


Amortized Analysis of Asynchronous Price Dynamics

Yun Kuen Cheung¹

Max Planck Institute for Informatics, Saarland Informatics Campus, Saarbrücken, Germany


ycheung@mpi-inf.mpg.de

 <https://orcid.org/0000-0002-9280-0149>

Richard Cole²

Courant Institute, NYU, New York, USA

cole@cs.nyu.edu

 <https://orcid.org/0000-0002-5885-0222>

Abstract

We extend a recently developed framework for analyzing asynchronous coordinate descent algorithms to show that an asynchronous version of tatonnement, a fundamental price dynamic widely studied in general equilibrium theory, converges toward a market equilibrium for Fisher markets with CES utilities or Leontief utilities, for which tatonnement is equivalent to coordinate descent.

2012 ACM Subject Classification Theory of computation → Market equilibria

Keywords and phrases Asynchronous Tatonnement, Fisher Market, Market Equilibrium, Amortized Analysis

Digital Object Identifier 10.4230/LIPIcs.ESA.2018.18

Related Version A full version is available at <https://arxiv.org/abs/1806.10952>.

Acknowledgements We thank several anonymous reviewers for their helpful suggestions.

1 Introduction

As is well known, it is PPAD-hard to compute equilibria for general games and markets [21, 9, 18, 8, 37, 10]. By viewing the players and the environment collectively as implicitly performing a computation, these hardness results indicate that, in general, a game or market cannot reach an equilibrium quickly (assuming no unexpected complexity results such as PPAD = FP). As a result, a lot of attention has been given to the design of polynomial-time algorithms to compute equilibria, either exactly or approximately, for specific families of games and markets. Most of these algorithms can be categorized as either simplex-like (e.g., Lemke-Howson [30]), numerical methods (e.g., the interior-point method [40] or the ellipsoid method [27]), or some carefully-crafted combinatorial algorithms (e.g., flow-based algorithms for computing an equilibrium of a market with agents having linear utility functions [22, 33, 24]).

¹ Part of the work done while this author was at the Courant Institute, NYU and at the Faculty of Computer Science, University of Vienna. He was supported in part by NSF Grant CCF-1217989, and the Vienna Science and Technology Fund (WWTF) project ICT10-002. Additionally the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant Agreement no. 340506.

² This work was supported in part by NSF grant CCF-1527568.



© Yun Kuen Cheung and Richard Cole;

licensed under Creative Commons License CC-BY

26th Annual European Symposium on Algorithms (ESA 2018).

Editors: Yossi Azar, Hannah Bast, and Grzegorz Herman; Article No. 18; pp. 18:1–18:15

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

However, it seems implausible that these algorithms describe the implicit computations in games or markets. In particular, many markets appear to have a highly distributed environment. This would appear to preclude computations which require centralized coordination, which is essential for the three categories of algorithms above. Consequently, in order to justify equilibrium concepts, we want natural algorithms which could plausibly be running (in an implicit form) in the associated distributed environments. Moreover, since it is preferable not to assume centralized timing or coordination, a desirable feature of such natural algorithms is robustness against asynchrony, which means such algorithms should remain effective even in situations where information transfer takes time and agents make decisions (i.e., perform computations) with possibly outdated information.

A first candidate for a natural algorithm in markets is tatonnement: it adjusts the price of a good upward if there is too much demand, and downward if too little. Indeed, tatonnement was proposed alongside the concept of a market equilibrium by Walras [38] in 1874. Since then, studies of market equilibria and tatonnement have received much attention in economics, operations research, and most recently in computer science; we list a small sample of the voluminous literature, focusing mainly on computer science works [2, 36, 23, 17, 19, 20, 14, 13, 35]. Underlying many of these works is the issue of what are plausible price adjustment mechanisms and in what types of markets they attain a market equilibrium.

The tatonnements studied in prior work have mostly been continuous, or discrete and synchronous. Cole and Fleischer [19] observed that real-world market dynamics are highly distributed and hence presumably asynchronous. They argued that any realistic price dynamics must involve out-of-equilibrium trade in order to induce the imbalances leading to price updates. Further, they argued that simple rules with relatively low information requirements were more plausible. The lowest imaginable level of information would be for each seller to only know the demand for the good it was selling, and for any price updating to occur in a non-coordinated manner, i.e., asynchronously. Accordingly, they introduced the *Ongoing market model*, a model of a repeating market incorporating update dynamics, and they analyzed the performance of an asynchronous tatonnement in this market. The market also incorporated warehouses (buffers) to cope with supply and demand imbalances.

Cheung, Cole and Devanur [13] showed that tatonnement is equivalent to coordinate descent on a convex function for several classes of Fisher markets, and consequently that a suitable synchronous tatonnement converges toward the market equilibrium in three general classes of markets: complementary-CES Fisher markets³, substitute-CES Fisher markets, and Leontief Fisher markets; Cheung [11] extended this to all nested-CES Fisher markets. In this paper, we show that this equivalence enables us to perform an amortized analysis to show that the corresponding asynchronous version of tatonnement converges toward the market equilibrium in these classes of markets; indeed, our analysis also covers Fisher markets in which some buyers have substitute-CES utility functions and others have complementary ones. We also note that the tatonnement for Leontief Fisher markets analyzed in [13] had an unnatural constraint on the step sizes; our analysis removes that constraint.

Finally, we remark that it is by no means obvious that the existence of a convergence result for synchronous updating implies an analogous result for asynchronous updating. An example of a setting where an asynchronous result has yet to be achieved is proportional response dynamics [41, 4, 15].

³ i.e., markets in which the buyers all have complementary CES utilities.

Technique of Analysis, and Comparison with the Companion Paper [16]. In a companion paper [16], we analyzed several versions of asynchronous coordinate descent. The analyses in both papers follow a common framework. We use an amortized analysis which relates the actual progress to the desired progress, where the desired progress is a constant fraction of the progress achieved with synchronous updating. The amortization is used to hide the difference between these two measures of progress by amortizing it over multiple updates. As we shall see, this difference is bounded by the squares of appropriate excess demand (resp. gradient) differences, and using Lipschitz gradient parameters, these can in turn be bounded by sum of the squares of recent changes to the prices (resp. coordinates). The final ingredient is to show that the progress is an upper bound on the square of the change to the updated price. Combining these ingredients yields a lower bound on the rate of progress.

In [16], it was assumed that the underlying convex function has some *global* finite Lipschitz gradient parameters, which is a common assumption in optimization and machine learning. The main focus there is on the maximum possible degree of parallelism which permits linear speedup, and on a number of challenges to devising rigorous and complete analyses which handle the subtle interplay between randomness (choices of coordinates) and asynchrony.

However, in the asynchronous tatonnement setting we analyze here, there are no global finite Lipschitz gradient parameters. Instead, we use *local* Lipschitz gradient parameters, as was done implicitly in [13]; the consequence is that the rate of convergence depends on the starting point. Also, the only acceptable degree of parallelism is the maximal one, i.e., all sellers are adjusting prices independently in parallel. The challenge is to devise an asynchronous analysis while keeping the price update rule reasonable, i.e., having the step size be an absolute constant which is independent of the number of goods. This calls for a somewhat different potential function and analysis from the one used for the asynchronous coordinate descent analysis in [16]; the analysis also differs quite substantially from the synchronous tatonnement analyses in [13].

Relevance to Theoretical Computer Science. Iterative procedures and dynamical systems are pervasive across multiple disciplines; a non-exhaustive list of such systems which have interested theorists includes bandwidth sharing (e.g., proportional response [39, 41, 15]), SDD linear system solvers [28, 29], distributed load balancing [26, 3], bird flocking [6], influence systems [7] and the spread of information memes across the Internet [31].

There have been many analyses of these systems, but one issue that has received relatively little attention is the timing of agents' actions. In most prior analyses, amenable timing schemes (e.g., synchronous or round robin updates) and perfect information retrieval were assumed, perhaps because they were more readily analyzed. However, typically these assumptions are unrealistic, and to better understand how these systems really behave, it is important to obtain asynchronous analyses of such systems. We believe the insight from our amortization framework may be useful in obtaining such analyses.

Other Related Work. In a similar spirit to our analysis, Cheung, Cole and Rastogi [14] analyzed asynchronous tatonnement in certain Fisher markets. This earlier work employed a potential function which drops continuously when there is no update and does not increase when an update is made. This approach could be followed for the current market setting, but in the current work, we instead use a discrete analysis which has more in common with our asynchronous coordinate descent analyses in [16]. Our work differs from [14] in two aspects. First, the update rule in [14] is more restricted: they use *average excess demand* for updates, while our update rule allows an *arbitrary value between the maximum and minimum excess*

demands. Second, while the high-level idea is similar, our potential function is substantially different from (and more sophisticated than) the one in [14], and the classes of markets covered by the two analyses are quite different.

In a recent work, Dvijotham et al. [25] study a different asynchronous dynamics. In their setting sellers are boundedly rational and buyers are myopic (i.e., best responding). More specifically, the base (zero) level for the sellers is to be best responding, and level $k + 1$ is obtained by best responding to level k sellers. They show that this system converges linearly to the market equilibrium in suitable Fisher markets including substitute-CES markets.

For the closely related topic of learning dynamics in games, where updates are based on the payoffs received by agents, again, the classical approach assumes synchronous or round-robin updates with up-to-date payoffs; models with stochastic update schedules were also studied previously (e.g., in [5, 1, 32]), while learning dynamics with delayed payoffs [34] were studied recently.

2 Preliminaries and Results

Fisher Market. In a Fisher market, there are n perfectly divisible goods and m buyers. Without loss of generality, the supply of each good is normalized to be one unit. Each buyer i has a utility function $u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}$, and a budget of size e_i . At any given price vector $\mathbf{p} \in \mathbb{R}_+^n$, each buyer purchases a maximum utility affordable collection of goods. More precisely, $\mathbf{x}_i \in \mathbb{R}_+^n$ is said to be a demand of buyer i if $\mathbf{x}_i \in \arg \max_{\mathbf{x}' : \mathbf{x}' \cdot \mathbf{p} \leq e_i} u_i(\mathbf{x}')$.

A price vector $\mathbf{p}^* \in \mathbb{R}_+^n$ is called a *market equilibrium* if at \mathbf{p}^* , there exists a demand \mathbf{x}_i of each buyer i such that

$$p_j^* > 0 \Rightarrow \sum_{i=1}^m x_{ij} = 1 \quad \text{and} \quad p_j^* = 0 \Rightarrow \sum_{i=1}^m x_{ij} \leq 1.$$

We note that in the markets we studied here, the demand at any price vector is unique. In these markets, we let $z_j := \sum_{i=1}^m x_{ij} - 1$ denote the excess demand for good j .

CES utilities. In this paper, each buyer i 's utility function is of the form

$$u_i(\mathbf{x}_i) = \left(\sum_{j=1}^n a_{ij} \cdot (x_{ij})^{\rho_i} \right)^{1/\rho_i},$$

for some $-\infty \leq \rho_i < 1$, where each a_{ij} is a non-negative number. $u_i(\mathbf{x}_i)$ is called a Constant Elasticity of Substitution (CES) utility function. They are a class of utility functions often used in economic analysis. The limit as $\rho_i \rightarrow -\infty$ is called a Leontief utility, usually written as $u_i(\mathbf{x}_i) = \min_j \frac{x_{ij}}{c_{ij}}$ ⁴; and the limit as $\rho_i \rightarrow 0$ is called a Cobb-Douglas utility, usually written as $\prod_j x_{ij}^{a_{ij}}$, with $\sum_j a_{ij} = 1$. The utilities with $\rho_i \leq 0$ capture goods that are complements, and those with $\rho_i \geq 0$ goods that are substitutes. Accordingly, when $\rho_i \leq 0$, we say the utility function is a complementary CES utility function, and when $\rho_i \geq 0$ we say it is a substitute CES utility function.

⁴ The utility function $u_i(\mathbf{x}) = \min_j \frac{x_{ij}}{c_{ij}}$ can be seen as the limit of $u_i(\mathbf{x}) = \left(\sum_j \left(\frac{x_{ij}}{c_{ij}} \right)^{\rho_i} \right)^{\frac{1}{\rho_i}}$ as $\rho_i \searrow -\infty$.

Directly Related Prior Results and Our Results. Cheung, Cole and Devanur [13] showed that tatonnement is equivalent to coordinate descent on a convex function ϕ for Fisher markets with buyers having complementary-CES or Leontief utility functions (and in a later version of the paper, substitute-CES utility functions too). To be specific, [13] showed that for the convex function

$$\phi(\mathbf{p}) = \sum_{k=1}^n p_k + \sum_{i=1}^m e_i \cdot \log \hat{u}_i(\mathbf{p}),$$

where $\hat{u}_i(\mathbf{p})$ is the optimal utility that buyer i attains at price vector \mathbf{p} with a unit of spending, we have that $\nabla_j \phi(\mathbf{p}) = -z_j(\mathbf{p})$. The corresponding update rule is

$$p'_j \leftarrow p_j \cdot [1 + \lambda \cdot \min\{z_j, 1\}], \quad (1)$$

where $\lambda > 0$ is a suitable constant. As the update rule is multiplicative, they assumed that the initial prices were positive.

As argued in [19], when the economic activity is occurring over time, it is natural to base each price update for a good on the excess demand observed by its seller since the time of the last price update to her good (possibly weighted toward more recent sales). This perceived excess demand can be written as the product of the length of the time interval with an instantaneous excess demand at some specific time in this interval, which yields the following modification of update rule (1).

$$p'_j \leftarrow p_j \cdot [1 + \lambda \cdot \min\{\tilde{z}_j, 1\} \cdot (t - \alpha_j(t))], \quad (2)$$

where $\alpha_j(t)$ denotes the time of the latest update to price j *strictly* before time t , \tilde{z}_j is a value between the minimum and maximum instantaneous excess demands during the time interval $(\alpha_j(t), t)$, and $\lambda > 0$ is a suitable constant. We assume that $t - \alpha_j(t) \leq 1$ for all $t \geq 0$ and for all goods j .

As we will see, having $\lambda \leq 1/25.5$ suffices. In comparison, in the synchronous version [13], $\lambda \leq 1/6$ suffices. This implies that the step sizes of the asynchronous tatonnement can be kept at a constant fraction of those used in its synchronous counterpart.

► **Theorem 1.** *For $\lambda \leq 1/25.5$, asynchronous tatonnement price updates using rule (2) converge linearly toward the market equilibrium in any complementary-CES market, and they converge in any Leontief Fisher market.*

► **Theorem 2.** *Let \mathcal{M} be a Fisher market in which buyers have CES utility functions. Suppose that $\rho := \max_i \rho_i < 1$ and $\min_i \rho_i > -\infty$ in \mathcal{M} . Let $E := \max\{1/(1 - \rho), 1\}$. Then for $\lambda \leq 1/(26E)$, asynchronous tatonnement price updates using rule (2) converge linearly toward the market equilibrium.*

Here, we focus on the result concerning complementary-CES Fisher markets; the analyses for the other cases are deferred to the full version. The analysis for Theorem 2 is just a small modification of the complementary case. For the Leontief Fisher markets, while the first part of the analysis is identical to the complementary case, this is not enough to demonstrate convergence, and to do so requires substantially more effort.

In an earlier version of this paper [12] on arXiv, we proved Theorem 1 (except that λ was slightly larger) using a potential function which decreases continuously over time, as was the case for the analyses in [20, 14] also. We believe the current analysis is considerably simpler. The main advantage of the prior analysis at this point is that we extended it to account for the warehouses in the Ongoing market model, albeit with a quite non-trivial argument. This seems possible with the potential function in the present paper too, but we suspect it would be of interest to at most a few specialists.

Standard Notation in Coordinate Descent. Let \mathbf{e}_j denote the unit vector along coordinate (in our context, price) j . A function F is L -Lipschitz-smooth if for any $\mathbf{p}, \Delta\mathbf{p} \in \mathbb{R}^n$, $\|\nabla F(\mathbf{p} + \Delta\mathbf{p}) - \nabla F(\mathbf{p})\| \leq L \cdot \|\Delta\mathbf{p}\|$. For any coordinates j, k , a function F is L_{jk} -Lipschitz-smooth if for any $\mathbf{p} \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $|\nabla_k F(\mathbf{p} + r \cdot \mathbf{e}_j) - \nabla_k F(\mathbf{p})| \leq L_{jk} \cdot |r|$. Also, as is standard, L_j denotes L_{jj} .

3 Key Ideas and Lemmas

For simplicity, we assume that at any particular time t , there is at most one update to one good. In general, since there is no coordination between price updates of different goods, it is possible that the prices of two goods are updated at the same *moment*; but by using any arbitrary tie-breaking (perturbation) rule, our analysis extends to such cases.

Recall update rule (2). For the purposes of our analysis, for each update we now need to know the elapsed time since the previous update to the same coordinate, or since time 0 if it is the first update to that coordinate; for the update at time τ , we denote this by Δt_τ . As explained in [19], in the Ongoing market model, all the sellers need to know is the size of their warehouse stock at the times of the current update and the previous update, which seem to be very natural information. We let $\alpha_j(\tau)$ denote the time of the most recent update to p_j *strictly* before time τ , or time 0 if there is no previous update to this price. We let $\alpha(\tau)$ denote the time of the most recent update to *any* price *strictly* before time τ , or time 0 if there is no previous update to any price. And we let $\delta_\tau := \tau - \alpha(\tau)$, the elapsed time since the most recent previous update to any price.

Suppose there is an update at time t . We let p_{k_t} denote the price updated at time t , we let $p_{k_t}^t$ denote its updated value, and $p_{k_t}^{t-}$ its value right before this update; note that $p_{k_t}^{t-} \equiv p_{k_t}^{\alpha(t)} \equiv p_{k_t}^{t-\delta_t}$. Let $\Delta p_{k_t}^t := p_{k_t}^t - p_{k_t}^{t-}$. Also, we let $\tilde{z}_{k_t}^t$ denote the value of the excess demand used in the update to price p_{k_t} at time t . Finally, we let $\Gamma_{k_t}^t := \max\{1, \tilde{z}_{k_t}^t\} / (\lambda p_{k_t}^{t-})$. Then update rule (2) can be rewritten in the following form:

$$p_{k_t}^t \leftarrow p_{k_t}^{t-} + \frac{1}{\Gamma_{k_t}^t} \cdot \tilde{z}_{k_t}^t \cdot \Delta t. \quad (3)$$

In our analysis, when we write $\sum_{\tau \in I}$, where I is a time interval, the summation is summing over all updates that occurred in time interval I .

Let z_k^t be the instantaneous excess demand for good k right before a price update at time t . We note that $z_k^t = -\nabla_k \phi(\mathbf{p}^{\alpha(t)})$. For each update τ , let $z_k^{\max, \tau}$ and $z_k^{\min, \tau}$ denote the maximum and minimum of accurate excess demand values of good k in the time interval $(\alpha_{k_\tau}(\tau), \tau)$.

We also need to define local Lipschitz parameters: $L_{jk}^{[\tau_a, \tau_b]}$ is an upper bound on the Lipschitz gradient parameter L_{jk} of the function ϕ within a rectangular hull of those prices which might appear in the time interval $[\tau_a, \tau_b]$. Observe that in update rule (2), since $|\min\{\tilde{z}_j, 1\}| \leq 1$ always, the above-mentioned rectangular hull is finitely bounded, and furthermore, it shrinks as λ gets smaller.

We use the following three lemmas. Lemma 3 is modified from a standard lemma in coordinate descent to accommodate local Lipschitz gradient continuity. Lemma 4 is a direct consequence of the Power-Mean inequality. Lemma 5 is a simple algebra exercise. See the full version for the missing proofs.

► **Lemma 3.** *Suppose there is an update to coordinate k_t at time t according to rule (2), and suppose that $\lambda \leq 1/10$. Recall that \tilde{z}_{k_t} is the value used in applying rule (2). Then*

$$\phi(\mathbf{p}^{\alpha(t)}) - \phi(\mathbf{p}^t) \geq \frac{\Gamma_{k_t}^t}{4} \cdot \frac{(\Delta p_{k_t}^t)^2}{\Delta t} - \frac{1}{\Gamma_{k_t}^t} \cdot (z_{k_t}^t - \tilde{z}_{k_t})^2 \cdot |\Delta t|.$$

► **Lemma 4.** *Suppose that w_1, w_2, \dots, w_ℓ and y_1, y_2, \dots, y_ℓ are non-negative numbers. Then*

$$\left(\sum_{j=1}^{\ell} w_j y_j \right)^2 \leq \left(\sum_{j=1}^{\ell} w_j \right) \left(\sum_{j=1}^{\ell} w_j \cdot (y_j)^2 \right).$$

► **Lemma 5.** *In Lemma 3, suppose that \tilde{z}'_{k_t} were used instead of \tilde{z}_{k_t} in update rule (3), but with $\Gamma_{k_t}^t$ unchanged. Let the new Δp_{k_t} value be $\Delta' p_{k_t}^t$. Then*

$$\Gamma_{k_t}^t \cdot \frac{(\Delta p_{k_t}^t)^2}{\Delta t} \geq \frac{\Gamma_{k_t}^t}{2} \cdot \frac{(\Delta' p_{k_t}^t)^2}{\Delta t} - \frac{1}{\Gamma_{k_t}^t} \cdot (\tilde{z}_{k_t} - \tilde{z}'_{k_t})^2 \cdot \Delta t.$$

In the RHS of the inequality in Lemma 3, we call the first term, $\frac{\Gamma_{k_t}^t}{4} \cdot \frac{(\Delta p_{k_t}^t)^2}{\Delta t}$, a progress term, and we call the second term, $-\frac{1}{\Gamma_{k_t}^t} \cdot (z_{k_t}^t - \tilde{z}_{k_t})^2 \cdot |\Delta t|$, an error term. The progress term is cut into two halves. The first half will be used to demonstrate progress of the convergence, while the second half will be saved to compensate for the error terms in subsequent updates. Accordingly, we design a potential function $\Phi(t)$ of the form $\Phi(t) := \phi(\mathbf{p}^t) + A(t)$, where $A(t) \geq 0$ for all t and $A(0) = 0$. We call $A(t)$ the *amortization bank*; its purpose is to save portions of the progress terms for future compensations.

By showing that $\Phi(t)$ reduces by a constant fraction ε in every $\mathcal{O}(1)$ time units, we can deduce that $\phi(\mathbf{p}^t) \leq \Phi(t) \leq \Phi(0) \cdot (1 - \varepsilon)^{\Theta(t)} = \phi(\mathbf{p}^\circ) \cdot (1 - \varepsilon)^{\Theta(t)}$, as desired.

We define the function $A(t)$ as follows:

$$A(t) = c_1 \sum_{\tau \in (t-1, t]} \sum_{k \neq k_\tau} (1 + \mathbb{1}[\mathcal{E}(k, \tau, t)]) \cdot L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^\tau} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau},$$

where $c_1 > 0$ is a constant we will determine later, and $\mathcal{E}(k, \tau, t)$ denotes the event that price k is *not* updated during the time interval $(\tau, t]$.

4 Analysis

Suppose there is an update at time $t \geq 2$. We let t_a denote the time of the latest update *strictly* before time $(t - 2)$, if any; otherwise, we let $t_a = 0$. We let t_b denote the time of the earliest update in the time interval $[t - 1, t]$. By Lemma 3,

$$\begin{aligned} & \sum_{\tau \in (t_a, t]} [\Phi(\alpha(\tau)) - \Phi(\tau)] \\ & \geq \sum_{\tau \in (t_a, t]} \left[\frac{\Gamma_{k_\tau}^\tau}{4} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{1}{\Gamma_{k_\tau}^\tau} (z_{k_\tau}^\tau - \tilde{z}_{k_\tau}^\tau)^2 \cdot \Delta t_\tau \right] + A(t_a) - A(t) \\ & \stackrel{(*)}{\geq} \sum_{\tau \in (t_a, t]} \left[\frac{\Gamma_{k_\tau}^\tau}{4} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{1}{\Gamma_{k_\tau}^\tau} (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \cdot \Delta t_\tau \right] + A(t_a) - A(t). \end{aligned} \quad (4)$$

Inequality (*) holds because in the tatonnement setting, both the accurate excess demand $z_{k_\tau}^\tau$ and the inaccurate excess demand $\tilde{z}_{k_\tau}^\tau$ must lie between $z_{k_\tau}^{\max, \tau}$ and $z_{k_\tau}^{\min, \tau}$.

For any $\nu \in (\alpha_{k_\tau}(\tau), \tau)$, let $\Delta' p_{k_\tau}^\nu$ denote the Δp_{k_τ} value if there were an update at time ν to price p_{k_τ} using the accurate excess demand $z_{k_\tau}^\nu$. By Lemma 5, for each $\tau \in (t_a, t]$,

$$\begin{aligned}
 & \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} \\
 & \geq \sum_{\nu \in (\max\{t_a, \alpha_{k_\tau}(\tau)\}, \tau]} \left[\frac{\Gamma_{k_\tau}^\tau}{16} \cdot \frac{(\Delta' p_{k_\tau}^\nu)^2}{\Delta t_\tau} \cdot \frac{\delta_\nu}{\Delta t_\tau} - \frac{1}{8\Gamma_{k_\tau}^\tau} (z_{k_\tau}^\nu - \tilde{z}_{k_\tau}^\tau)^2 \cdot \delta_\nu \right] \\
 & \geq \frac{\Gamma_{k_\tau}^\tau}{16} \left(\sum_{\nu \in (\max\{t_a, \alpha_{k_\tau}(\tau)\}, \tau]} \frac{(\Delta' p_{k_\tau}^\nu)^2}{(\Delta t_\tau)^2} \cdot \delta_\nu \right) - \frac{1}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2. \tag{5}
 \end{aligned}$$

For any k and any time ν , let $\beta_k(\nu)$ denote the time of the earliest update to price k on or after time ν . Combining (4) and (5) yields

$$\begin{aligned}
 & \sum_{\tau \in (t_a, t]} [\Phi(\alpha(\tau)) - \Phi(\tau)] \\
 & \geq \sum_{\tau \in (t_a, t]} \sum_{\nu \in (\max\{t_a, \alpha_{k_\tau}(\tau)\}, \tau]} \frac{\Gamma_{k_\tau}^\tau}{16} \cdot \frac{(\Delta' p_{k_\tau}^\nu)^2}{(\Delta t_\tau)^2} \cdot \delta_\nu \\
 & \quad + \left[\sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} + A(t_a) - A(t) \right] - \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \\
 & \geq \frac{1}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \sum_{k=1}^n \frac{\Gamma_k^{\beta_k(\nu)} \cdot (\Delta' p_k^\nu)^2}{(\Delta t_{\beta_k(\nu)})^2} - \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \\
 & \quad + \left[\sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} + A(t_a) - A(t) \right] \\
 & \geq \frac{1}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \left[\sum_{k=1}^n \frac{\Gamma_k^{\beta_k(\nu)} (\Delta' p_k^\nu)^2}{(\Delta t_{\beta_k(\nu)})^2} + c_2 \cdot A(\alpha(\nu)) \right] - \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \\
 & \quad + \left[\sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} + A(t_a) - A(t) - \frac{c_2}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot A(\alpha(\nu)) \right],
 \end{aligned}$$

for some small constant $c_2 > 0$ we will determine later.

In [13], it was proved that the function ϕ is strongly convex in any region bounded away from zero prices, and that the maximum Γ value throughout the tatonnement is upper bounded by a finite constant which depends on the starting price \mathbf{p}° .⁵ We denote the finite upper bound on all Γ 's by $\bar{\Gamma}$, and the strong convexity parameter of ϕ by μ_ϕ , which also depends on the starting prices. We let $\varepsilon := \mu_\phi / \bar{\Gamma}$. Then it is a standard fact in optimization that

$$\sum_{k=1}^n \frac{\Gamma_k^{\beta_k(\nu)} (\Delta' p_k^\nu)^2}{(\Delta t_{\beta_k(\nu)})^2} = \sum_{k=1}^n \frac{1}{\Gamma_k^{\beta_k(\nu)}} \cdot (z_k^\nu)^2 \geq \sum_{k=1}^n \frac{1}{\bar{\Gamma}} \cdot (z_k^\nu)^2 \geq \varepsilon \cdot \phi(\mathbf{p}^{\alpha(\nu)}).$$

⁵ Their argument concerned the synchronous setting, but it can be reused without change for the asynchronous setting.

Setting $\varepsilon' = \min\{\varepsilon, c_2\}/16$ yields

$$\begin{aligned} & \sum_{\tau \in (t_a, t]} [\Phi(\alpha(\tau)) - \Phi(\tau)] \\ & \geq \varepsilon' \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot \Phi(\alpha(\nu)) - \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \\ & \quad + \left[\sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} + A(t_a) - A(t) - \frac{c_2}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot A(\alpha(\nu)) \right]. \quad (6) \end{aligned}$$

In the subsections below, we will prove that for a suitable choice of the Γ parameters and c_1, c_2 , the final two terms of (6), in sum, are non-negative. Also, we will show that Φ is decreasing over time. With these, the above inequality implies that

$$\Phi(t_a) - \Phi(t) = \sum_{\tau \in (t_a, t]} [\Phi(\alpha(\tau)) - \Phi(\tau)] \geq \varepsilon' \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot \Phi(\alpha(\nu)) \geq \varepsilon' \cdot \Phi(t),$$

and hence $\Phi(t) \leq \Phi(t_a)/(1 + \varepsilon')$. By iterating this (note that $t > t_a \geq t - 3$), we obtain

$$\phi(\mathbf{p}^t) \leq \Phi(t) \leq (1 + \varepsilon')^{-(t/3-1)} \cdot \Phi(0) = (1 + \varepsilon')^{-(t/3-1)} \cdot \phi(\mathbf{p}^0),$$

thus demonstrating linear convergence.

4.1 Φ is a Decreasing Function

For any time τ at which there is an update, by Lemma 3 and the definition of A , we have

$$\begin{aligned} & \Phi(\alpha(\tau)) - \Phi(\tau) \\ & \geq \frac{\Gamma_{k_\tau}^\tau}{4} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{1}{\Gamma_{k_\tau}^\tau} (z_{k_\tau}^\tau - \tilde{z}_{k_\tau}^\tau)^2 \cdot \Delta t_\tau + c_1 \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \nu+1]} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\ & \quad - 2c_1 \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_{k_\tau}^\tau}{p_{k_\tau}^{\tau-}} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau}. \end{aligned}$$

Next,

$$\begin{aligned} & (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \\ & \leq \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \tau]} |\Delta p_{k_\nu}^\nu| \right)^2 \\ & = \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} \left(L_{k_\nu, k_\tau}^{[\nu, \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \right) \cdot \left(\frac{|\Delta p_{k_\nu}^\nu|}{\Delta t_\nu} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \right) \right)^2 \\ & \leq \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \right) \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \tau]} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \right) \\ & \hspace{20em} \text{(by Lemma 4)} \\ & \leq \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \right) \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \nu+1]} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \right). \quad (7) \end{aligned}$$

18:10 Amortized Analysis of Asynchronous Price Dynamics

Combining the above two equations and recalling that $\Delta t_\tau \leq 1$ yields

$$\begin{aligned}
& \Phi(\alpha(\tau)) - \Phi(\tau) \\
& \geq \left(\frac{\Gamma_{k_\tau}^\tau}{4} - 2c_1 \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \right) \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} \\
& \quad + \left(c_1 - \frac{1}{\Gamma_{k_\tau}^\tau} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \right) \\
& \quad \cdot \left(\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \nu+1]} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \right). \tag{8}
\end{aligned}$$

Thus, for Φ to be decreasing, we impose the following conditions (the second one is stronger than what is needed at this point):

$$\Gamma_{k_\tau}^\tau \geq 8c_1 \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \quad \text{and} \quad \Gamma_{k_\tau}^\tau \geq \frac{2}{c_1} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}}. \tag{9}$$

4.2 The Sum of the Last Two Terms in (6) is Non-negative

It remains to show that the sum of the last two terms in (6) is non-negative, i.e.,

$$\begin{aligned}
& \sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{c_2}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot A(\alpha(\nu)) + A(t_a) - A(t) \\
& \geq \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2. \tag{10}
\end{aligned}$$

We first simplify the LHS using the definition of A :

$$\begin{aligned}
& \sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{c_2}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot A(\alpha(\nu)) + A(t_a) - A(t) \\
& \geq \sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \cdot \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{c_2}{16} \sum_{\nu \in (t_a-1, t_b]} 4c_1 \sum_{k \neq k_\nu} L_{k_\nu, k}^{[\nu, \nu+1]} \cdot \frac{p_k^\nu}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\
& \quad + c_1 \sum_{\nu \in (t_a-1, t_a]} \sum_{k \neq k_\nu} L_{k_\nu, k}^{[\nu, \nu+1]} \cdot \frac{p_k^\nu}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\
& \quad - 2c_1 \sum_{\nu \in (t-1, t]} \sum_{k \neq k_\nu} L_{k_\nu, k}^{[\nu, \nu+1]} \cdot \frac{p_k^\nu}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\
& \geq \left(c_1 - \frac{c_1 c_2}{4} \right) \sum_{\nu \in (t_a-1, t_a]} \sum_{k \neq k_\nu} L_{k_\nu, k}^{[\nu, \nu+1]} \cdot \frac{p_k^\nu}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\
& \quad + \sum_{\tau \in (t_a, t]} \left[\frac{\Gamma_{k_\tau}^\tau}{8} - \left(\frac{c_1 c_2}{4} + 2c_1 \right) \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \right] \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau}.
\end{aligned}$$

By imposing the requirement that $\Gamma_{k_\tau}^\tau \geq 8c_3 \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}}$, for some constant

$c_3 > 0$ which we will determine later, we obtain

$$\begin{aligned} & \sum_{\tau \in (t_a, t]} \frac{\Gamma_{k_\tau}^\tau}{8} \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau} - \frac{c_2}{16} \sum_{\nu \in (t_a, t_b]} \delta_\nu \cdot A(\alpha(\nu)) + A(t_a) - A(t) \\ & \geq \min \left\{ c_1 - \frac{c_1 c_2}{4}, c_3 - \frac{c_1 c_2}{4} - 2c_1 \right\} \cdot \sum_{\tau \in (t_a - 1, t]} \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \frac{(\Delta p_{k_\tau}^\tau)^2}{\Delta t_\tau}. \end{aligned}$$

On the other hand, by (7) and by the second condition imposed in (9),

$$\begin{aligned} & \sum_{\tau \in (t_a, t]} \frac{9}{8\Gamma_{k_\tau}^\tau} \cdot (z_{k_\tau}^{\max, \tau} - z_{k_\tau}^{\min, \tau})^2 \leq \frac{9c_1}{16} \sum_{\tau \in (t_a, t]} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \nu+1]} \cdot \frac{p_{k_\tau}^{\tau-}}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\ & = \frac{9c_1}{16} \sum_{\tau \in (t_a, t]} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\nu, \nu+1]} \cdot \frac{p_{k_\tau}^\nu}{p_{k_\nu}^{\nu-}} \cdot \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu} \\ & \leq \frac{9c_1}{16} \sum_{\nu \in (t_a - 1, t]} \sum_{k \neq k_\nu} L_{k_\nu, k}^{[\nu, \nu+1]} \cdot \frac{p_k^\nu}{p_{k_\nu}^{\nu-}} \frac{(\Delta p_{k_\nu}^\nu)^2}{\Delta t_\nu}. \end{aligned}$$

By the above two inequalities, to satisfy (10), it suffices to have $\frac{9c_1}{16} \leq \min \left\{ c_1 - \frac{c_1 c_2}{4}, c_3 - \frac{c_1 c_2}{4} - 2c_1 \right\}$. There are multiple possible choices for c_1, c_2, c_3 . We choose $c_3 = 21c_1/8$ and $c_2 = 1/4$. To summarize, we need the Γ parameters to satisfy

$$\Gamma_{k_\tau}^\tau \geq 21c_1 \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \quad \text{and} \quad \Gamma_{k_\tau}^\tau \geq \frac{2}{c_1} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau)} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}}. \quad (11)$$

Our remaining tasks are to derive upper bounds on the two summations in (11).

4.3 Upper Bounds on the Local Lipschitz Parameters, and Determining the Γ 's

Suppose in a Fisher market with buyers having CES utility functions, each buyer i has a budget of e_i , and her CES utility function has parameter ρ_i . For each i , let $\theta_i := \rho_i / (\rho_i - 1)$. As we have discussed in Section 2, at any given price vector $\mathbf{p} \in \mathbb{R}_+^n$, buyer i computes the demand-maximizing bundle of goods costing at most e_i ; we let $x_{i\ell}(\mathbf{p})$ denote buyer i 's demand for good ℓ at price vector \mathbf{p} .

In a Fisher market with buyers having complementary-CES utility functions, the following properties are well-known. (See [14].)

1. For any $k \neq j$,

$$\left| \frac{\partial^2 \phi}{\partial p_j \partial p_k}(\mathbf{p}) \right| = \sum_{i=1}^m \frac{\theta_i x_{ij}(\mathbf{p}) x_{ik}(\mathbf{p})}{e_i} \leq \sum_{i=1}^m \frac{x_{ij}(\mathbf{p}) x_{ik}(\mathbf{p})}{e_i}.$$

2. Given positive price vector \mathbf{p} , for any $0 < r_1 < r_2$, let \mathbf{p}' be prices such that for all ℓ , $r_1 p_\ell \leq p'_\ell \leq r_2 p_\ell$. Then for all ℓ , $\frac{1}{r_2} x_\ell(\mathbf{p}) \leq x_\ell(\mathbf{p}') \leq \frac{1}{r_1} x_\ell(\mathbf{p})$.

► **Lemma 6.** *If the parameter λ in update rule (2) is at most $1/10$, then*

$$\sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} \leq e^{4\lambda(\lambda+1)} \cdot \frac{x_{k_\tau}(\mathbf{p}^{\tau-})}{p_{k_\tau}^{\tau-}}$$

18:12 Amortized Analysis of Asynchronous Price Dynamics

and

$$\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau]} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \leq 2e^{8\lambda(\lambda+1)} \cdot \frac{x_{k_\tau}(\mathbf{p}^{\tau-})}{p_{k_\tau}^{\tau-}}.$$

Proof. Let $\lambda' = 2\lambda(\lambda + 1)$. Since $\lambda \leq 1/10$, it is easy to observe that for any $\nu \in [\tau, \tau + 1]$ and for any k (including coordinate k_τ),

$$e^{-\lambda'} \cdot p_k^{\tau-} \leq p_k^\nu \leq e^{\lambda'} \cdot p_k^{\tau-}, \quad (12)$$

on noting that the Δt_ν terms span up to 2 time units.

Accordingly, let $\tilde{P} := \left\{ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \mid \forall k \in [n], e^{-\lambda'} \cdot p_k^{\tau-} \leq \tilde{p}_k \leq e^{\lambda'} \cdot p_k^{\tau-} \right\}$. Then,

$$\begin{aligned} \sum_{k \neq k_\tau} L_{k_\tau, k}^{[\tau, \tau+1]} \cdot \frac{p_k^\tau}{p_{k_\tau}^{\tau-}} &\leq \frac{1}{p_{k_\tau}^{\tau-}} \sum_{k \neq k_\tau} \left(\max_{\tilde{p} \in \tilde{P}} \left| \frac{\partial^2 \phi}{\partial p_{k_\tau} \partial p_k}(\tilde{p}) \right| \right) \cdot p_k^\tau \\ &\leq \frac{1}{p_{k_\tau}^{\tau-}} \sum_{k \neq k_\tau} \sum_{i=1}^m \frac{(e^{\lambda'} x_{ik_\tau}(\mathbf{p}^{\tau-})) \cdot (e^{\lambda'} x_{ik}(\mathbf{p}^{\tau-}))}{e_i} \cdot p_k^{\tau-} \quad (\text{by Properties 1 and 2}) \end{aligned} \quad (13)$$

$$\begin{aligned} &\leq \frac{e^{2\lambda'}}{p_{k_\tau}^{\tau-}} \sum_{i=1}^m x_{ik_\tau}(\mathbf{p}^{\tau-}) \sum_{k \neq k_\tau} \frac{x_{ik}(\mathbf{p}^{\tau-}) \cdot p_k^{\tau-}}{e_i} \\ &\leq \frac{e^{2\lambda'}}{p_{k_\tau}^{\tau-}} \sum_{i=1}^m x_{ik_\tau}(\mathbf{p}^{\tau-}) \quad (\text{the second summation } \leq 1, \text{ due to the budget constraint}) \\ &= e^{2\lambda'} \cdot \frac{x_{k_\tau}(\mathbf{p}^{\tau-})}{p_{k_\tau}^{\tau-}}. \end{aligned} \quad (14)$$

For the time range $\nu \in [\alpha_{k_\tau}(\tau), \tau]$, inequality (12) also holds. Thus,

$$\begin{aligned} &\sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau]} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot \frac{p_{k_\nu}^{\nu-}}{p_{k_\tau}^{\tau-}} \\ &\leq \frac{1}{p_{k_\tau}^{\tau-}} \sum_{\nu \in (\alpha_{k_\tau}(\tau), \tau]} L_{k_\nu, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot \Delta t_\nu \cdot e^{2\lambda'} \cdot p_{k_\nu}^{\tau-} \\ &\leq \frac{e^{2\lambda'}}{p_{k_\tau}^{\tau-}} \sum_{k \neq k_\tau} L_{k, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot p_k^{\tau-} \cdot \sum_{\substack{\nu \in (\alpha_{k_\tau}(\tau), \tau) \\ k_\nu = k}} \Delta t_\nu \\ &\leq \frac{2e^{2\lambda'}}{p_{k_\tau}^{\tau-}} \sum_{k \neq k_\tau} L_{k, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot p_k^{\tau-}. \quad (\text{observe that the } \sum_{\nu} \Delta t_\nu \text{ term is at most 2}) \end{aligned}$$

The summation $\sum_{k \neq k_\tau} L_{k, k_\tau}^{[\alpha_{k_\tau}(\tau), \tau]} \cdot p_k^{\tau-}$ above can be bounded as in (14), yielding an upper bound of $e^{2\lambda'} \cdot x_{k_\tau}(\mathbf{p}^{\tau-})$. \blacktriangleleft

To conclude, by (11) and Lemma 6, it suffices to have:

$$\frac{1}{\lambda p_{k_\tau}^{\tau-}} \cdot \max\{1, \tilde{z}_{k_\tau}\} = \Gamma_{k_\tau}^\tau \geq \max \left\{ 21c_1 e^{4\lambda(\lambda+1)}, \frac{4}{c_1} \cdot e^{8\lambda(\lambda+1)} \right\} \cdot \frac{x_{k_\tau}(\mathbf{p}^{\tau-})}{p_{k_\tau}^{\tau-}},$$

or equivalently, $\lambda \cdot \max \left\{ 21c_1 e^{4\lambda(\lambda+1)}, \frac{4}{c_1} \cdot e^{8\lambda(\lambda+1)} \right\} \leq \frac{\max\{1, \bar{z}_{k_T}\}}{x_{k_T}(\mathbf{P}^{\tau-})}$. The minimum possible value of the RHS above is $1/(2e^{2\lambda(\lambda+1)})$. Thus, we need that

$$\lambda \cdot \max \left\{ 42c_1 e^{6\lambda(\lambda+1)}, \frac{8}{c_1} \cdot e^{10\lambda(\lambda+1)} \right\} \leq 1.$$

We choose c_1 such that the two parameters in the max are equal, i.e., $c_1 = \frac{2}{\sqrt{21}} \cdot e^{2\lambda(\lambda+1)}$. Then the above inequality reduces to $4\sqrt{21} \cdot \lambda \cdot e^{8\lambda(\lambda+1)} \leq 1$; $\lambda \leq 1/25.5$ suffices.

References

- 1 Carlos Alós-Ferrer and Nick Netzer. The logit-response dynamics. *Games and Economic Behavior*, 68(2):413–427, 2010.
- 2 Kenneth J. Arrow, H. D. Block, and Leonid Hurwicz. On the stability of competitive equilibrium, ii. *Econometrica*, 27(1):82–109, 1959.
- 3 Petra Berenbrink, Tom Friedetzky, Leslie Ann Goldberg, Paul W. Goldberg, Zengjian Hu, and Russell A. Martin. Distributed selfish load balancing. *SIAM J. Comput.*, 37(4):1163–1181, 2007. doi:10.1137/060660345.
- 4 Benjamin Birnbaum, Nikhil R. Devanur, and Lin Xiao. Distributed algorithms via gradient descent for fisher markets. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, EC '11, pages 127–136. ACM, 2011. doi:10.1145/1993574.1993594.
- 5 Lawrence E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387–424, 1993.
- 6 Bernard Chazelle. Natural algorithms. In *SODA*, pages 422–431, 2009.
- 7 Bernard Chazelle. The dynamics of influence systems. In *FOCS*, pages 311–320, 2012.
- 8 X. Chen, D. Dai, Y. Du, and S. H. Teng. Settling the complexity of arrow-debreu equilibria in markets with additively separable utilities. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 273–282, 2009. doi:10.1109/FOCS.2009.29.
- 9 Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player nash equilibria. *J. ACM*, 56(3):14:1–14:57, may 2009.
- 10 Xi Chen, Dimitris Pappas, and Mihalis Yannakakis. The complexity of non-monotone markets. *J. ACM*, 64(3):20:1–20:56, 2017. doi:10.1145/3064810.
- 11 Yun Kuen Cheung. *Analyzing Tatonnement Dynamics in Economics Markets*. PhD thesis, Courant Institute of Mathematical Sciences, New York University, Proquest Dissertations Publishing, 2014. Available at https://cs.nyu.edu/media/publications/cheung_yunkuen.pdf.
- 12 Yun Kuen Cheung and Richard Cole. Amortized analysis on asynchronous gradient descent. *CoRR*, abs/1412.0159, 2014.
- 13 Yun Kuen Cheung, Richard Cole, and Nikhil Devanur. Tatonnement beyond gross substitutes? Gradient descent to the rescue. In *STOC*, pages 191–200, 2013. Full version available at https://cims.nyu.edu/~ykcheung/publication/STOC13_full_paper.pdf. doi:10.1145/2488608.2488633.
- 14 Yun Kuen Cheung, Richard Cole, and Ashish Rastogi. Tatonnement in ongoing markets of complementary goods. In *EC*, pages 337–354, 2012. doi:10.1145/2229012.2229039.
- 15 Yun Kuen Cheung, Richard Cole, and Yixin Tao. Dynamics of distributed updating in fisher markets. In *Proceedings of the 2018 ACM Conference on Economics and Computation, Ithaca, NY, USA, June 18–22, 2018*, EC '18, pages 351–368, 2018. doi:10.1145/3219166.3219189.
- 16 Yun Kuen Cheung, Richard Cole, and Yixin Tao. A unified approach to analyzing asynchronous coordinate descent — standard and partitioned. *Submitted*, 2018.

- 17 Bruno Codenotti, Benton McCune, and Kasturi Varadarajan. Market equilibrium via the excess demand function. In *STOC*, pages 74–83, 2005. doi:10.1145/1060590.1060601.
- 18 Bruno Codenotti, Amin Saberi, Kasturi Varadarajan, and Yinyu Ye. Leontief economies encode nonzero sum two-player games. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm*, SODA '06, pages 659–667. Society for Industrial and Applied Mathematics, 2006.
- 19 Richard Cole and Lisa Fleischer. Fast-converging tatonnement algorithms for one-time and ongoing market problems. In *STOC*, pages 315–324, 2008. doi:10.1145/1374376.1374422.
- 20 Richard Cole, Lisa Fleischer, and Ashish Rastogi. Discrete price updates yield fast convergence in ongoing markets with finite warehouses. *CoRR*, 2010.
- 21 Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. The complexity of computing a nash equilibrium. *SIAM J. Comput.*, 39(1):195–259, 2009.
- 22 Nikhil R. Devanur, Christos H. Papadimitriou, Amin Saberi, and Vijay V. Vazirani. Market equilibrium via a primal-dual algorithm for a convex program. *J. ACM*, 55(5):22:1–22:18, 2008.
- 23 Akitaka Dohtani. Global stability of the competitive economy involving complementary relations among commodities. *Journal of Mathematical Economics*, 22(1):73–83, 1993.
- 24 Ran Duan and Kurt Mehlhorn. A combinatorial polynomial algorithm for the linear arrow-debreu market. *Inf. Comput.*, 243:112–132, 2015. doi:10.1016/j.ic.2014.12.009.
- 25 Krishnamurthy Dvijotham, Yuval Rabani, and Leonard J. Schulman. Convergence of incentive-driven dynamics in fisher markets. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '17, pages 554–567. Society for Industrial and Applied Mathematics, 2017.
- 26 Eyal Even-Dar, Alexander Kesselman, and Yishay Mansour. Convergence time to nash equilibrium in load balancing. *ACM Trans. Algorithms*, 3(3):32, 2007. doi:10.1145/1273340.1273348.
- 27 K. Jain. A polynomial time algorithm for computing the arrow-debreu market equilibrium for linear utilities. In *Forty Fifth Annual IEEE Symposium on Foundations of Computer Science*, FOCS'04, pages pp. 286–294, Rome, Italy, 2004.
- 28 Jonathan A. Kelner, Lorenzo Orecchia, Aaron Sidford, and Zeyuan Allen Zhu. A simple, combinatorial algorithm for solving SDD systems in nearly-linear time. In *STOC*, pages 911–920, 2013.
- 29 Yin Tat Lee and Aaron Sidford. Efficient accelerated coordinate descent methods and faster algorithms for solving linear systems. In *FOCS*, pages 147–156, 2013.
- 30 C. E. Lemke and J. T. Howson Jr. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):413–423, 1964.
- 31 Jure Leskovec, Lars Backstrom, and Jon M. Kleinberg. Meme-tracking and the dynamics of the news cycle. In *KDD*, pages 497–506, 2009.
- 32 Jason R. Marden and Jeff S. Shamma. Revisiting log-linear learning: Asynchrony, completeness and payoff-based implementation. *Games and Economic Behavior*, 75(2):788–808, 2012.
- 33 James B. Orlin. Improved algorithms for computing Fisher's market clearing prices. In *Proceedings of the Forty Second Annual ACM Symposium on Theory of Computing*, STOC'10, pages 291–300, 2010.
- 34 Siddharth Pal and Richard J. La. Simple learning in weakly acyclic games and convergence to Nash equilibria, 2015. URL: <http://www.ece.umd.edu/~hyong1a/PAPERS/ALLERTON15.pdf>.
- 35 Christos H. Papadimitriou and Mihalis Yannakakis. An impossibility theorem for price-adjustment mechanisms. *PNAS*, 5(107):1854–1859, 2010.

- 36 Hirofumi Uzawa. Walras' tatonnement in the theory of exchange. *Review of Economic Studies*, 27(3):182–194, 1960.
- 37 Vijay V. Vazirani and Mihalis Yannakakis. Market equilibrium under separable, piecewise-linear, concave utilities. *J. ACM*, 58(3):10:1–10:25, 2011.
- 38 Léon Walras. *Eléments d'Économie Politique Pure*. Corbaz, 1874. (Translated as: *Elements of Pure Economics*. Homewood, IL: Irwin, 1954.).
- 39 Fang Wu and Li Zhang. Proportional response dynamics leads to market equilibrium. In *Proceedings of the Thirty-ninth Annual ACM Symposium on Theory of Computing*, STOC '07, pages 354–363. ACM, 2007. doi:10.1145/1250790.1250844.
- 40 Yinyu Ye. A path to the arrow-debreu competitive market equilibrium. *Math. Program.*, 111(1-2):315–348, 2008. doi:10.1007/s10107-006-0065-5.
- 41 Li Zhang. Proportional response dynamics in the fisher market. *Theor. Comput. Sci.*, 412(24):2691–2698, 2011.