = \frac{3(2+v)}{3\sqrt{3+3v+v^2}}$, which equals zero when $v = -2$, and has a pair of conjugate singularities at $-\frac{3}{2} \pm \frac{\sqrt{3}}{2} i$ with modulus $\sqrt{3} < 2$, so the Lagrangean form (12) is not sub-critical (and thus the saddle-point at -2 is not dominant). Indeed, by the closed-form expression (13) of ϕ, we have

$$t = \Delta F(t) \quad \Longrightarrow \quad t = \frac{3\Delta^2 - \Delta\sqrt{3(4 - \Delta^2)}}{2(1 - \Delta^2)}.$$

This implies that

$$c_{2k+1} = -\sqrt{3}\left(1 - 2\sum_{1 \leqslant j \leqslant k} \binom{2j-2}{j-1} \frac{16^{-j}}{j}\right) = -\frac{3}{2} + O\left(k^{-\frac{3}{2}} 16^{-k}\right).$$

Thus they can be replaced by $-\frac{3}{2}$ for moderate values of k (depending on the desired degree of precision).

Schröder numbers (A001003).

In this case, we have $f(z) = \frac{1-z}{1-2z}$ and

$$\phi(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4}, \tag{15}$$

implying that $r = 1 - \frac{1}{\sqrt{2}} < \frac{1}{2}$, $\rho = \frac{r}{f(r)} = 3 - 2\sqrt{2}$, and ϕ has the dominant singularity at $3 - 2\sqrt{2}$. Furthermore, $F(v) = -r\sqrt{\frac{1}{\sqrt{2}} - v}$, and therefore the equation $vF'(v) = F(v)$ has a solution at $v = \sqrt{2} > \frac{1}{\sqrt{2}}$ lying outside the circle where F is analytic. So the Lagrangean form $t = \Delta F(t)$ is not sub-critical. On the other hand, t can be solved in terms of Δ as

$$t = \Delta F(t) \quad \Longrightarrow \quad t = -\frac{(\sqrt{2}-1)^2\Delta^2 + (\sqrt{2}-1)\Delta\sqrt{(\sqrt{2}-1)^2\Delta^2 + 4\sqrt{2}}}{4}.$$

Thus

$$c_{2k+1} = [\Delta^{2k+1}]t = \frac{(-1)^{k+1}}{2^{\frac{3}{4}}} \binom{k - \frac{3}{2}}{k} 2^{-\frac{5}{2}k} (\sqrt{2}-1)^{2k+1}.$$

We get the same $k^{-\frac{3}{2}}$-asymptotics for the coefficients c_k even though $t = \Delta F(t)$ is not sub-critical. Note that c_{2k+1} are asymptotic to $c'k^{-\frac{3}{2}}\rho_c^{-k}$ for large k, where $\rho_c \approx 32.97$, meaning that they converge exponentially to zero. By the closed-form expression for Schröder numbers, we have the asymptotic expansion

$$[z^n]\phi(z) = \frac{1}{n}\sum_{0\leqslant j\leqslant n-2}\binom{n-2}{j}\binom{n}{j+1}2^{n-2-j} \sim (3+2\sqrt{2})^n\sum_{k\geqslant 0}c_{2k+1}\binom{n-k-\frac{3}{2}}{n}.$$

Again, the right-hand side is preferable for large-n numerics and the numerical fits are very good even for small values of n. For example, for $n=10$, $\left|\frac{[z^n]\phi(z)}{(3+2\sqrt{2})^n} - c_1\binom{n-\frac{3}{2}}{n}\right| \leqslant 6.2\times 10^{-6}$.

The same approach applies to many other sequences with f a polynomial or a rational form. Indeed, several hundred of sequences were found in OEIS whose generating functions satisfy the Lagrangean form (1) with polynomial or rational f. Some of these will be compiled and discussed in the journal version.

5 Applications II: $[z^n]G(\phi(z))$ and the $n^{-\frac{5}{2}}$-asymptotics

A map is an *embedding of a connected planar multigraph on the sphere*, up to orientation preserving homeomorphism. Asymptotic enumeration of planar maps often features a universal $n^{-\frac{5}{2}}$-behavior, in contrast to $n^{-\frac{3}{2}}$ for that of trees; see [1, 2] for more information and references. Given a class $\mathscr{M}$ of maps, let m_n denote the number of maps in $\mathscr{M}$ with n edges. Let $M(z) := \sum_{n\geqslant 0} m_n z^n$ be the generating function of m_n, which is specified by the Lagrangean form

$$M(z) = G(\phi(z)), \qquad \phi(z) = zf(\phi(z)). \tag{16}$$

As the number of different types of maps is huge (see, e.g., [2, 4, 10, 11, 12, 18]), we content ourselves in this extended abstract only with the discussion of Table 2 in [1] (a total of 14 examples reformatted below with a correction for $\mathscr{M}_2$) for representative asymptotic patterns, focusing on the calculations of the asymptotic coefficients e_k, a missing facet in most previous publications. See [1] for precise definitions of the diverse terms used here (such as non-separable, bridgeless, singular, irreducible, etc.).

Incremental maps

We first categorize the 14 examples into two groups according to the availability for the counting function $m_n = r_n m_{n-1}$ for some rational function r_n—such a counting formula entails a Markovian property and in turn an incremental construction procedure. This group includes (see the following tables) the six examples $\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3, \mathscr{B}_1, \mathscr{T}_1$ and $\mathscr{T}_2$. Note that $\mathscr{M}_2 = \mathscr{T}_2$; see [3]. Since the application of our expansions is straightforward, we omit the details of the expansions.

type of maps	$\mathscr{M}_1$ general	$\mathscr{M}_2$ bridgeless $= \mathscr{T}_2$ triangulations	$\mathscr{M}_3$ non-separable	$\mathscr{B}_1$ bipartite	$\mathscr{T}_1$ singular triangulations
$f(z)$	$3(1+z)^2$	$(1+z)^4$	$(1+z)^3$	$2(1+z)^2$	$2(1+z^3)$
$[z^n]\phi(z)$ or $\phi(z)$	$\frac{3^n}{n+1}\binom{2n}{n}$	$\frac{1}{3n+1}\binom{4n}{n}$	$\frac{1}{2n+1}\binom{3n}{n}$	$\frac{2^n}{n+1}\binom{2n}{n}$	$\frac{2^n}{2n+1}\binom{3n}{n}$
OEIS(ϕ)	A005159	A002293	A001764	A151374	A153231

type of maps	$\mathscr{M}_1$ general	$\mathscr{M}_2$ bridgeless $= \mathscr{T}_2$ triangulations	$\mathscr{M}_3$ non-separable	$\mathscr{B}_1$ bipartite	$\mathscr{T}_1$ singular triangulations
$G(z)$	$\frac{z(2-z)}{3}$	$(1-z)(1+z)^2$	$\frac{z(2+z-z^2)}{(1+z)^3}$	$\frac{z(2-z)}{4}$	$\frac{z(1-z)}{2}$
$[z^n]G(\phi(z))$	$\frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$	$\frac{2\cdot(4n+1)!}{(n+1)!(3n+2)!}$	$\frac{2(3n)!}{(n+1)!(2n+1)!}$	$\frac{3\cdot 2^{n-1}(2n)!}{n!(n+2)!}$	$\frac{2^n(3n)!}{(n+1)!(2n+1)!}$
OEIS$(G(\phi))$	A000168	A000260	A000139	A000257	A000309
r	1	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$
ρ	$\frac{1}{12}$	$\frac{27}{256}$	$\frac{4}{27}$	8	$\frac{2}{27}$
$F(t)$	$-(2+t)$	$-\frac{\sqrt{3}(4+3t)^2}{9\sqrt{32+16t+3t^2}}$	$-\frac{(3+2t)^{\frac{3}{2}}}{2\sqrt{9+2t}}$	$-(2+t)$	$-\frac{(3+2t)^{\frac{3}{2}}}{2\sqrt{9+2t}}$
$G'(r+v)$	$-\frac{2}{3}v$	$-v(4+3v)$	$\frac{-32v}{(3+2v)^3}$	$-\frac{1}{2}v$	$-v$

Non-incremental maps

The remaining eight maps are further divided into two sub-groups:
(1) $[z^n]f(z) \geqslant 0$: $\mathscr{M}_4$, $\mathscr{M}_6$, and $\mathscr{T}_3$;
(2) $[z^n]f(z)$ contains negative coefficients: $\mathscr{M}_5, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4$, and $\mathscr{B}_5$.
Our theorems in the introduction are directly applicable to the first sub-group, and can be readily modified for the second for which the condition C1 (nonnegativity of coefficients) fails.

Non-incremental maps with $[z^n]f(z) \geqslant 0$

Similar to the incremental maps given above, we summarize the major properties of the three non-incremental maps in the following table.

type of maps	$\mathscr{M}_4$ simple	$\mathscr{M}_6$ 3-connected	$\mathscr{T}_3$ irreducible triangulations
$f(z)$	$\frac{(3+z)^2}{3-z}$	$\frac{1}{1-z}$	$\frac{1}{(1-z)^2}$
$[z^n]\phi(z)$ or $\phi(z)$	$\frac{3-\sqrt{1-8z}}{2(1+z)}$	$\frac{1}{n}\binom{2n-2}{n-1}$	$\frac{(3n+1)!}{(n+1)!(2n+1)!}$
OEIS(ϕ)	A062062	A000108	A006013
$G(z)$	$\frac{z(9-3z-z^2)}{27}$	$\frac{z^5(1-z-z^2)}{(1+z)^3(1+z-z^2)}$	$\frac{z(1-z-z^2)}{(1-z)(1+z)^2}$
OEIS$(G(\phi))$	A022558	A000287	A000256
r	1	$\frac{1}{2}$	$\frac{1}{3}$
ρ	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{4}{27}$
$F(t)$	$-\frac{4+t}{3}$	$-\frac{1}{2}$	$\frac{-2}{3\sqrt{3(1-t)}}$
$G'(r+v)$	$\frac{-v(4+v)}{9}$	$-\frac{2v(1+2v)^4\begin{pmatrix}167+176v-24v^2\\ -64v^3-16v^4\end{pmatrix}}{(3+2v)^4(5-4v^2)^2}$	$\frac{-729v}{(2-3v)^2(4+3v)^3}$

We consider only the simple maps $\mathscr{M}_4$ with $f(z) = \frac{(3+z)^2}{3-z}$ and $G(z) = \frac{z(9-3z-z^2)}{27}$. In this case, ϕ is given by

$$\phi(z) = \frac{3}{2} \cdot \frac{1 - 2z - \sqrt{1-8z}}{1+z},$$

implying that

$$G(\phi(z)) = -1 + \frac{4}{1+z} + \frac{18}{(1+z)^2} - \frac{27}{2(1+z)^3} + \frac{(1-8z)^{\frac{3}{2}}}{2(1+z)^3}, \tag{17}$$

which then gives

$$[z^n]G(\phi(z)) = (-1)^n \left(\frac{1}{2} - \frac{9}{4}n - \frac{27}{4}n^2\right) + \frac{1}{2} \sum_{0 \leqslant j \leqslant n} \binom{j+2}{2} (-1)^j \binom{n-j-\frac{5}{2}}{n-j} 8^{n-j},$$

for $n \geqslant 1$. On the other hand, we also have, by Lagrange inversion formula,

$$[z^n]G(\phi(z)) = 2 \sum_{0 \leqslant j < n} \frac{(2n+1)!(2n-j-2)!(n-2j+1)}{n!j!(n-1-j)!(2n-j+2)!} \qquad (n \geqslant 1).$$

The main difference is that the former expands at $z = \frac{1}{8}$, while the latter at the origin.

On the other hand, without relying on the explicit forms, we also have (with $\Delta = \sqrt{1-z}$)

$$F(t) = -\frac{4+t}{3} \quad \Longrightarrow \quad t = -\frac{4\Delta}{3+\Delta} \Longrightarrow G(\phi(\tfrac{z}{8})) = -1 + \frac{32}{3+\Delta^2} - \frac{64}{(3+\Delta)^3}.$$

We then obtain the same singular expansion as above, which is convergent in the region with $|1-8z| < 9$. It follows that

$$[z^n]G(\phi(z)) \sim 8^n \sum_{k \geqslant 0} e_{2k+3} \binom{n-k-\frac{5}{2}}{n}, \quad \text{where} \quad e_{2k+3} = \frac{1}{2}\left(\frac{8}{9}\right)^3 \binom{k+2}{2} 9^{-k}.$$

Non-incremental maps with $[z^n]f(z) \leqslant 0$

The remaining five cases are listed below.

type of maps	$\mathscr{M}_5$ non-separable simple	$\mathscr{B}_2$ bipartite simple	$\mathscr{B}_3$ bipartite bridgeless	$\mathscr{B}_4$ bipartite non-separable	$\mathscr{B}_5$ bipartite non-separable simple
$f(z)$	$\frac{(1+z)^6}{(1+2z)^2}$	$\frac{8(1+z)^2}{4+2z-z^2}$	$\frac{(2+z)^6}{32(1+z)^2}$	$\frac{32(1+z)^2}{(4+2z-z^2)^2}$	$\frac{128(1+z)^2}{(4+2z-z^2)^3}$
$G(z)$	$\frac{z(1+z-z^2)}{(1+z)^3}$	$\frac{z(2-z)}{4}$	$\frac{z^2(8+4z-4z^2-z^3)}{32(1+z)^2}$	$\frac{z(2-z)}{4}$	$\frac{z(2-z)}{4}$
OEIS($G(\phi)$)	–	–	–	A069728	A298358
r	$\frac{1}{2}$	1	1	1	1
ρ	$\frac{128}{729}$	$\frac{5}{32}$	$\frac{128}{729}$	$\frac{25}{128}$	$\frac{25}{512}$
$F(t)$	$(\star)_1$	$-\frac{\sqrt{5}(2+t)}{\sqrt{9+4t}}$	$(\star)_2$	$-t$	$(\star)_3$
$G'(r+v)$	$\frac{-64v(1+v)}{9}$	$-\frac{1}{2}v$	$-\frac{v(1+v)(v+3)^2(7+3v)}{32(2+v)^3}$	$-\frac{1}{2}v$	$-\frac{1}{2}v$

Here $(\star)_1 := \frac{-(3+2t)^3}{\sqrt{1215+2862t+2160t^2+576t^3+64t^4}}$, $(\star)_2 := \frac{-2(3+t)^3}{\sqrt{1215+1431t+540t^2+72t^3+4t^4}}$, and $(\star)_3 := \frac{-5\sqrt{5}(2+t)}{\sqrt{425+300t-60t^2-60t^3+4t^4+4t^5}}$.

We now show how to extend the same analysis to the cases when $[z^n]f(z)$ contains negative coefficients. From the viewpoint of the saddle-point method, a sufficient condition replacing the condition C1 is as follows (see also [6]):

C1' (Concentration of $|f(z)|$)

$$f(r) > 0 \text{ for } 0 < r < R \text{ and } |f(re^{i\theta})| < f(r) \text{ for } 0 < |\theta| < \pi. \tag{18}$$

Briefly, this condition implies, by the saddle-point method, that the major contribution to the integral representation of $[t^{n-1}]G'(r+t)f(t)^n$ comes from a small neighborhood of $t = r$, and in turn that the asymptotic expansion (8) holds.

$\mathscr{B}_2$: simple bipartite maps

In this case, f and G are given by $f(z) = \frac{8(1+z)^2}{4+2z-z^2}$ and $G(z) := \frac{1}{4}z(2-z)$; thus $r = 1$, $\rho = \frac{5}{32}$, and

$$F(t) = -\frac{\sqrt{5}(2+t)}{\sqrt{9+4t}}.$$

We can check the condition C1' by elementary calculus and then derive the expansion (8); in particular, we have

$$\frac{1}{\sqrt{5}}\left(\frac{5}{32}\right)^n [z^n]G(\phi(z)) \sim \frac{50}{243}\binom{n-\frac{5}{2}}{n} + \frac{1100}{59049}\binom{n-\frac{7}{2}}{n} + \cdots.$$

Whether the left-hand side is easy to compute or not is irrelevant here. Also we can compute e_{2k+3} by the following closed-form expression

$$e_{2k+3} = \frac{2\cdot 5^{k+\frac{3}{2}}}{(2k+3)3^{2k+3}} \sum_{0\leqslant j\leqslant 2k+1} \binom{k+j+\frac{1}{2}}{j}\binom{2k+3}{j+2}\left(-\frac{8}{9}\right)^j \qquad (k \geqslant 0).$$

The same technique applies to $\mathscr{B}_4$ and $\mathscr{B}_5$, but not to $\mathscr{M}_5$ and $\mathscr{B}_3$ because the condition C1' fails when z is near $-\frac{1}{2}$ (for $\mathscr{M}_5$) and near -1 (for $\mathscr{B}_3$) because of polar singularities there. However, we can modify suitably the integration contour so as to avoid the polar singularities and prove that the contribution comes principally from $z \sim r$ in the corresponding Cauchy integral. So we still get the expansions (8) for both cases; we omit the details here.

6 Applications III: Other examples

Due to a space constraint, we mention in this section only two interesting examples for which our expansions apply.

- The number of rooted 3-connected bicubic maps of $2n$ vertices (see [15, 16] and A298358) is given by

$$m_n = \frac{3}{n-1}[z^{n-1}]g(z)f(z)^n,$$

where $f(z) = \frac{(1+2z)^2}{(1+z-z^2)^3}$ and $g(z) = \frac{z^3(1-2z)}{(1+2z)(1+z-z^2)}$. By modifying our expansions (the condition C1' holds), we have

$$m_n \sim \frac{3n}{n-1}\left(\frac{512}{125}\right)^n \sum_{k\geqslant 0} e_{2k+3}\binom{n-k-\frac{5}{2}}{n},$$

where

$$e_k := \frac{1}{k}[t^{k-1}]g\left(\tfrac{1}{2}+t\right)\left(\frac{-5\sqrt{5}(1+t)}{\sqrt{425-240t^2+64t^4+600t-480t^3+128t^5}}\right)^k.$$

- The number of labeled rooted trees of subsets of an n-set (see [5] or A005172) is given by $[z^n]G(\phi(z))$, where $\phi = zf(\phi)$ with $f(z) = \frac{z}{1-e^z+\log(2e^z-1)}$ and $G(z) = e^z - 1$. Interestingly, all coefficients $[z^n]f(z)$ are positive for $1 \leqslant n \leqslant 47$, but negative coefficients appear from $n = 48$ on. By checking the conditions C1', C2 and C3, we then obtain (8) with

$$F(t) = -\frac{\sqrt{2\log 2 - 1}\, t}{\sqrt{2\log 2 + 3(e^t-1) - 2\log(3e^t-1)}}.$$

Then we deduce that (with $r = \log\frac{3}{2}$)

$$G(r+t) = \tfrac{2}{3}\left(T\left(e^{-1-(\log 2-\frac{1}{2})\Delta^2}\right)-1\right),$$

where $T(z) = \sum_{n\geqslant 1}\frac{n^{n-1}}{n!}z^n$ denotes the generating function for Cayley trees.
- All examples treated in [16] can be dealt with by our expansions. Consider, as in [16], the asymptotics of the Stirling numbers of the second kind:

$$\left\{{2n \atop n}\right\} = \frac{(2n)!}{n!}[z^{2n}](e^z-1)^n = \frac{(2n)!}{(n-1)!}\cdot\frac{1}{n}[z^{n-1}]z^{-1}f(z)^n,$$

where $f(z) := \frac{e^z-1}{z}$. Although $G = \log z$ is not analytic at $z = 0$, we can still apply the same expansion (8) and obtain

$$\frac{(n-1)!}{(2n)!}\left\{{2n \atop n}\right\} \sim \rho^{-n}\sum_{j\geqslant 0} e_{2j+1}\binom{n-j-\frac{3}{2}}{n},$$

where $r = 2 - T(2e^{-2})$, $\rho = \frac{r}{f(r)}$, and $e_k = k^{-1}[t^{k-1}](r+t)^{-1}F(t)^k$ for $k \geqslant 1$. The Stirling numbers of the first kind with $f = -z^{-1}\log(1-z)$ is similar.
Other examples in [16] include the relations

$$\frac{1}{n\cdot(n-1)!^2}\sum_{0\leqslant k<n}\left\{{n-1 \atop k}\right\}(n-1+k)! = \frac{1}{n}[z^{n-1}](2-e^z)^{-n}$$
$$\frac{1}{n}\sum_{0\leqslant k<n}\binom{n}{k}\binom{2n-2-k}{n-1}(-1)^k 2^{n-k} = \frac{1}{n}[z^{n-1}]\frac{(1+z)^n(1-2z)^n}{(1+z)^2},$$

and the following table for the form $[z^n]g(z)f(z)^n$:

g	f	g	f
1	$(1-\frac{3}{8}z+\frac{1}{24}z^2)^{-1}$	1	$(1-\sqrt{3}z+z^2)^{-1}$
1	$(1-z-z^2)^{-1}$	1	$\frac{1}{2}(1+e^z)$
$z(1-z)^{-1}$	$e^{\alpha z}$	$z(1-z)^{-\alpha-1}$	e^z
$(1-\alpha z)^{-1}$	e^z	$(1-z)^{-1}$	$(1+z)^\alpha$
$-\log(1-3z)$	$(1+z)^4$	$-z(1-z)^{-1}\log(1-z)$	e^z

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