Vanishing of Cohomology Groups of Random Simplicial Complexes

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Abstract

We consider k-dimensional random simplicial complexes that are generated from the binomial random $(k+1)$-uniform hypergraph by taking the downward-closure, where $k \geq 2$. For each $1 \leq j \leq k-1$, we determine when all cohomology groups with coefficients in $\mathbb{F}_2$ from dimension one up to j vanish and the zero-th cohomology group is isomorphic to $\mathbb{F}_2$. This property is not monotone, but nevertheless we show that it has a single sharp threshold. Moreover, we prove a hitting time result, relating the vanishing of these cohomology groups to the disappearance of the last minimal obstruction. Furthermore, we study the asymptotic distribution of the dimension of the j-th cohomology group inside the critical window. As a corollary, we deduce a hitting time result for a different model of random simplicial complexes introduced in [Linial and Meshulam, Combinatorica, 2006], a result which has only been known for dimension two [Kahle and Pittel, Random Structures Algorithms, 2016].

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1 Introduction

1.1 Motivation

In their seminal paper [12], Erdős and Rényi introduced the uniform random graph and addressed the problem of determining the probability of this graph being connected. Nowadays, this classical result is usually stated for the binomial model, in which each edge is present with a given probability p independently: the connectedness of the binomial random graph $G(n,p)$ on n vertices undergoes a *phase transition* around the sharp threshold $p = \frac{\log n}{n}$ [24], where log denotes the natural logarithm.

▶ **Theorem 1.1.** *Let ω be any function of n which tends to infinity as $n \to \infty$. Then* with high probability,[1] *the following holds.*

(i) *If $p = \frac{\log n - \omega}{n}$, then $G(n,p)$ is not connected.*
(ii) *If $p = \frac{\log n + \omega}{n}$, then $G(n,p)$ is connected.*

As an even stronger result, Erdős and Rényi [12] determined the limiting probability for connectedness around the point of the phase transition. Subsequently, Bollobás and Thomason [7] proved a *hitting time* result, stating that whp the random graph *process* becomes connected at the very same time at which the last isolated vertex—the smallest obstruction for connectedness—disappears.

Since then, various higher-dimensional analogues of both random graphs and connectedness have been analysed and in particular two different approaches have received considerable attention. A first natural generalisation is the random k-uniform hypergraph $G_p = G(k;n,p)$ in which each $(k+1)$-tuple of vertices forms a hyperedge with probability p independently. There are several natural ways of defining connectedness of G_p, which have been extensively studied [4, 5, 6, 8, 9, 10, 11, 15, 16, 22, 23].

A more recent approach concerns random simplicial complexes, of which a first model for the 2-dimensional case was introduced by Linial and Meshulam [17]. They considered $\mathbb{F}_2$-*homological* 1-*connectivity* of the random 2-complex as the vanishing of its first homology group with coefficients in the two-element field $\mathbb{F}_2$, which is equivalent to the vanishing of the first cohomology group. More precisely, the model $\mathcal{Y}_p = \mathcal{Y}(k;n,p)$ considered by Linial and Meshulam [17] for $k=2$ and subsequently by Meshulam and Wallach [20] for general $k \geq 2$ is defined as follows. Starting from the full $(k-1)$-*dimensional skeleton* on $[n] := \{1,\dots,n\}$, that is, all simplices from dimension zero up to $k-1$, each $(k+1)$-set forms a k-simplex with probability p independently. They showed that the vanishing of the $(k-1)$-th cohomology group $H^{k-1}(\mathcal{Y}_p;\mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$ has a sharp threshold at $p = \frac{k \log n}{n}$.

▶ **Theorem 1.2** ([17, 20])**.** *Let ω be any function of n which tends to infinity as $n \to \infty$. Then with high probability, the following holds.*

(i) *If $p = \frac{k \log n - \omega}{n}$, then $H^{k-1}(\mathcal{Y}_p;\mathbb{F}_2) \neq 0$.*
(ii) *If $p = \frac{k \log n + \omega}{n}$, then $H^{k-1}(\mathcal{Y}_p;\mathbb{F}_2) = 0$.*

Later, Kahle and Pittel [15] derived a hitting time result for the case $k=2$ and determined the limiting probability of $H^{k-1}(\mathcal{Y}_p;\mathbb{F}_2) = 0$ for general $k \geq 2$ and p in the critical window.

In this paper, we aim to bridge the gap between random hypergraphs and random simplicial complexes. We consider random simplicial k-complexes that arise as the downward-closure of random $(k+1)$-uniform hypergraphs. Unlike $\mathcal{Y}_p$, in this model the presence of the

[1] With probability tending to 1 as n tends to infinity, whp for short.

full $(k-1)$-dimensional skeleton is not guaranteed, thus the vanishing of the cohomology groups of dimensions lower than $k-1$ does not hold trivially. Therefore, for each $j \in [k-1]$, we introduce $\mathbb{F}_2$-*cohomological* j-*connectedness* as the vanishing of *all* cohomology groups with coefficients in $\mathbb{F}_2$ from dimension one up to j and the zero-th cohomology group being isomorphic to $\mathbb{F}_2$.

Although this notion of connectedness is not monotone, we prove that nevertheless $\mathbb{F}_2$-cohomological j-connectedness has a sharp threshold. Furthermore, we derive a hitting time result and determine the limiting probability for $\mathbb{F}_2$-cohomological j-connectedness in the critical window. As a corollary, we deduce a hitting time result for $\mathcal{Y}_p$ in *general dimension*, thus extending the hitting time result of Kahle and Pittel [15].

1.2 Model

Throughout the paper let $k \geq 2$ be a fixed integer. For a positive integer ℓ, let $[\ell] := \{1, \ldots, \ell\}$.

▶ **Definition 1.3.** A family $\mathcal{G}$ of non-empty finite subsets of a vertex set V is called a *simplicial complex* if it is downward-closed, i.e. if every non-empty set A that is contained in a set $B \in \mathcal{G}$ also lies in $\mathcal{G}$, and if the singleton $\{v\}$ is in $\mathcal{G}$ for every $v \in V$.

The elements of a simplicial complex $\mathcal{G}$ of cardinality $k+1$ are called k-*simplices* of $\mathcal{G}$. If $\mathcal{G}$ has no $(k+1)$-simplices, then we call it k-*dimensional*, or k-*complex*. If $\mathcal{G}$ is a k-complex, then for each $j = 0, \ldots, k-1$ the j-*skeleton* of $\mathcal{G}$ is the j-complex formed by all i-simplices in $\mathcal{G}$ with $0 \leq i \leq j$.

We aim to define a model of random k-complexes starting from the binomial random $(k+1)$-uniform hypergraph $G_p = G(k; n, p)$ on vertex set $[n]$: the 0-simplices are the vertices of G_p, the k-simplices are the hyperedges of G_p, but there is more than one way to guarantee the downward-closure property, to obtain a simplicial complex. In the model $\mathcal{Y}_p$ considered by Meshulam and Wallach in [20], the full $(k-1)$-skeleton on $[n]$ is always included. In contrast, we shall only include those simplices that are *necessary* to ensure the downward-closure property.

▶ **Definition 1.4.** We denote by $\mathcal{G}_p = \mathcal{G}(k; n, p)$ the random k-dimensional simplicial complex on vertex set $[n]$ such that

- the 0-simplices are the singletons of $[n]$;
- the k-simplices are the hyperedges of G_p;
- for each $j \in [k-1]$, the j-simplices are exactly the $(j+1)$-subsets of hyperedges of G_p.

In other words, $\mathcal{G}_p$ is the random k-complex on $[n]$ obtained from G_p by taking the downward-closure of each hyperedge.

Given a simplicial complex $\mathcal{G}$, let $H^i(\mathcal{G}; \mathbb{F}_2)$ be its i-th cohomology group with coefficients in $\mathbb{F}_2$ (see Section 2.1 for the definition). Connectedness of $\mathcal{G}_p$ in the topological sense—which we call *topological connectedness* in order to distinguish it from other notions of connectedness—is equivalent to $H^0(\mathcal{G}_p; \mathbb{F}_2)$ being (isomorphic to) $\mathbb{F}_2$. We therefore define a notion of connectedness as follows.

▶ **Definition 1.5.** For a positive integer j, a simplicial complex $\mathcal{G}$ is called $\mathbb{F}_2$-*cohomologically* j-*connected* (j-*cohom-connected* for short) if

- $H^0(\mathcal{G}; \mathbb{F}_2) = \mathbb{F}_2$;
- $H^i(\mathcal{G}; \mathbb{F}_2) = 0$ for all $i \in [j]$.

One might define an analogous version of connectedness via the vanishing of *homology* groups, which would be equivalent to $\mathbb{F}_2$-cohomological j-connectedness by the Universal Coefficient Theorem (see e.g. [21]).

A significant difference between $\mathcal{G}_p$ and $\mathcal{Y}_p$ is that for $\mathcal{Y}_p$ the only requirement for $\mathbb{F}_2$-cohomologically $(k-1)$-connectedness is the vanishing of the $(k-1)$-th cohomology group, since the presence of the full $(k-1)$-skeleton guarantees topological connectedness and the vanishing of the j-th cohomology groups for all $j \in [k-2]$.

Moreover, it is important to observe that $\mathbb{F}_2$-cohomological j-connectedness is *not* a monotone increasing property of $\mathcal{G}_p$: adding a k-simplex to a j-cohom-connected complex might yield a complex without this property (see Example 2.3). Thus, the existence of a single threshold for j-cohom-connectedness is *not* guaranteed, but one of our main results shows that such a threshold indeed exists.

1.3 Main results

The main contributions of this paper are fourfold. Firstly, we prove (Theorem 1.8) that for each $j \in [k-1]$, $\mathbb{F}_2$-cohomological j-connectedness of $\mathcal{G}_p$ undergoes a phase transition at around probability

$$p_j := \frac{(j+1)\log n + \log\log n}{(k-j+1)n^{k-j}}(k-j)!. \tag{1}$$

Secondly, we prove a hitting time result (also Theorem 1.8), which relates the j-cohom-connectedness threshold to the disappearance of all copies of the *minimal obstruction* M_j (Definition 1.7). Thirdly, our results directly imply an analogous hitting time result for $\mathcal{Y}_p$ (Corollary 1.9), which Kahle and Pittel [15] proved for $k=2$. Lastly, we analyse the critical window around the threshold p_j, showing that inside the window the dimension of the j-th cohomology group converges in distribution to a Poisson random variable (Theorem 1.10).

Before defining the minimal obstruction M_j, we need the following concept.

▶ **Definition 1.6.** Given a k-simplex K in a k-complex $\mathcal{G}$, a collection $\mathcal{F} = \{P_0, \dots, P_{k-j}\}$ of j-simplices forms a *j-flower in K* if $K = \bigcup_i P_i$ and $C := \bigcap_i P_i$ satisfies $|C| = j$. We call the j-simplices P_i the *petals* and the set C the *centre* of the j-flower $\mathcal{F}$.

Observe that for each k-simplex K and each $(j-1)$-simplex $C \subseteq K$, there is a unique j-flower in K with centre C, namely

$$\mathcal{F}(K, C) := \{C \cup \{w\} \mid w \in K \setminus C\}.$$

When j is clear from the context, we simply refer to a j-flower as a *flower*. A *j-cycle* is a set J of j-simplices such that every $(j-1)$-simplex is contained in an *even* number of j-simplices in J.

▶ **Definition 1.7.** A *copy of M_j* (see Figure 1) in a k-complex $\mathcal{G}$ is a triple (K, C, J) where

(M1) K is a k-simplex;

(M2) C is a $(j-1)$-simplex in K and each petal of the flower $\mathcal{F} = \mathcal{F}(K, C)$ is contained in *no other* k-simplex of $\mathcal{G}$;

(M3) J is a j-cycle that contains exactly one petal of $\mathcal{F}$, i.e. there exists a vertex $w_0 \in K \setminus C$ such that

$$J \cap \mathcal{F} = \Big\{C \cup \{w_0\}\Big\}.$$

We will see (Lemma 2.2) that a copy of M_j can be interpreted as a minimal obstruction for $\mathbb{F}_2$-cohomological j-connectedness.

The random k-complex $\mathcal{G}_p$ can be viewed as a *process*, by assigning a *birth time* to each k-simplex. More precisely, for each $(k+1)$-set of vertices in $[n]$ independently, sample a birth

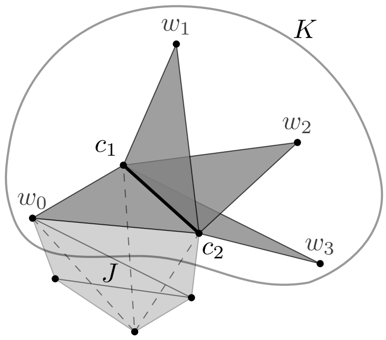

Figure 1 A copy of M_j for $k = 5$, $j = 2$. The centre $C = \{c_1, c_2\}$ lies in all petals $P_i = C \cup \{w_i\}$, $i = 0, \dots, 3$ (dark grey), which are contained in no other k-simplex except K. The j-cycle J (light grey) intersects the flower $\mathcal{F}(K, C) = \{P_0, P_1, P_2, P_3\}$ only in the petal $P_0 = C \cup \{w_0\}$.

time uniformly at random from $[0, 1]$.[2] Then $\mathcal{G}_p$ is exactly the complex generated by the $(k+1)$-sets with birth times at most p, by taking the downward-closure. If p is gradually increased from 0 to 1, we may interpret $\mathcal{G}_p$ as a process. Thus, we can define p_{M_j} as the birth time of the k-simplex whose appearance causes the last copy of M_j to disappear. More formally, let

$$p_{M_j} := \sup\{p \in [0, 1] \mid \mathcal{G}_p \text{ contains a copy of } M_j\}. \tag{2}$$

Our first main result is that the value p_{M_j} is the hitting time for j-cohom-connectedness of $\mathcal{G}_p$ and is "close" to p_j defined in (1), implying that p_j is in fact a sharp threshold for $\mathbb{F}_2$-cohomological j-connectedness.

▶ **Theorem 1.8.** *Let $k \geq 2$ be an integer and let ω be any function of n which tends to infinity as $n \to \infty$. For each $j \in [k-1]$, with high probability the following statements hold.*

(i) $\frac{(j+1)\log n + \log\log n - \omega}{(k-j+1)n^{k-j}}(k-j)! < p_{M_j} < \frac{(j+1)\log n + \log\log n + \omega}{(k-j+1)n^{k-j}}(k-j)!$.

(ii) *For all $p < p_{M_j}$, $\mathcal{G}_p$ is not $\mathbb{F}_2$-cohomologically j-connected, i.e.*

$$H^0(\mathcal{G}_p; \mathbb{F}_2) \neq \mathbb{F}_2 \text{ or } H^i(\mathcal{G}_p; \mathbb{F}_2) \neq 0 \text{ for some } i \in [j].$$

(iii) *For all $p \geq p_{M_j}$, $\mathcal{G}_p$ is $\mathbb{F}_2$-cohomologically j-connected, i.e.*

$$H^0(\mathcal{G}_p; \mathbb{F}_2) = \mathbb{F}_2 \text{ and } H^i(\mathcal{G}_p; \mathbb{F}_2) = 0 \text{ for all } i \in [j].$$

For the case $j = k-1$, Theorem 1.8 gives a threshold $p_{k-1} = \frac{k\log n + \log\log n}{2n}$ for $\mathbb{F}_2$-cohomologically $(k-1)$-connectedness, which is about half as large as the threshold $\frac{k\log n}{n}$ in Theorem 1.2 for $\mathcal{Y}_p$. The reason for this is that the minimal obstructions are different: in $\mathcal{Y}_p$ the minimal obstruction is a $(k-1)$-simplex which is not contained in any k-simplex of the complex (such a $(k-1)$-simplex is called *isolated*). By definition, isolated $(k-1)$-simplices do not exist in $\mathcal{G}_p$, because $\mathcal{G}_p$ contains only those $(k-1)$-simplices that lie in some k-simplex.

Observe that Theorem 1.8 ii and iii provide a hitting time result for the process described above. A similar result was proved by Kahle and Pittel [15] for $\mathcal{Y}_p$, but only for the 2-dimensional case. As a corollary of Theorem 1.8, we can now derive a hitting time result for $\mathcal{Y}_p$ for general $k \geq 2$. To this end, let

$$p_{\text{isol}} := \sup\{p \in [0, 1] \mid \mathcal{Y}_p \text{ contains isolated } (k-1)\text{-simplices}\} \tag{3}$$

[2] With probability 1 no two $(k+1)$-sets have the same birth time.

be the birth time of the k-simplex whose appearance causes the last isolated $(k-1)$-simplex to disappear and let

$$p_{\text{conn}} := \sup\{p \in [0,1] \mid H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \neq 0 \ \} \tag{4}$$

be the time when $\mathcal{Y}_p$ becomes $\mathbb{F}_2$-cohomological $(k-1)$-connected.

▶ **Corollary 1.9.** *Let $k \geq 2$ be an integer. Then, with high probability $p_{conn} = p_{isol}$.*

Our last main result gives an explicit expression for the limiting probability of the random complex $\mathcal{G}_p$ being $\mathbb{F}_2$-cohomologically j-connected inside the critical window given by the threshold p_j. More generally, we prove that the *dimension* of the j-th cohomology group with coefficients in $\mathbb{F}_2$ converges in distribution to a Poisson random variable.

▶ **Theorem 1.10.** *Let $k \geq 2$ be an integer, $j \in [k-1]$ and $c \in \mathbb{R}$ be a constant. Suppose that $c_n \in \mathbb{R}$ are such that $c_n \xrightarrow{n\to\infty} c$. If*

$$p = \frac{(j+1)\log n + \log\log n + c_n}{(k-j+1)n^{k-j}}(k-j)!,$$

then $\dim\left(H^j(\mathcal{G}_p;\mathbb{F}_2)\right)$ *converges in distribution to a Poisson random variable with expectation*

$$\lambda_j := \frac{(j+1)e^{-c}}{(k-j+1)^2 j!},$$

while whp $H^0(\mathcal{G}_p;\mathbb{F}_2) = \mathbb{F}_2$ and $H^i(\mathcal{G}_p;\mathbb{F}_2) = 0$ for all $i \in [j-1]$. In particular, we have

$$\mathbb{P}\left(\mathcal{G}_p \text{ is } j\text{-cohom-connected}\right) \xrightarrow{n\to\infty} e^{-\lambda_j}.$$

Note that a similar result for $\mathcal{Y}_p$ was proved by Kahle and Pittel [15].

1.4 Related work

The vanishing of $H^{k-1}(\mathcal{Y}_p;\mathbb{F}_2)$ considered in [17] and [20] is a monotone property due to the presence of the full $(k-1)$-dimensional skeleton. This fact in particular makes it possible to use a simple second moment argument to prove the subcritical case (i.e. statement (i)) of Theorem 1.2.

In contrast, $\mathcal{G}_p$ does *not* contain the full $(k-1)$-dimensional skeleton. As a consequence, we need to consider *all* cohomology groups up to dimension j, for each $j \in [k-1]$. Moreover, our notion of $\mathbb{F}_2$-cohomological j-connectedness is *not* a monotone property, which makes the subcritical case far from trivial. In fact, it does not suffice to prove that $\mathcal{G}_p$ is not j-cohom-connected at $p_- = \frac{(j+1)\log n + \log\log n - \omega}{(k-j+1)n^{k-j}}(k-j)!$; rather we need to show that whp $\mathcal{G}_p$ is not j-cohom-connected for *any* p up to and including p_-.

The proof of the supercritical case $p \geq p_{M_j}$ is also more challenging than for $\mathcal{Y}_p$. We are forced to derive better bounds for the number of *bad functions* (see Definition 2.1), due to the fact that for $j = k-1$, the threshold in Theorem 1.8 is about half as large as the corresponding threshold in [20].

2 Preliminaries

2.1 Cohomology terminology

We formally introduce cohomology with coefficients in $\mathbb{F}_2$ for a simplicial complex. The following notions are all standard, except the definition of a bad function (Definition 2.1).

Given a simplicial k-complex $\mathcal{G}$, for each $j \in \{0,\dots,k\}$ denote by $C^j(\mathcal{G})$ the set of *j-cochains*, that is, the set of 0-1 functions on the j-simplices. The *support* of a function in $C^j(\mathcal{G})$ is the set of j-simplices mapped to 1. Each $C^j(\mathcal{G})$ forms a group with respect to pointwise addition modulo 2. We define the *coboundary operators* $\delta^j\colon C^j(\mathcal{G}) \to C^{j+1}(\mathcal{G})$ for $j = 0,\dots,k-1$ as follows. For $f \in C^j(\mathcal{G})$, the 0-1 function $\delta^j f$ assigns to each $(j+1)$-simplex σ the value

$$\delta^j f(\sigma) := \sum_{\tau\subset\sigma,\ |\tau|=j+1} f(\tau) \pmod 2.$$

In addition, we denote by δ^{-1} the unique group homomorphism $\delta^{-1}\colon \{0\} \to C^0(\mathcal{G})$. The j-cochains in $\operatorname{im}\delta^{j-1}$ and $\ker\delta^j$ are called *j-coboundaries* and *j-cocycles*, respectively. A straightforward calculation shows that each coboundary operator is a group homomorphism and that every j-coboundary is also a j-cocycle, i.e. $\operatorname{im}\delta^{j-1} \subseteq \ker\delta^j$. Therefore, we can define the j-th cohomology group of $\mathcal{G}$ with coefficients in $\mathbb{F}_2$ as the quotient group

$$H^j(\mathcal{G};\mathbb{F}_2) := \ker\delta^j / \operatorname{im}\delta^{j-1}.$$

By definition, $H^j(\mathcal{G};\mathbb{F}_2)$ vanishes if and only if every j-cocycle is a j-coboundary. This motivates the following definition of a *bad function*.

▶ **Definition 2.1.** We say that a function $f \in C^j(\mathcal{G})$ is *bad* if

(i) f is a j-cocycle, i.e. it assigns an even number of 1's to the j-simplices on the boundary of each $(j+1)$-simplex;

(ii) f is not a j-coboundary, i.e. it is not induced by a 0-1 function on the $(j-1)$-simplices.

Thus, $H^j(\mathcal{G};\mathbb{F}_2)$ vanishes if and only if no bad function in $C^j(\mathcal{G})$ exists.

Recall that a set J of j-simplices is a j-cycle if every $(j-1)$-simplex lies in an even number of j-simplices in J. It is easy to see that if f is a j-cocycle and J is a j-cycle such that $f|_J$ has support of odd size, then f is not a j-coboundary and thus is a bad function.

2.2 Minimal obstructions

Let us explain why M_j (Definition 1.7) can be interpreted as the minimal obstruction to j-cohom-connectedness. Given a copy (K,C,J) of M_j in a k-complex $\mathcal{G}$, define a function $f \in C^j(\mathcal{G})$ that takes value 1 on the petals of the flower $\mathcal{F}(K,C)$ and 0 everywhere else. Since each petal lies in K but in no further k-simplices, every $(j+1)$-simplex contains either two petals or none. In particular, f is even on the boundary of every $(j+1)$-simplex. However, J would be a j-cycle containing precisely one j-simplex (namely $C \cup \{w_0\}$) on which f takes value 1, ensuring that f is bad. The support of f has size $k-j+1$.

▶ **Lemma 2.2.** *Let $\mathcal{G}$ be a k-complex and let S be a non-empty support of a j-cocycle. Then either S is the flower of an M_j (and thus $|S| = k-j+1$) or $|S| \geq k-j+2$.*

Both the presence of a copy of M_j and j-cohom-connectedness are *not* monotone, as the following example shows.

▶ **Example 2.3.** Let $\mathcal{G}$ be the 2-complex on vertex set $\{1,2,3,4,5\}$ generated by the 3-uniform hypergraph with hyperedges $\{1,2,3\}$ and $\{1,4,5\}$, see Figure 2. Then $\mathcal{G}$ is 1-cohom-connected and thus contains no copies of M_1. Adding to $\mathcal{G}$ the 2-simplex $\{2,3,4\}$ (and its downward-closure) creates several copies of M_1 and thus yields a complex $\mathcal{G}'$ which is not 1-cohom-connected. If we further add the 2-simplex $\{1,3,4\}$ to $\mathcal{G}'$, we obtain a 2-complex $\mathcal{G}''$ which is 1-cohom-connected and thus contains no copies of M_1.

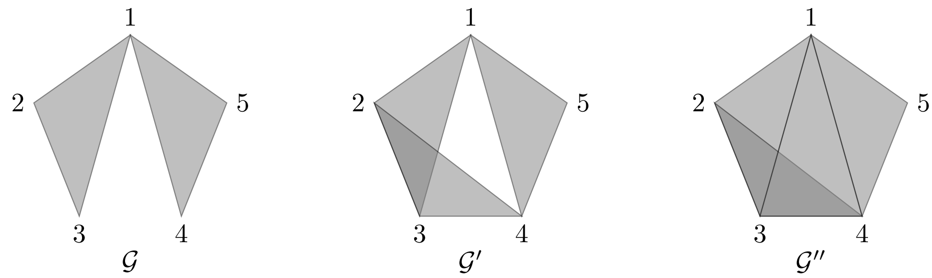

Figure 2 Adding simplices might create new copies of M_j or destroy existing ones.

3 Subcritical regime

3.1 Overview

In this section we study the subcritical case $p < p_{M_j}$ and state results necessary for the proofs of statements i and ii of Theorem 1.8.

Define

$$p_T := \sup\{p \in [0,1] \mid \mathcal{G}_p \text{ is not topologically connected}\}$$

as the birth time of the k-simplex whose appearance causes the complex $\mathcal{G}_p$ to become topologically connected. In addition, we will need the probabilities

$$p_0^- := \frac{\log n}{n^k},$$
$$p_j^- := \left(1 - \frac{1}{\sqrt{\log n}}\right) \frac{(j+1)\log n}{(k-j+1)n^{k-j}}(k-j)! \quad \text{for each } j \in [k-1].$$

Observe that $H^0(\mathcal{G}_p;\mathbb{F}_2) \neq \mathbb{F}_2$ in $[0,p_T)$ by definition. In order to prove Theorem 1.8 ii, we aim to show that whp $H^j(\mathcal{G}_p;\mathbb{F}_2) \neq 0$ in $[p_{j-1}^-, p_{M_j})$ for all $j \in [k-1]$ and that

$$[0,p_T) \cup \bigcup_{i=1}^{j} [p_{i-1}^-, p_{M_i}) = [0, p_{M_j}),$$

which we prove by showing that $p_T > p_0^-$ and $p_{M_j} > p_j^- > p_{j-1}^-$ for all $j \in [k-1]$ whp. To cover the interval $[p_{j-1}^-, p_{M_j})$, we in fact prove the existence of *just three copies* of M_j such that whp for all p in this interval, at least one of these copies is present in $\mathcal{G}_p$.

▶ **Lemma 3.1.** *Let $j \in [k-1]$. With high probability, there exist three triples (K_ℓ, C_ℓ, J_ℓ), $\ell = 1,2,3$, such that for all $p \in [p_{j-1}^-, p_{M_j})$, (K_ℓ, C_ℓ, J_ℓ) forms a copy of M_j in $\mathcal{G}_p$ for some ℓ. In particular, whp $H^j(\mathcal{G}_p;\mathbb{F}_2) \neq 0$ for all $p \in [p_{j-1}^-, p_{M_j})$.*

3.2 Topological connectedness

Topological connectedness of $\mathcal{G}_p$ is equivalent to vertex-connectedness of the random $(k+1)$-uniform hypergraph, whose (sharp) threshold follows e.g. as a special case of [8] or [22].

▶ **Lemma 3.2.** *Let ω be any function of n which tends to infinity as $n \to \infty$. Then with high probability*

$$\frac{\log n - \omega}{n^k} k! < p_T < \frac{\log n + \omega}{n^k} k!$$

and thus in particular $p_T > p_0^-$.

3.3 Finding obstructions

In order to prove Lemma 3.1, we make use of a simplified version of the obstruction M_j.

▶ **Definition 3.3.** *copy of* M_j^- in a k-complex $\mathcal{G}$ is a pair (K, C) such that

(M1) K is a k-simplex;

(M2) C is a $(j-1)$-simplex in K such that each petal of the flower $\mathcal{F}(K, C)$ is contained in no other k-simplex of $\mathcal{G}$.

In other words, a copy of M_j^- can be viewed as a copy of M_j without the condition (M3) of Definition 1.7, i.e. without the j-cycle J containing one of the petals. Therefore, if (K, C, J) is a copy of M_j in $\mathcal{G}_p$, then (K, C) is a copy of M_j^-. Vice versa, the following lemma ensures that whp for p at least

$$p_j^{(1)} := \frac{1}{10(j+1)\binom{k+1}{j+1}n^{k-j}},$$

whp every copy of M_j^- gives rise to a copy of M_j, allowing us to consider just copies of M_j^- as obstructions to j-cohom-connectedness. In other words, the existence of copies of M_j^- and M_j are essentially equivalent for $p \geq p_j^{(1)}$.

▶ **Lemma 3.4.** *There exists a positive constant γ such that with high probability for every $p \geq p_j^{(1)}$, each j-simplex σ in $\mathcal{G}_p$ lies in at least γn many j-cycles in $\mathcal{G}_p$ that meet only in σ. In particular, whp for all $p \geq p_j^{(1)}$, every copy of M_j^- in $\mathcal{G}_p$ is part of a copy of M_j.*

3.4 Excluding obstructions and determining the hitting time

A second moment argument shows that at time

$$\bar{p}_j := \frac{(j+1)\log n + \frac{1}{2}\log\log n}{(k-j+1)n^{k-j}}(k-j)!, \tag{5}$$

whp $\mathcal{G}_{\bar{p}_j}$ contains (a growing number of) copies of M_j^-, and thus whp also copies of M_j by Lemma 3.4. Define $\bar{p}_{M_j}$ as the first birth time p larger than $\bar{p}_j$ such that there are no copies of M_j in $\mathcal{G}_p$. By definition of p_{M_j}, conditioned on the high probability event $M_j \subset \mathcal{G}_{\bar{p}_j}$, we have $\bar{p}_{M_j} \leq p_{M_j}$. In the next lemma we show that they are in fact equal whp.

To do so, we need the following definition.

▶ **Definition 3.5.** Given a k-complex $\mathcal{G}$, a k-simplex K is a *local obstacle* if K contains at least $k-j+1$ many j-simplices which are not contained in any other k-simplex of $\mathcal{G}$.

Observe that each M_j^- is in particular a local obstacle. Moreover, whp each copy of M_j^- in $\mathcal{G}_p$ for $p \geq \bar{p}_j$ gives rise to copies of M_j by Lemma 3.4.

▶ **Lemma 3.6.** *With high probability, for all $p \geq \bar{p}_j$ every local obstacle that exists in $\mathcal{G}_p$ also exists in $\mathcal{G}_{\bar{p}_j}$. In particular, we have $p_{M_j} = \bar{p}_{M_j}$ whp.*

▶ **Corollary 3.7.** *Whp for all $p \geq p_{M_j}$, there are no copies of M_j^- in $\mathcal{G}_p$.*

By first and second moment arguments, we can now easily derive that p_{M_j} is "close to" p_j. Observe that the following corollary is exactly Theorem 1.8 i.

▶ **Corollary 3.8.** *Let ω be any function of n which tends to infinity as n tends to infinity. Then whp*

$$\frac{(j+1)\log n + \log\log n - \omega}{(k-j+1)n^{k-j}}(k-j)! < p_{M_j} < \frac{(j+1)\log n + \log\log n + \omega}{(k-j+1)n^{k-j}}(k-j)!.$$

3.5 Covering the interval

Our strategy to derive Lemma 3.1 is to divide the interval $[p_{j-1}^-, p_{M_j})$ into three subintervals $[p_{j-1}^-, p_j^{(1)}]$, $[p_j^{(1)}, p_j^-]$, $[p_j^-, p_{M_j})$, each of which we cover by one copy of M_j. We first use a second moment argument to show that at time p_{j-1}^-, whp there are "many" copies of M_j. With high probability, at least one copy (K_1, C_1, J_1) survives until probability $p_j^{(1)}$.

In order to find a copy of M_j that covers the interval $[p_j^{(1)}, p_j^-]$, we show that whp "many" copies of M_j^- exist at time p_j^-, of which one whp was already present at the beginning of the interval. Together with the fact that whp each M_j^- gives rise to a copy of M_j (Lemma 3.4), this implies that whp one copy (K_2, C_2, J_2) of M_j exists throughout this interval.

For the remaining interval $[p_j^-, p_{M_j})$, consider a copy (K_3, C_3) of M_j^- that vanishes at time p_{M_j}. Corollary 3.8 implies that whp $p_j^- = (1 - o(1))p_{M_j}$, and thus (K_3, C_3) whp was already present at time p_j^-. Now Lemma 3.4 ensures the existence of a j-cycle J_3 such that (K_3, C_3, J_3) is a copy of M_j throughout the range $[p_j^-, p_{M_j})$.

4 Critical window and supercritical regime

In this section, we study obstructions around the point of the claimed phase transition and in the supercritical regime, that is, for $p = (1 + o(1))p_j$ and $p \geq p_{M_j}$, respectively. The results of this section will form the foundation of the proof of Theorem 1.8 iii. Furthermore, they will play a crucial role in the proof of Theorem 1.10.

By the definition of p_{M_j}, there are no copies of M_j in $\mathcal{G}_p$ (and also no copies of M_j^- by Corollary 3.7) for any $p \geq p_{M_j}$. It remains to show that there are no other obstructions either. In fact, we shall even prove (Lemma 4.2) that from slightly before p_{M_j} onwards, any j-cocycles are generated by copies of M_j^-. To make this more precise, we need the following notation.

▶ **Definition 4.1.** We say that a j-cochain $f_{K,C}$ *arises from* a copy (K, C) of M_j^- in a k-complex $\mathcal{G}$ if its support is the j-flower $\mathcal{F}(K, C)$. Observe that then $f_{K,C}$ is a j-cocycle.

We say that a j-cocycle f in $\mathcal{G}$ is *generated by copies of* M_j^- if it lies in the same cohomology class as a sum of cocycles that arise from copies of M_j^-. We denote by $\mathcal{N}_{\mathcal{G}}$ the set of j-cocycles that are *not* generated by copies of M_j^-.

We show that whp for all $p \geq p_{M_j}$, $\mathcal{N}_{\mathcal{G}_p} = \emptyset$, which will in particular imply that there are no non-empty j-cocycles in $\mathcal{G}_p$. Furthermore, a similar argument will enable us to directly relate the number of copies of M_j^- with the dimension of $H^j(\mathcal{G}_p; \mathbb{F}_2)$ (cf. Theorem 1.10).

▶ **Lemma 4.2.** *For every $p \geq p_j^-$, we have $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ with high probability. Moreover, with high probability $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ for all $p \geq p_{M_j}$ simultaneously.*

In order to prove Lemma 4.2, we first show that a smallest support of elements of $\mathcal{N}_{\mathcal{G}}$ would have to have a property we call *traversability.*

▶ **Definition 4.3.** Let $\mathcal{G}$ be a k-complex and $S \subseteq \mathcal{G}$ be a collection of j-simplices. For $\sigma_1, \sigma_2 \in S$, we write $\sigma_1 \sim \sigma_2$ if σ_1 and σ_2 lie in a common k-simplex.[3] We say that S is *traversable* if the transitive closure of $\sim$ is $S \times S$.

In other words, a set of j-simplices in a k-complex is traversable if it *cannot* be partitioned into two non-empty subsets such that no k-simplex contains j-simplices in both subsets.

[3] Observe that this relation is reflexive, because every j-simplex is contained in at least one k-simplex.

▶ **Lemma 4.4.** *Let $\mathcal{G}$ be a k-complex and f be an element of $\mathcal{N}_\mathcal{G}$ with smallest support S. Then S is traversable.*

We then show that whp no such smallest support can exist in $\mathcal{G}_p$. For "small" support size and probability around p_j, a standard application of the first moment method suffices.

▶ **Lemma 4.5.** *For $p = (1+o(1))p_j$ and for any constant $d \geq k-j+2$, with high probability $\mathcal{G}_p$ has no j-cocycle with traversable support of size s with $k-j+2 \leq s \leq d$.*

For larger size, we make use of traversability to define a breadth-first search process that finds all possible supports. Using this process, we can bound the number of possible smallest supports of elements of $\mathcal{N}_{\mathcal{G}_p}$ more carefully, thus allowing us to prove that whp *for all relevant p simultaneously*, such a smallest support cannot be "large".

▶ **Lemma 4.6.** *There exists a positive constant $\bar{d}$ such that with high probability for all $p \geq p_j^-$, the smallest support of a j-cocycle in $\mathcal{N}_{\mathcal{G}_p}$ has size $s < \bar{d}$.*

In particular, for any fixed $p = (1+o(1))p_j$, whp the smallest support of elements of $\mathcal{N}_{\mathcal{G}_p}$ is not "small" by Lemma 4.5 and not "large" by Lemma 4.6, which means that $\mathcal{N}_{\mathcal{G}_p} = \emptyset$ whp.

Finally, we complete the argument by proving that any new element of $\mathcal{N}_{\mathcal{G}_p}$ with "small" support that might appear if we increase p would have to give rise to a "new" local obstacle. But Lemma 3.6 already tells us that whp no new local obstacles appear. This concludes the proof of Lemma 4.2.

5 Proofs of main results

5.1 Proof of Theorem 1.8

Corollary 3.8 states that for any function ω of n which tends to infinity as $n \to \infty$, whp

$$\frac{(j+1)\log n + \log\log n - \omega}{(k-j+1)n^{k-j}}(k-j)! < p_{M_j} < \frac{(j+1)\log n + \log\log n + \omega}{(k-j+1)n^{k-j}}(k-j)!,$$

which is precisely Theorem 1.8 i.

To prove ii, recall that Lemma 3.1 tells us that for each $i \in [j-1]$, whp $H^i(\mathcal{G}_p;\mathbb{F}_2) \neq 0$ for all $p \in [p_{i-1}^-, p_{M_i})$. By i, whp

$$p_{M_i} > \left(1 - \frac{1}{\sqrt{\log n}}\right)\frac{(i+1)\log n}{(k-i+1)n^{k-i}}(k-i)! = p_i^-,$$

and thus whp $\mathcal{G}_p$ is not j-cohom-connected throughout $\bigcup_{i=1}^{j}[p_{i-1}^-, p_{M_i}) = [p_0^-, p_{M_j})$.

Now observe that by Lemma 3.2 whp $p_T > p_0^-$ and that $\mathcal{G}_p$ is not topologically connected in $[0, p_T)$ by definition of p_T. Therefore, whp $\mathcal{G}_p$ is not j-cohom-connected in

$$[0, p_{M_j}) = [0, p_T) \cup [p_0^-, p_{M_j}),$$

as required.

It remains to prove iii. By Corollary 3.7, we know that for all $p \geq p_{M_j}$, there are no copies of M_j^- in $\mathcal{G}_p$. Thus, if $H^j(\mathcal{G}_p;\mathbb{F}_2) \neq 0$, then any representative of a non-zero cohomology class cannot arise from copies of M_j^- and therefore lies in $\mathcal{N}_{\mathcal{G}_p}$ (Definition 4.1). But by Lemma 4.2, whp each such $\mathcal{N}_{\mathcal{G}_p}$ is empty and thus whp $H^j(\mathcal{G}_p;\mathbb{F}_2) = 0$ for all $p \geq p_{M_j}$. Analogously, whp all cohomology groups $H^i(\mathcal{G}_p;\mathbb{F}_2)$ for $i \in [j-1]$ vanish, because whp $p_{M_i} < p_{M_j}$ by i. Finally, by i and Lemma 3.2 whp $p_T < p_{M_j}$, meaning that whp $\mathcal{G}_p$ is topologically connected for all $p \geq p_{M_j}$. This implies that whp each such $\mathcal{G}_p$ is $\mathbb{F}_2$-cohomologically j-connected. ◀

5.2 Proof of Corollary 1.9

Let ω be any function of n which tends to infinity as $n \to \infty$. It follows by a simple first and second moment argument (see e.g. [20]) that whp

$$\frac{k \log n - \omega}{n} < p_{\text{isol}} < \frac{k \log n + \omega}{n}. \tag{6}$$

In order to prove that $p_{\text{conn}} = p_{\text{isol}}$ whp, suppose that a $(k-1)$-simplex σ is isolated in $\mathcal{Y}_p$ for some p. The indicator function f_σ of σ is a $(k-1)$-cocycle, because σ is isolated. But f_σ is *not* a $(k-1)$-coboundary, because σ lies in (many) $(k-1)$-cycles due to the presence of the full $(k-1)$-dimensional skeleton. In particular, $H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) \neq 0$. By the definitions of p_{conn} and p_{isol}, this implies that $p_{\text{conn}} \geq p_{\text{isol}}$.

For the opposite direction, fix the birth times of all k-simplices. Then for all $p \geq p_{\text{isol}}$, we have $\mathcal{Y}_p = \mathcal{G}_p$ and therefore $\mathcal{Y}_p$ is $\mathbb{F}_2$-cohomological $(k-1)$-connected whp for every $p \geq \max(p_{\text{isol}}, p_{M_{k-1}})$ by Theorem 1.8 iii. By (6) and Theorem 1.8 i for $j = k-1$, whp for any (slowly) growing function ω

$$p_{\text{isol}} > \frac{k \log n - \omega}{n} > \frac{k \log n + \log \log n + \omega}{2n} > p_{M_{k-1}},$$

hence whp for all $p \geq p_{\text{isol}}$ we have $H^{k-1}(\mathcal{Y}_p; \mathbb{F}_2) = 0$. This means that whp $p_{\text{conn}} \leq p_{\text{isol}}$ and thus $p_{\text{conn}} = p_{\text{isol}}$, as required. ◀

5.3 Proof of Theorem 1.10

We are interested in the asymptotic distribution of $D_j := \dim \left(H^j(\mathcal{G}_p; \mathbb{F}_2) \right)$ for

$$p = \frac{(j+1) \log n + \log \log n + c_n}{(k-j+1) n^{k-j}} (k-j)!, \qquad \text{where } c_n \xrightarrow{n \to \infty} c \in \mathbb{R}.$$

Denote by X_- the number of copies of M_j^- in $\mathcal{G}_p$. Standard calculations show that

$$\mathbb{E}(X_-) = (1 + o(1)) \lambda_j, \qquad \text{where } \lambda_j = \frac{(j+1) e^{-c}}{(k-j+1)^2 j!}.$$

Moreover, we show that for each fixed integer $t \geq 1$

$$\mathbb{E} \binom{X_-}{t} = (1 + o(1)) \frac{\lambda_j^t}{t!}.$$

These equalities are precisely what is necessary to apply the method of moments (see e.g. [13]) in order to show that X_- converges in distribution to a Poisson random variable with expectation λ_j, which we denote by $X_- \xrightarrow{d} \text{Po}(\lambda_j)$.

It remains to show that $X_- = D_j$ whp. To this end, denote by $f_1, \dots, f_{X_-}$ the j-cocycles arising from the copies of M_j^- in $\mathcal{G}_p$. Lemma 4.2 states that whp the cohomology classes of $f_1, \dots, f_{X_-}$ generate $H^j(\mathcal{G}_p; \mathbb{F}_2)$, which means that $X_- \geq D_j$ whp.

In order to prove the opposite direction, we show that the cohomology classes of $f_1, \dots, f_{X_-}$ are linearly independent. Observe first that whp $X_- = o(n)$ by Markov's inequality, because X_- has bounded expectation. Let $I \subseteq [X_-]$ be non-empty and let S be the support of $\sum_{i \in I} f_i$. Whp no two f_i's can have their supports contained in the same k-simplex K, because otherwise their union would be a traversable support of size s with $k - j + 2 \leq s \leq 2(k - j + 1)$, but such supports whp do not exist by Lemma 4.5.

Thus, whp the f_i's have disjoint support by property (M2) of an M_j^- (Definition 3.3), and in particular $S \neq \emptyset$. Pick $\sigma \in S$. Lemma 3.4 tells us that whp there are $\Theta(n)$ many j-cycles in $\mathcal{G}_p$ that contain σ and are otherwise disjoint. But at most $|S| \leq (k-j+1)|I| = o(n)$ of these j-cycles can contain another j-simplex in S, which means that whp there are j-cycles that meet S only in σ, showing that $\sum_{i\in I} f_i$ is not a j-coboundary. Therefore the cohomology classes of $f_1, \ldots, f_{X_-}$ are linearly independent whp. This shows that $X_- \leq D_j$ and thus $X_- = D_j$ whp, as desired.

Together with $X_- \xrightarrow{d} \mathrm{Po}(\lambda_j)$, this proves that $D_j \xrightarrow{d} \mathrm{Po}(\lambda_j)$. By Theorem 1.8 (for $j-1$ instead of j), whp $H^0(\mathcal{G}_p;\mathbb{F}_2) = \mathbb{F}_2$ and $H^i(\mathcal{G}_p;\mathbb{F}_2) = 0$ for all $i \in [j-1]$. In particular,

$$\begin{aligned}\mathbb{P}(\mathcal{G}_p \text{ is } j\text{-cohom-connected}) &= \mathbb{P}\big(H^j(\mathcal{G}_p;\mathbb{F}_2) = 0\big) + o(1)\\ &= (1+o(1))\mathbb{P}\big(\mathrm{Po}(\lambda_j) = 0\big)\\ &= (1+o(1))e^{-\lambda_j}.\end{aligned}$$

This concludes the proof of Theorem 1.10. ◀

6 Concluding remarks

The vanishing of cohomology groups with coefficients in $\mathbb{F}_2$ is just one possible way of defining the concept of "connectedness" of $\mathcal{G}_p$. An obvious alternative would be to consider coefficients from other groups or fields. For $\mathcal{Y}_p$, such notions of connectedness have been studied for coefficients in any finite abelian group, in $\mathbb{Z}$, or in any field [1, 2, 14, 18, 19, 20].

A rather strong notion of connectedness would be to require the homotopy groups $\pi_1(\mathcal{G}_p), \ldots, \pi_j(\mathcal{G}_p)$ to vanish. For the 2-dimensional case, the vanishing of $\pi_1(\mathcal{Y}_p)$ was studied by Babson, Hoffman and Kahle [3]. In particular, they showed that whp $\pi_1(\mathcal{Y}_p) \neq 0$ at the time that $H^1(\mathcal{Y}_p;\mathbb{F}_2)$ becomes zero. From that time on, the models $\mathcal{Y}_p$ and $\mathcal{G}_p$ coincide. As $\pi_1(\mathcal{G}_p) \neq 0$ follows immediately from $H^1(\mathcal{G}_p;\mathbb{F}_2) \neq 0$, the range that should be of particular interest with respect to $\pi_1(\mathcal{G}_p)$ in the 2-dimensional case is

$$\frac{\log n + \frac{1}{2}\log\log n}{n} \leq p \leq \frac{2\log n + \omega}{n}.$$

A natural conjecture would be that whp $\pi_1(\mathcal{G}_p) \neq 0$ in this range.

Theorem 1.9 provides a limit result for the dimensions $D_j = \dim(H^j(\mathcal{G}_p;\mathbb{F}_2))$ around the point of the phase transition. It would be interesting to know the behaviour of D_j also for earlier regimes. More precisely, we know by Theorem 1.8 that whp $D_j \neq 0$ in the interval $[p_{j-1}^-, p_{M_j})$. Can we say more about the value of D_j in this interval? How far below p_{j-1}^- do we have $D_j > 0$ whp?

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