Communication Complexity of Correlated Equilibrium with Small Support

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Abstract

We define a two-player $N \times N$ game called the 2-cycle game, that has a unique pure Nash equilibrium which is also the only correlated equilibrium of the game. In this game, every $1/\mathrm{poly}(N)$-approximate correlated equilibrium is concentrated on the pure Nash equilibrium. We show that the randomized communication complexity of finding any $1/\mathrm{poly}(N)$-approximate correlated equilibrium of the game is $\Omega(N)$. For small approximation values, our lower bound answers an open question of Babichenko and Rubinstein (STOC 2017).

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1 Introduction

> If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.
>
> *Roger Myerson*

One of the most famous solution concepts in game theory is Nash equilibrium [28]. Roughly speaking, a Nash equilibrium is a set of mixed strategies, one per player, from which no player has an incentive to deviate. A well-studied computational problem in algorithmic game theory is that of finding a Nash equilibrium of a (non-cooperative) game. Since finding a Nash equilibrium is considered hard (in particular, it is a PPAD-complete problem), researchers studied the problem of finding an approximate Nash equilibrium, where intuitively, no player can benefit much by deviating from his mixed strategy. The complexity of finding an approximate Nash equilibrium has been studied in several models of computation, including computational complexity, query complexity and communication

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complexity. For surveys on algorithmic game theory in general and equilibria in particular see for example [29, 32, 13, 34].

When the players reach an equilibrium they learn to predict correctly the actions of the other players. To better understand how players might learn the actions and payoffs of other players, there is an extensive study of learning dynamics and their convergence to equilibria, see for example [22, 37, 19]. One natural class of dynamics in which approximate equilibria concepts are studied is that of uncoupled dynamics [17, 18], where each player knows his own utilities and not those of the other players. The rate of convergence of uncoupled dynamics to an approximate equilibrium is closely related to the communication complexity of finding the approximate equilibrium [8].

Communication complexity is a central model in complexity theory that has been extensively studied. In the two-player randomized model of Yao [36], each player gets an input and their goal is to solve a communication task that depends on both inputs. The players can use both common and private random coins and are allowed to err with some small probability. The communication complexity of a protocol is the total number of bits communicated by the two players. The communication complexity of a communication task is the minimum number of bits that the players need to communicate in order to solve the task with high probability, where the minimum is taken over all protocols. For surveys on communication complexity see for example [25, 26, 33].

An important generalization of Nash equilibrium is correlated equilibrium [1, 2]. Whereas in a Nash equilibrium the players choose their strategies independently, in a correlated equilibrium the players can coordinate their decisions, choosing a joint strategy. There are two notions of correlated equilibrium which we call *correlated equilibrium* (CE) and *rule correlated equilibrium* (RCE)[3]. In a CE no player can benefit from replacing one action with another, whereas in a RCE no player can benefit from simultaneously replacing every action with another action (using a switching rule). While the above two notions are equivalent, approximate CE and approximate RCE are not equivalent, but are closely related.

The communication task of finding an approximate (rule) correlated equilibrium is as follows. The actions sets and the approximation value are known to both players. Each player gets a utility function that specify her payoffs for every pair of actions (given as a truth table). At the end of the communication both players should know the same correlated mixed strategy which is an approximate (rule) correlated equilibrium.

In the multi-party setting, [16, 30, 21] showed protocols for finding an exact CE of N-player binary action games with $\text{poly}(N)$ bits of communication (note that the input size per player is 2^N). In the two-player setting, every $N \times N$ game has a trivial $1/N$-approximate CE (the uniform distribution over all pairs of actions, which can be found with zero communication). However, there is no trivial approximate RCE, even for constant approximation values. Babichenko and Rubinstein [6] raised the following questions:

> *Does a* $\text{polylog}(N)$ *communication protocol for finding an approximate* RCE *of two-player* $N \times N$ *games exist? Is there a* $\text{poly}(N)$ *communication complexity lower bound?*

1.1 Our Main Result

We show a communication complexity lower bound for finding a $1/\text{poly}(N)$-approximate CE of a two-player $N \times N$ game that we call the 2-cycle game. Since every approximate RCE is an approximate CE, the same lower bound holds for RCE.

[3] In most literature, both notions of correlated equilibrium are referred to by the same name. In this paper, we choose to distinguish between them.

▶ **Theorem 1.** *Let $n \geq 3$ be an odd integer, let $N = 2n$ and $\varepsilon \leq \frac{1}{4N^3}$. Then, every randomized communication protocol for finding an ε-approximate correlated equilibrium of the 2-cycle $N \times N$ game, with error probability at most $\frac{1}{3}$, has communication complexity at least $\Omega(N)$.*

As far as we know, there were no communication complexity lower bounds for (exact) CE or RCE of two-player games prior to this work. Note that Theorem 1 implies a lower bound of $\Omega(N)$ on the number of queries to the utility functions, for any randomized query algorithm that finds a $1/\text{poly}(N)$-approximate CE (or RCE) of the 2-cycle $N \times N$ game, with probability at least $2/3$.

After the first version of this paper was published online two similar results were proved. Babichenko [4] showed that any $1/\text{poly}(N)$-approximate RCE in the generalized matching-pennies game reveals the entire input of one of the players, thus proved the same lower bound as in Theorem 1. Ko and Schvartzman [24] independently showed that for any $\Omega(1/N) < \varepsilon < 1/10$, the communication complexity of finding an ε-approximate RCE is $\Omega(\varepsilon^{-1/2} \log N)$. In our opinion, the main difference between our lower bound and the above results is that their games have no approximate RCE with succinct representation, while our game has a unique pure Nash equilibrium (which is also a RCE with one pair of actions in its support). We will elaborate on the importance of proving lower bounds for equilibria with succinct representations later in this section.

It remains a very interesting open problem to determine the communication complexity of finding a *constant*-approximate RCE of two-player games. Currently, there is an exponential gap between the best known lower and upper bounds on the communication complexity of finding a constant-approximate RCE of two-player $N \times N$ games, where the best known lower bound is logarithmic in N [24].

In a recent breakthrough, Babichenko and Rubinstein [6] proved the first non-trivial lower bound on the randomized communication complexity of finding an approximate Nash equilibrium. More precisely, they proved a lower bound of $\Omega\left(N^{\varepsilon_0}\right)$ on the randomized communication complexity of finding an ε-approximate Nash equilibrium of a two-player $N \times N$ game, for every $\varepsilon \leq \varepsilon_0$, where ε_0 is some small constant. Theorem 1 implies a randomized communication complexity lower bound of $\Omega(N)$ for finding a $1/\text{poly}(N)$-approximate Nash equilibrium of the 2-cycle game[4]. This is a slightly stronger lower bound but for much smaller approximation values. Our proof is more simple and straightforward, as it does not go through several intermediate problems.

The 2-cycle game is a very simple game, in the sense that it is a win-lose, sparse game, in which each player has a unique best response to every action. For the class of win-lose, sparse games, our lower bound is tight up to logarithmic factors, as a player can send his entire utility matrix using $O(N \log N)$ bits of communication. However, our lower bound does not hold for much larger approximation values, since there are examples of approximate equilibria of the 2-cycle game for larger approximation values, that can be found with small amount of communication (see Appendix A for details).

Correlated Equilibrium with Succinct Representation

In the communication model, for a problem to be meaningful, we would like the output size to be much smaller than the number of bits needed to solve it. Specifically, for the problem of finding an approximate RCE (or CE) in $N \times N$ games, we would like the output to have a

[4] We are able to improve the approximation parameter from $1/4N^3$ in Theorem 1 above to $1/16N^2$ in the case of approximate Nash equilibrium.

succinct representation of size polylog(N) bits. A natural notion of succinct representation is the support size of an equilibrium, thus we could define the communication problem associated with finding a RCE to be finding an approximate RCE of polylog(N) support size.

For $1/\text{poly}(N)$ approximation values, there are two-player $N \times N$ games for which every approximate RCE is of poly(N) support size. In contrast, Babichenko, Barman and Peretz [5] showed that every two-player game has a constant-approximate RCE of support size $O(\log N)$. Therefore, finding an approximate RCE of polylog(N) support size is a *total* search problem (i.e., a solution always exists) for constant approximation values[5], but is not total for $1/\text{poly}(N)$ approximation values.

As a step before understanding the total problem of finding a constant-approximate RCE, we consider *promise* problems, where we are guaranteed to have a RCE with a small support. Our lower bound for finding an approximate RCE implies that even for games in which we are promised to have a RCE with a small support, finding an approximate RCE remains hard.

1.2 Sampling from a Correlated Equilibrium

We introduce another natural communication task in the context of joint strategies, which is the task of sampling from a CE. Intuitively, in the task of sampling from a CE (or RCE)[6], the players are required to output each pair of actions with probability that is close to the probability of this pair of actions under some CE. More formally, at the end of the communication, each player outputs a *single* action, such that the distribution of the protocol on pairs of actions is close (say, Δ-close in ℓ_1 distance, for some small Δ) to some joint distribution which is a CE of the game. The above problem was suggested by Moni Naor [27].

The problem of finding a CE in the communication model requires that both players know at the end of the communication the entire joint strategy, which might be large. We believe that the easier task of sampling from a CE is interesting in real-life scenarios, since by sampling from an equilibrium of the game the players can act according to that equilibrium. Sampling communication tasks were studied in many different variants in different contexts, such as compression of randomized protocols, simulation of randomized protocols, agreement distillation, sketching algorithms, approximation algorithms based on rounding linear programming relaxations, the study of parallel repetition and cryptography.

When a game has a CE with a small support size, by sampling from this equilibrium the players can recover (learn) the equilibrium with high probability. However, it might be the case that the game has an approximate RCE with a large support from which sampling is easy, while finding (even approximately) the entire joint strategy is hard. In particular, a poly-logarithmic number of samples might not be enough to recover the equilibrium. For example, if one of the players knows a CE of the game, she can sample a pair of actions according to the equilibrium and send the other player his action. However, if the equilibrium she knows has no succinct representation, communicating it might be hard.

The 2-cycle game has a unique exact CE which is the pure Nash equilibrium of the game, and every $1/\text{poly}(N)$-approximate CE of the game is *concentrated* on the pure Nash equilibrium. Thus, by sampling a pair of actions from a $1/\text{poly}(N)$-approximate CE, the players can recover the pure Nash equilibrium with high probability. That is, not only finding a $1/\text{poly}(N)$-approximate CE of the game is hard, but sampling from such an equilibrium is also hard. Since every approximate RCE is an approximate CE, the following lower bound holds also for RCE.

[5] In fact, [5] showed that the problem is total for $1/\text{polylog}(N)$ approximation values.
[6] This problem can be naturally extended to sampling from an approximate correlated equilibrium.

▶ **Theorem 2.** *Let $n \geq 3$ be an odd integer, let $N = 2n$, $\Delta \in [0, 2/3)$ and $\varepsilon \leq 1/4N^3(2/3 - \Delta)$. Let μ be an ε-approximate correlated equilibrium of the 2-cycle $N \times N$ game. Then, every randomized communication protocol for sampling from a distribution that is Δ-close in ℓ_1 distance to μ, with error probability at most $\frac{1}{3}$, has communication complexity at least $\Omega(N)$.*

It remains a very interesting open problem to determine the communication complexity of sampling from a *constant*-approximate RCE of two-player games.

1.3 Proof Overview

In the 2-cycle $N \times N$ game, the utility functions are constructed from two subsets of $[n]$ where $n = N/2$. The two subsets have exactly one element in common. Each player is given one of these subsets and computes a directed graph. The two graphs have a common vertex set of size N. The actions of each player are the N vertices. In each graph, every vertex has a unique out-neighbor. Intuitively, each player wants to play the unique out-neighbor (according to his graph) of the vertex played by the other player.

The construction of the utility functions of the 2-cycle game was inspired by ideas of [35] of constructing utility functions from inputs to the fixed-point problem, that is, from continuous functions on a compact convex space. We construct the utility functions in the same way, but from discrete functions, i.e., the unique out-neighbor functions in directed graphs.

To understand how the equilibria of the game look like, we examine the union of the two graphs. An element in the intersection of the subsets creates a directed 2-cycle in the union of the two graphs, with one edge from each graph. Given the two vertices of this 2-cycle, one can recover the index of the element in the intersection of the subsets. The game has a unique (exact) equilibrium which is the two vertices of the 2-cycle (that is, a pure Nash equilibrium). Since it is hard to find the element in the intersection of the subsets, finding an equilibrium of the game is also hard.

The heart of the proof is to show that every $1/\text{poly}(N)$-approximate equilibrium is concentrated on the pure Nash equilibrium. For ease of presentation, we focus on the special case of approximate Nash equilibria. Let (a^*, b^*) be a $1/\text{poly}(N)$-approximate Nash equilibrium of the game. We say that a function $f : [N] \to [0, 1]$ is concentrated on $i \in [N]$ if $f(i) > f(j)$ for every $j \in [N] \setminus \{i\}$. We show that a^*, b^* are concentrated on u^* and v^* respectively, where (u^*, v^*) is the pure Nash equilibrium of the game. Hence given (a^*, b^*), the players can recover the pure Nash equilibrium of the game with no communication. Intuitively, for any vertex v, $a^*(v)$ cannot be large unless one of its neighbors (in the graph of player A) has large probability according to b^*. Similarly, $b^*(v)$ cannot be large unless one of v's neighbors (in the graph of player B) has large probability according to a^*. We use this property to bound a^* and b^* on all the vertices other than u^* and v^* respectively, one by one, moving along alternating edges from the two graphs.

It is interesting to see what happens when the construction of the game is used on two subsets that do not intersect. In this case, the union of the graphs has no 2-cycle and the game has no pure Nash equilibrium. The game has a unique exact Nash equilibrium (a^*, b^*), where a^* is uniform on half of the vertices that correspond to one subset and b^* is uniform on half of the vertices that correspond to the other subset. We note that this is not an equilibrium of the game when the subsets do intersect. For the actual 2-cycle game, constructed from intersecting subsets, we show that every $1/\text{poly}(N)$-approximate equilibrium reveals the pure Nash equilibrium of the game. Thus changing a single bit in the representation of the subsets affects every $1/\text{poly}(N)$-approximate equilibrium of the game. On a more technical note, we bound a^* and b^* on all vertices other than the ones in the 2-cycle, one by one, starting with a vertex v such that all incoming edges to v are from vertices with small probability according to b^*. Such a vertex does not exist if there is no element in the intersection of the subsets.

1.4 Related Works

We overview previous works related to the computation of correlated equilibria of two-player $N \times N$ games.

Computational complexity

An exact correlated equilibrium can be computed for two-player games in polynomial time by a linear program [20]. Additionally, the decision version of finding correlated equilibria with particular properties have also been considered in literature (for examples see [12, 7]).

Query complexity

Fearnley et al. [10] showed a deterministic query algorithm that finds a $1/2$-approximate Nash equilibrium by making $O(N)$ queries and Fearnley and Savani [11] showed a randomized query algorithm that finds a 0.382-approximate Nash equilibrium by making $O(N \log N)$ queries. For coarse correlated equilibrium, Goldberg and Roth [15] provided a randomized query algorithm that finds a constant approximate coarse correlated equilibrium by making $O(N \log N)$ queries.

Communication complexity

Goldberg and Pastink [14] showed a communication protocol that finds a 0.438-approximate Nash equilibrium by exchanging polylog(N) bits of communication, and Czumaj et al. [9] showed a communication protocol that finds a 0.382-approximate Nash equilibrium with similar communication.

2 Preliminaries

2.1 General Notation

For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$. For two bit strings $x, y \in \{0,1\}^*$, let xy be the concatenation of x and y. For a bit string $x \in \{0,1\}^n$ and an index $i \in [n]$, x_i is the $i^{th} + 1$ bit in x and $\bar{x}$ is the negated bit string, that is $\bar{x}_i$ is the negation of x_i. For a function $\mu : \Omega \to [0,1]$, where Ω is some finite set, and a subset $S \subseteq \Omega$, let

$$\mu(S) = \sum_{z \in S} \mu(z).$$

Define $\mu(\emptyset) = 0$ and $\max_{z \in \emptyset} \mu(z) = 0$. For $u \in \Omega$ we say that μ is *concentrated on* u if

$$\mu(u) > \mu(v) \qquad \forall\ v \in \Omega \setminus \{u\}.$$

For a function $\mu : \mathcal{U} \times \mathcal{V} \to [0,1]$, where $\mathcal{U}, \mathcal{V}$ are some finite sets, a subset $S \subseteq \mathcal{U}$ and $v \in \mathcal{V}$, let

$$\mu(S, v) = \sum_{u \in S} \mu(u, v).$$

Similarly, for a subset $S \subseteq \mathcal{V}$ and $u \in \mathcal{U}$ let $\mu(u, S) = \sum_{v \in S} \mu(u, v)$. Define $\mu(\emptyset, v) = \mu(u, \emptyset) = 0$.

2.2 Approximate Correlated Equilibrium

A win-lose, finite game for two players A and B is given by two utility functions $u_A : \mathcal{U}\times\mathcal{V} \to \{0,1\}$ and $u_B : \mathcal{U}\times\mathcal{V} \to \{0,1\}$, where $\mathcal{U}$ and $\mathcal{V}$ are finite sets of actions. We say that the game is an $N \times N$ game, where $N = \max\{|\mathcal{U}|, |\mathcal{V}|\}$. A *mixed strategy* for player A is a distribution over $\mathcal{U}$ and a mixed strategy for player B is a distribution over $\mathcal{V}$. A mixed strategy is called *pure* if it has only one action in its support. A *correlated mixed strategy* is a distribution over $\mathcal{U}\times\mathcal{V}$. A *switching rule* for player A is a mapping from $\mathcal{U}$ to $\mathcal{U}$ and a switching rule for player B is a mapping from $\mathcal{V}$ to $\mathcal{V}$.

▶ **Definition 3** (Approximate Correlated Equilibrium)**.** Let $\varepsilon \in [0,1)$. An ε-approximate correlated equilibrium of a two-player game is a correlated mixed strategy μ such that the following two conditions hold:

1. For all actions $u, u' \in \mathcal{U}$,
$$\sum_{v\in\mathcal{V}} \mu(u,v)\cdot(u_A(u',v) - u_A(u,v)) \le \varepsilon.$$
2. For all actions $v, v' \in \mathcal{V}$,
$$\sum_{u\in\mathcal{U}} \mu(u,v)\cdot(u_B(u,v') - u_B(u,v)) \le \varepsilon.$$

▶ **Definition 4** (Approximate Rule Correlated Equilibrium)**.** Let $\varepsilon \in [0,1)$. An ε-approximate rule correlated equilibrium of a two-player game is a correlated mixed strategy μ such that the following two conditions hold:

1. For every switching rule f for player A,
$$\mathbb{E}_{(u,v)\sim\mu}\left[u_A(f(u),v) - u_A(u,v)\right] \le \varepsilon.$$
2. For every switching rule f for player B,
$$\mathbb{E}_{(u,v)\sim\mu}\left[u_B(u,f(v)) - u_B(u,v)\right] \le \varepsilon.$$

When the approximation value is zero the two notions above coincide. In general, every approximate rule correlated equilibrium is an approximate correlated equilibrium.

▶ **Proposition 5.** *Fix an N action two-player game and let $\varepsilon \in [0,1)$. Then, every ε-approximate rule correlated equilibrium of the game is an ε-approximate correlated equilibrium of the game. In the other direction, every ε-approximate correlated equilibrium of the game is an $(\varepsilon\cdot N)$-approximate rule correlated equilibrium of the game.*

The communication task of finding an ε-approximate (rule) correlated equilibrium is as follows. Consider a win-lose, finite game for two players A and B, given by two utility functions $u_A : \mathcal{U}\times\mathcal{V} \to \{0,1\}$ and $u_B : \mathcal{U}\times\mathcal{V} \to \{0,1\}$.

Inputs: The actions sets $\mathcal{U}, \mathcal{V}$ and the approximation value ε are known to both players. Player A gets the utility function u_A and player B gets the utility function u_B. The utility functions are given as truth tables of size $|\mathcal{U}| \times |\mathcal{V}|$ each.

At the end of the communication: Both players know the same correlated mixed strategy μ over $\mathcal{U}\times\mathcal{V}$, such that μ is an ε-approximate (rule) correlated equilibrium.

▶ Remark. Note that the communication complexity of a communication protocol for finding an ε-approximate (rule) correlated equilibrium is the total number of bits exchanged between the two players, which might be smaller than the number of bits required to describe the correlated mixed strategy μ to an observer with no prior information.

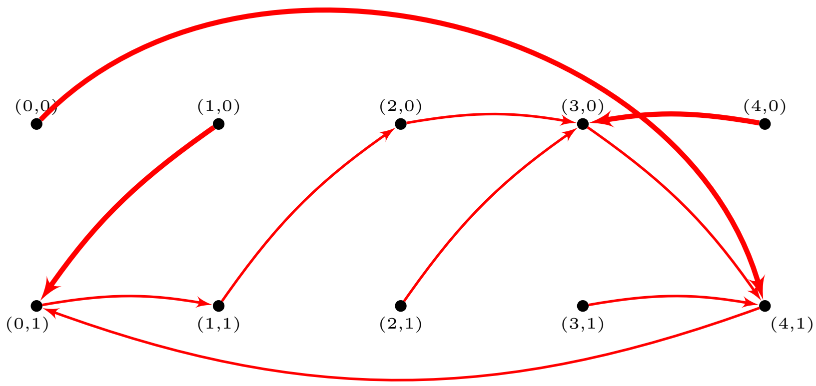

Figure 1 The graph G_A built from the 5 bit string 11001. The thick edges are the edges going back (of the form $((i,0),(i-1,x_{i-1}))$).

3 The 2-Cycle Game

Let $n \geq 3$ be an odd integer. The 2-cycle game is a win-lose, $N \times N$ game, where $N = 2n$. It is constructed from two n-bit strings $x, y \in \{0,1\}^n$ for which there exists exactly one index $i \in [n]$, such that $x_i > y_i$. Throughout the paper, all operations (adding and subtracting) are done modulo n.

The graphs

Given a string $x \in \{0,1\}^n$, player A computes the graph G_A on the set of vertices $V = [n] \times \{0,1\}$ with the following set of directed edges (an edge (u,v) is directed from u into v):

$$\begin{aligned} E_A = &\Big\{((i,1),(i+1,x_{i+1})) : i \in [n]\Big\} \\ &\cup \Big\{((i,0),(i+1,x_{i+1})) : i \in [n], x_i = 0\Big\} \\ &\cup \Big\{((i,0),(i-1,x_{i-1})) : i \in [n], x_i = 1\Big\}. \end{aligned}$$

See an example of such a graph in Figure 1.

Given a string $y \in \{0,1\}^n$, player B computes the graph G_B on the same set of vertices V with the following set of directed edges:

$$E_B = \Big\{((i,z),(i+1,y_{i+1})) : i \in [n], z \in \{0,1\}\Big\}.$$

See an example of such a graph in Figure 2.

The actions and utility functions

The sets of actions are $\mathcal{U} = \mathcal{V} = V$. Intuitively, each player wants to play the unique out-neighbor (according to his graph) of the vertex played by the other player. For example, if player B plays vertex v then player A wants to play the vertex u such that $(v,u) \in E_A$.

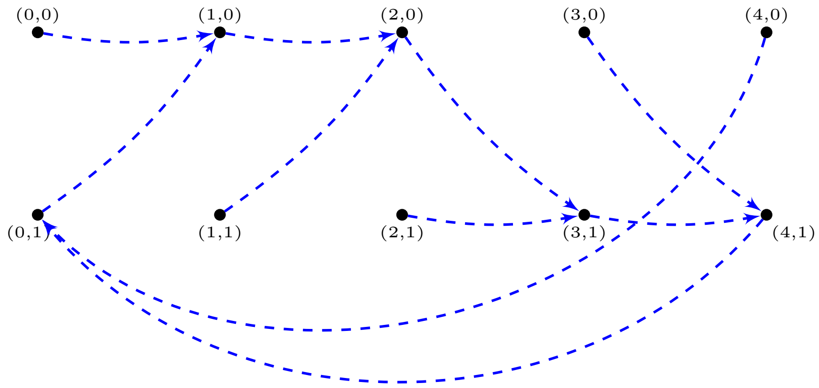

Figure 2 The graph G_B built from the 5 bit string 10011.

Formally, the utility function $u_A : V^2 \to \{0,1\}$ of player A is defined for every pair of actions $(u,v) \in V^2$ as

$$u_A(u,v) = \begin{cases} 1 & \text{if } (v,u) \in E_A \\ 0 & \text{otherwise} \end{cases} .$$

The utility function $u_B : V^2 \to \{0,1\}$ of player B is defined for every pair of actions $(u,v) \in V^2$ as

$$u_B(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E_B \\ 0 & \text{otherwise} \end{cases} .$$

Notations and basic properties

For two vertices $u, v \in V$, (u,v) is a *2-cycle* if $(v,u) \in E_A$ and $(u,v) \in E_B$. For a vertex $u \in V$, define

$$N_A(u) = \{v \in V \ : \ (v,u) \in E_A\}$$
$$N_B(u) = \{v \in V \ : \ (v,u) \in E_B\}.$$

That is, $N_A(u)$ is the set of incoming neighbors to u in E_A, and $N_B(u)$ is the set of incoming neighbors to u in E_B. Let $d_A(u) = |N_A(u)|$ and $d_B(u) = |N_B(u)|$. For a subset $S \subseteq V$, define

$$N_A(S) = \cup_{v \in S} N_A(v)$$
$$N_B(S) = \cup_{v \in S} N_B(v).$$

Edges in E_A of the form $((i,0),(i-1,x_{i-1}))$ for $i \in [n]$ are called *back-edges*. Let x, y be the strings from which the game was constructed. Note that u_A determines x, and u_B determines y. For an index $i \in [n]$ we say that i is *disputed* if $x_i > y_i$. Otherwise, we say that i is *undisputed*. Define i^* to be the unique disputed index. We denote the following key vertices:

$$u^* = (i^* - 1, x_{i^*-1})$$
$$v^* = (i^*, 0) = (i^*, y_{i^*}).$$

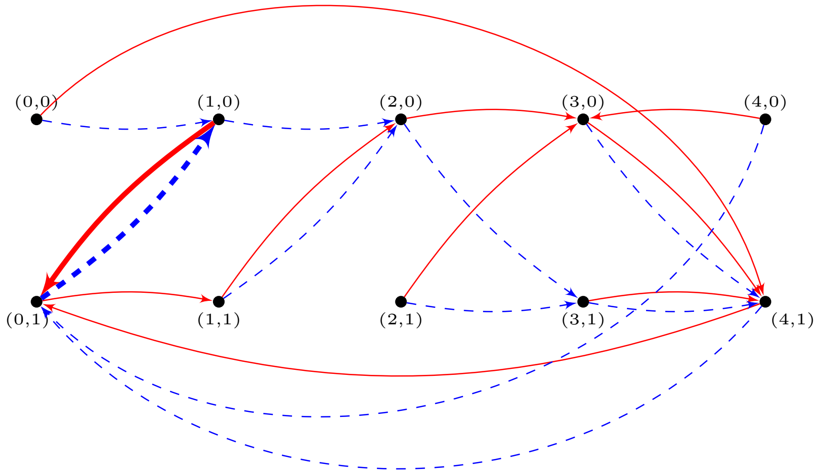

Figure 3 The 2-cycle in the union of the graphs G_A from Figure 1 and G_B from Figure 2.

To simplify notations, for a function f taking inputs from the set V and a vertex $v = (i, z) \in V$, we write $f(i, z)$ instead of $f((i, z))$.

The following are some useful, basic properties of the 2-cycle game.

▶ **Proposition 6** (Out-degree). *For every $v \in V$, there exists exactly one $u \in V$ such that $u_A(u, v) = 1$. Similarly, for every $u \in V$, there exists exactly one $v \in V$ such that $u_B(u, v) = 1$.*

▶ **Proposition 7** (Max in-degree). *For every $v \in V$, it holds that $d_A(v) \leq 3$ and $d_B(v) \leq 2$.*

To understand how the equilibria of the game look like, we will examine the union of the graphs G_A and G_B. The union of the graphs contains a unique 2-cycle, with one edge from G_A and one from G_B. We will see that this 2-cycle corresponds to a pure Nash equilibrium of the game. The 2-cycle in the union of the graphs G_A from Figure 1 and G_B from Figure 2 appears in Figure 3.

▶ **Proposition 8** (A 2-cycle). *Let $(v, u) \in E_A$ be a back-edge. If $v \neq v^*$ then $d_B(v) = 0$. Otherwise, $u = u^*$ and (u^*, v^*) is a 2-cycle.*

Proof. Let $u = (i, z_A) \in V$, for some $z_A \in \{0, 1\}$ and assume there exits $v = (i + 1, z_B) \in N_A(u)$, for some $z_B \in \{0, 1\}$. By the definition of E_A,

$$z_A = x_i, \quad x_{i+1} = 1 \quad \text{and} \quad z_B = 0.$$

If $v \neq v^*$, then $y_{i+1} = 1$ and by the definition of E_B, $d_B(v) = 0$. Otherwise $v = v^*$ and $x_{i^*} > y_{i^*}$. Since $v = v^*$ it holds that $u = u^*$. Since $x_{i^*} > y_{i^*}$ it holds that $y_{i+1} = 0$ and by the definition of E_B, $(u, v) \in E_B$. ◀

3.1 Pure Nash Equilibrium

By Claim 9 below, the 2-cycle game has a unique pure Nash equilibrium. Together with Proposition 8, the pure Nash equilibrium of the game corresponds to the 2-cycle in the union of the two graphs.

▶ **Claim 9.** *The 2-cycle game has exactly one pure Nash equilibrium* (u^*, v^*).

Proof. By Proposition 8, (u^*, v^*) is a 2-cycle. That is, $u_A(u^*, v^*) = 1$ and $u_B(u^*, v^*) = 1$. Since the maximum payoff for either player for any pair of actions is at most 1, it is easy to see that (u^*, v^*) is a pure Nash equilibrium of the game.

Let $u, v \in V$ such that $u \neq u^*$ or $v \neq v^*$. Let a' be the mixed strategy for player A in which she always plays u, and b' be the mixed strategy for player B in which he always plays v. By Proposition 8, either $(v, u) \notin E_A$ or $(u, v) \notin E_B$. By proposition 6, there exist $u', v' \in V$ such that $(v, u') \in E_A$ and $(u, v') \in E_B$. If $(v, u) \notin E_A$ then let a be the mixed strategy for player A in which she always plays u'. We get that

$$\mathbb{E}_{u''\sim a, v''\sim b'}[u_A(u'', v'')] = u_A(u', v) = 1 \text{ and } \mathbb{E}_{u''\sim a', v''\sim b'}[u_A(u'', v'')] = u_A(u, v) = 0.$$

Otherwise $(u, v) \notin E_B$, then let b be the mixed strategy for player B in which he always plays v'. We get that

$$\mathbb{E}_{u''\sim a', v''\sim b}[u_B(u'', v'')] = u_B(u, v') = 1 \text{ and } \mathbb{E}_{u''\sim a', v''\sim b'}[u_B(u'', v'')] = u_B(u, v) = 0.$$

Therefore, (u, v) is not a pure Nash equilibrium. ◀

The following theorem states that finding the pure Nash equilibrium (equivalently, the 2-cycle) of the 2-cycle game is hard. The proof is by a reduction from the following search variant of unique set disjointness: Player A gets a bit string $x \in \{0, 1\}^n$ and player B gets a bit string $y \in \{0, 1\}^n$. They are promised that there exists exactly one index $i^* \in [n]$ such that $x_{i^*} > y_{i^*}$. Their goal is to find the index i^*. It is well known that the randomized communication complexity of solving this problem with constant error probability is $\Omega(n)$ [3, 23, 31]. This problem is called the *universal monotone relation*. For more details on the universal monotone relation and its connection to unique set disjointness see [25]. Note that a lower bound for finding a pure Nash equilibrium of a different game is already known due to [8].

▶ **Theorem 10.** *Every randomized communication protocol for finding the pure Nash equilibrium of the 2-cycle* $N \times N$ *game, with error probability at most* $\frac{1}{3}$*, has communication complexity at least* $\Omega(N)$.

Proof. Let $x, y \in \{0, 1\}^n$ be the inputs to the search variant of unique set disjointness described above. Consider the 2-cycle $N \times N$ game which is constructed from these inputs, given by the utility functions u_A, u_B. Assume towards a contradiction that there exists a communication protocol π for finding the pure Nash equilibrium of the 2-cycle game with error probability at most $1/3$ and communication complexity $o(N)$. The players run π on u_A, u_B and with probability at least $2/3$, at the end of the communication, player A knows u and player B knows v, such that (u, v) is the pure Nash equilibrium of the game. By Claim 9, $u = u^*$ and $v = v^*$. Given u^*, v^* to the players A and B respectively, both players know the index i^*, which is a contradiction. ◀

4 From Approximate Equilibrium to the Pure Nash

In this section we prove Theorem 1. Let $n \geq 3$ be an odd integer, let $N = 2n$ and $\varepsilon \leq 1/4N^3$. Let μ be an ε-approximate correlated equilibrium of the 2-cycle $N \times N$ game, and let x, y be the strings from which the game was constructed. Recall that the pure Nash equilibrium of the game is denoted (u^*, v^*), where $u^* = (i^* - 1, x_{i^*-1})$ and $v^* = (i^*, 0) = (i^*, y_{i^*})$

(see Claim 9). The following theorem implies that μ is concentrated on (u^*, v^*), that is $\mu(u^*, v^*) > \mu(u, v)$ for every $u, v \in V$ such that $u \neq u^*$ or $v \neq v^*$. Therefore given μ, the players know the pure Nash equilibrium with no communication, and Theorem 1 follows from Theorem 10.

▶ **Theorem 11.** *$\mu(u, v) \leq (3N + 1)\varepsilon < 1/N^2$ for every $u, v \in V$ such that $u \neq u^*$ or $v \neq v^*$.*

Next we prove Theorem 11. Let $u, v \in V$ and denote $a(u) = \mu(u, N_A(u))$, $b(v) = \mu(N_B(v), v)$. By Proposition 6, there exists $v' \in V$ such that $u \in N_B(v')$, therefore

$$\mu(u, v) \leq \mu(N_B(v'), v) \leq b(v) + \varepsilon, \tag{1}$$

where the second step follows from the definition of approximate correlated equilibrium (see Definition 3). Similarly it holds that $\mu(u, v) \leq a(u) + \varepsilon$. For every $i \in [n]$, it holds that $N_A(i, \bar{x}_i) = \emptyset$ and $N_B(i, \bar{y}_i) = \emptyset$, therefore $a(i, \bar{x}_i) = b(i, \bar{y}_i) = 0$. We will bound $a(i, x_i)$ for every $i \in [n] \setminus \{i^* - 1\}$ and $b(i, y_i)$ for every $i \in [n] \setminus \{i^*\}$.

▶ **Claim 12.** *For every $i \in [n] \setminus \{i^* - 1\}$, if $(i, y_{i-1}) \in N_A(i, x_i)$ then*

$$a(i, x_i) \leq b(i - 1, y_{i-1}) + 3\varepsilon,$$

otherwise, $a(i, x_i) \leq 3\varepsilon$. For every $i \in [n] \setminus \{i^*\}$,

$$b(i, y_i) \leq a(i - 1, x_{i-1}) + 2\varepsilon.$$

Proof. Let $i \in [n] \setminus \{i^* - 1\}$. By Equation (1), for every $v \in V$ it holds that $\mu((i, x_i), v) \leq b(v) + \varepsilon$. Summing for every $v \in N_A(i, x_i)$ we get that $a(i, x_i) \leq b(N_A(i, x_i)) + 3\varepsilon$, where we bounded the right-hand side using a bound on the maximum in-degree, see Proposition 7. If there exists a back-edge $(v, (i, x_i)) \in E_A$ than by Proposition 8, $d_B(v) = 0$ (that is $N_B(v) = \emptyset$), and $b(v) = 0$. Therefore back-edges do not contribute to the bound on $a(i, x_i)$. It remains to consider edges from $(i - 1, y_{i-1})$ and $(i - 1, \bar{y}_{i-1})$. If $(i, y_{i-1}) \in N_A(i, x_i)$ then

$$a(i, x_i) \leq b(i - 1, y_{i-1}) + b(i - 1, \bar{y}_{i-1}) + 3\varepsilon = b(i - 1, y_{i-1}) + 3\varepsilon,$$

otherwise, $a(i, x_i) \leq b(i - 1, \bar{y}_{i-1}) + 3\varepsilon = 3\varepsilon$. Similarly for every $i \in [n] \setminus \{i^*\}$,

$$\begin{aligned} b(i, y_i) &\leq a(N_B(i, y_i)) + 2\varepsilon \\ &= a(i - 1, x_{i-1}) + a(i - 1, \bar{x}_{i-1}) + 2\varepsilon \\ &= a(i - 1, x_{i-1}) + 2\varepsilon. \end{aligned}$$ ◀

Using Claim 12 we can bound $a(i, x_i)$ for every $i \in [n] \setminus \{i^* - 1\}$ and $b(i, y_i)$ for every $i \in [n] \setminus \{i^*\}$ as follows. Let $\delta = 3\varepsilon$. We start with $(i^* + 1, x_{i^*+1})$. Since $x_{i^*} = 1$ and $y_{i^*} = 0$, it holds that $(i^*, y_{i*}) \notin N_A(i^* + 1, x_{i^*+1})$. Therefore by Claim 12,

$$a(i^* + 1, x_{i^*+1}) \leq \delta.$$

Once we bound $a(v)$ (or $b(v)$) for some vertex v, we can apply Claim 12 again to bound the value of b (respectively a) on a neighbor of v. We get that

$$b(i^* + 2, y_{i^*+2}) \leq a(i^* + 1, x_{i^*+1}) + \delta \leq 2\delta,$$

then

$$a(i^* + 3, x_{i^*+3}) \leq b(i^* + 2, y_{i^*+2}) + \delta \leq 3\delta$$

and so on. After we apply Claim 12 n times, since n is odd, we get that $a(i^*, x_{i^*}) \leq n\delta$. We apply Claim 12 $(n-2)$ more times, until get that

$$b(i^* - 2, y_{i^*-2}) \leq n\delta + (n-2)\delta \leq 2n\delta.$$

This concludes the proof as we showed that every $a(i, x_i)$ for $i \in [n] \setminus \{i^* - 1\}$ and every $b(i, y_i)$ for $i \in [n] \setminus \{i^*\}$ is at most $2n\delta = 3N\varepsilon$.

4.1 Sampling from a Correlated Equilibrium

Theorem 2 immediately follows from the fact that the correlated equilibria are concentrated on the pure Nash equilibrium: Let $n \geq 3$ be an odd integer, let $N = 2n$, $\Delta \in [0, 2/3)$ and $\varepsilon \leq 1/4N^3(2/3 - \Delta)$. Let μ be an ε-approximate correlated equilibrium of the 2-cycle $N \times N$ game, and let x, y be the strings from which the game was constructed. Recall that the pure Nash equilibrium of the game is denoted (u^*, v^*), where $u^* = (i^* - 1, x_{i^*-1})$ and $v^* = (i^*, 0) = (i^*, y_{i^*})$ (see Claim 9). By Theorem 11 above, $\mu(u, v) \leq (3N+1)\varepsilon$ for every $u, v \in V$ such that $u \neq u^*$ or $v \neq v^*$. Thus,

$$\mu(u^*, v^*) \geq 1 - (N^2 - 1)(3N+1)\varepsilon > 1 - 4N^3\varepsilon \geq \frac{1}{3} + \Delta.$$

If the players can sample from a distribution that is Δ-close in ℓ_1 distance to μ, using $o(N)$ communication bits, then they can find (u^*, v^*) after $O(1)$ attempts with high probability, using $o(N)$ communication bits, in contradiction to Theorem 10.

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A Trivial Approximate Equilibria of The 2-Cycle Game

In this section, we provide trivial approximate correlated equilibrium of the 2-cycle game from which it is not possible to recover the disputed index.

Let us suppose that for all $i \in \left[\frac{n}{2}+3\right]$, we have $x_i = y_i = 0$.

We define a joint distribution μ as follows

$$\mu((i,z_A),(j,z_B)) = \begin{cases} \frac{16\alpha}{n^2} \text{ if } z_A, z_B = 0 \text{ and } \frac{n}{4}+4 \le i,j \le \frac{n}{2}+2, \\ \frac{16\alpha}{n^2} \text{ if } z_A, z_B = 0, \frac{n}{4}+2 \le j \le \frac{n}{2}+2 \text{ and } i = \frac{n}{4}+3, \\ \frac{16\alpha}{n^2} \text{ if } z_A, z_B = 0, \frac{n}{4}+2 \le i \le \frac{n}{2}+2 \text{ and } j = \frac{n}{4}+3, \\ \frac{16\alpha}{n^2} - \frac{64\alpha\cdot(n/4-i+3)}{n^3} \text{ if } z_A, z_B = 0, 2 \le i,j \le \frac{n}{4}+2 \text{ and } i-j=1, \\ \frac{16\alpha}{n^2} - \frac{64\alpha\cdot(n/4-j+3)}{n^3} \text{ if } z_A, z_B = 0, 2 \le i,j \le \frac{n}{4}+2 \text{ and } j-i=1, \\ 0 \text{ otherwise}, \end{cases}$$

where α is some normalizing constant less than 2 such that $\sum_{(u,v)\in V^2} \mu(u,v) = 1$.

Let $\varepsilon = 64\alpha/n^3$. For every action $u = (i, z_A)$ of Alice such that $z_A \neq 0$, we have that $\mu(u,v) = 0$ for all $v \in V$. Similarly for every action $v = (j, z_B)$ of Bob such that $z_B \neq 0$, we have that $\mu(u,v) = 0$ for all $u \in V$. Also, for every action $u = (i, z_A)$ of Alice such that $i \in \{n/2+3,\dots,n\} \cup \{1\}$, we have that $\mu(u,v) = 0$ for all $v \in V$. And, similarly for every action $v = (j, z_B)$ of Bob such that $j \in \{n/2+3,\dots,n\} \cup \{1\}$, we have that $\mu(u,v) = 0$ for all $u \in V$. Since μ is symmetric[7], it follows that in order to show that μ is an ε-approximate correlated equilibrium we only need to consider a vertex $u = (i,0)$ when $i \in \left[\frac{n}{2}+2\right]$.

First, we consider when $i \le \frac{n}{4}+2$. Let $u' \in V$. We have

$$\sum_{v\in V} \mu(u,v)\cdot(u_A(u',v) - u_A(u,v)) = \mu(u, N_A(u')) - \mu(u, N_A(u)) = \mu(u, N_A(u')) - \frac{16\alpha}{n^2} + \frac{64\alpha\cdot(n/4-i+3)}{n^3}.$$

[7] i.e., $\mu(u,v) = \mu(v,u)$ for all $u,v \in V$.

Now if $v = (j, z_B) \in N_A(u')$ and $|j - i| \neq 1$ then, we have $\mu(u, v) = 0$. Thus, we assume $j - i = 1$, as we suppose $u \neq u'$. Then, we have

$$\mu(u, N_A(u')) \leq \frac{16\alpha}{n^2} - \frac{64\alpha \cdot (n/4 - i - 1 + 3)}{n^3} = \frac{16\alpha}{n^2} - \frac{64\alpha \cdot (n/4 - i + 3)}{n^3} + \frac{64\alpha}{n^3}.$$

This implies,

$$\sum_{v \in V} \mu(u, v) \cdot (u_A(u', v) - u_A(u, v)) \leq \frac{64\alpha}{n^3} = \varepsilon.$$

Next, we consider when $\frac{n}{4} + 4 \leq i \leq \frac{n}{2} + 2$. Let $u' \in V$. We have

$$\sum_{v \in V} \mu(u, v) \cdot (u_A(u', v) - u_A(u, v)) = \mu(u, N_A(u')) - \mu(u, N_A(u)) = \mu(u, N_A(u')) - \frac{16\alpha}{n^2}.$$

Now if $v = (j, z_B) \in N_A(u')$ and $j \geq \frac{n}{2} + 3$ then, we have $\mu(u, v) = 0$. Also if $j \leq \frac{n}{4} + 2$ then, we have $\mu(u, v) = 0$. Thus, we assume $j \in [n/4 + 3, n/4 + 2]$ and $\beta = 0$. Then, we have

$$\mu(u, N_A(u')) \leq \frac{16\alpha}{n^2}.$$

This implies,

$$\sum_{v \in V} \mu(u, v) \cdot (u_A(u', v) - u_A(u, v)) \leq 0.$$

Finally, we consider when $i = \frac{n}{4} + 3$. Let $u' = (i', z'_A) \in V$. We have

$$\sum_{v \in V} \mu(u, v) \cdot (u_A(u', v) - u_A(u, v)) = \mu(u, N_A(u')) - \frac{16\alpha}{n^2} + \frac{64\alpha}{n^3}.$$

Now if $v = (j, z_B) \in N_A(u')$ and $j \geq \frac{n}{2} + 3$ then, we have $\mu(u, v) = 0$. Also if $j \leq \frac{n}{4} + 2$ and $|j - i| \neq 1$ then, we have $\mu(u, v) = 0$. Since $u \neq u'$ we have that $j \in [n/4 + 3, n/4 + 2]$ and $\beta = 0$. Then we have

$$\mu(u, N_A(u')) \leq \frac{16\alpha}{n^2}.$$

This implies,

$$\sum_{v \in V} \mu(u, v) \cdot (u_A(u', v) - u_A(u, v)) \leq \frac{64\alpha}{n^3} = \varepsilon.$$

Thus, μ is an ε-approximate correlated equilibrium and an $(\varepsilon \cdot N)$-approximate rule correlated equilibrium.