Effective Divergence Analysis for Linear Recurrence Sequences

Shaull Almagor
Department of Computer Science, Oxford University, UK
shaull.almagor@cs.ox.ac.uk

Brynmor Chapman
MIT CSAIL
brynmor@mit.edu

Mehran Hosseini
Department of Computer Science, Oxford University, UK
mehran.hosseini@cs.ox.ac.uk

Joël Ouaknine[1]
Max Planck Institute for Software Systems, Germany &
Department of Computer Science, Oxford University, UK
joel@mpi-sws.org

James Worrell[2]
Department of Computer Science, Oxford University, UK
jbw@cs.ox.ac.uk

Abstract

We study the growth behaviour of rational linear recurrence sequences. We show that for low-order sequences, divergence is decidable in polynomial time. We also exhibit a polynomial-time algorithm which takes as input a divergent rational linear recurrence sequence and computes effective fine-grained lower bounds on the growth rate of the sequence.

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1 Introduction

Linear recurrence sequences (LRS), such as the Fibonacci numbers, permeate a wide range of scientific fields, from economics and theoretical biology to computer science and mathematics. In computer-aided verification, for example, LRS techniques play a key rôle in the termination analysis of a large class of simple while loops – see [17] for a short survey on this topic. Likewise, the ergodic behaviour of Markov chains in probability theory [1], or the stability of supply-and-demand price equilibria in laggy markets in economics (the so-called "cobweb

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model") [3] can be analysed through an examination of the asymptotic behaviour of certain types of LRS; in particular, instability of price equilibria corresponds precisely to *divergence* of the associated LRS.

In this paper, we undertake a systematic and fine-grained analysis of the growth behaviour of rational linear recurrence sequences from the point of view of effectiveness and complexity. In order to describe our main results, we first require some preliminary definitions. A sequence of real numbers $\mathbf{u} = \langle u \rangle_{n=1}^{\infty}$ is said to satisfy a *linear recurrence of order* k if there are real numbers $a_1, \ldots, a_{k+1}$ such that

$$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_{k-1} u_{n+1} + a_k u_n + a_{k+1} \tag{1}$$

for all $n \in \mathbb{N}$. Such a recurrence is said to be *homogeneous* if $a_{k+1} = 0$ and *inhomogeneous* if $a_{k+1} \neq 0$. The *characteristic polynomial* of the recurrence is

$$p(x) := x^k - a_1 x^{k-1} - \cdots - a_{k-1} x - a_k \,.$$

The zeros of p are called the *characteristic roots*. A characteristic root of maximum modulus is said to be *dominant* and its modulus is the *dominant modulus*. The *multiplicity* of a characteristic root γ is the maximal $m \in \mathbb{N}$ such that $(x-\gamma)^m$ divides $p(x)$.

An LRS is said to be *rational* if it consists of rational numbers, *integral* if it consists of integers, and *algebraic* if it consists of algebraic numbers. An LRS is *simple* if all of its characteristic roots have multiplicity 1, and is *non-degenerate* if no ratio of two distinct characteristic roots is a root of unity.[3]

We say that an LRS $\mathbf{u}$ *diverges to* ∞ if $\lim_{n\to\infty} u_n = \infty$ (technically speaking: for all $T \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $u_n \geq T$). We also say that $\mathbf{u}$ is *absolutely divergent* (or *diverges in absolute value*) if $\lim_{n\to\infty} |u_n| = \infty$.

The LRS $\mathbf{u}$ is said to be *positive* if $u_n \geq 0$ for all $n \geq 1$, and *ultimately positive* if there is some $N \in \mathbb{N}$ such that $u_n \geq 0$ for all $n \geq N$.

A celebrated result from the 1930s, the Skolem-Mahler-Lech theorem (see [9]), implies that all non-degenerate integral LRS are absolutely divergent. This statement is however *non-effective* in a very basic sense: given a finite representation of a non-degenerate integral LRS $\mathbf{u}$, there is no known algorithm to compute a bound N such that $u_n \neq 0$ for $n \geq N$. It is also worth pointing out that the divergence assertion fails in general for non-integral LRS.

The question of the so-called *rate of absolute divergence* for non-degenerate integral LRS was subsequently extensively studied; see [9, Sec. 2.4] for an account of some of the key results accumulated over the last several decades. To begin with, a fairly straightforward fact is the following: if $\mathbf{u}$ is an algebraic LRS of order k with dominant modulus ρ, then there is an effectively computable constant a such that, for all $n \geq 1$, $|u_n| \leq a\rho^n n^k$. In the 1970s, a conjecture was formulated to the effect that any non-degenerate integral LRS has, essentially, the maximal possible growth rate (see the next theorem for a precise statement). The conjecture was finally settled positively independently by Evertse [10] and by van der Poorten and Schlickewei [21]:

▶ **Theorem 1.** *For any non-degenerate algebraic LRS* $\mathbf{u}$ *of dominant modulus* $\rho > 1$*, and any* $\varepsilon > 0$*, there exists a constant* N *such that, for all* $n \geq N$*, we have* $|u_n| \geq \rho^{(1-\varepsilon)n}$.

[3] For most practical purposes – and certainly for all of the computational tasks considered in this paper – LRS can be assumed to be non-degenerate, since any degenerate LRS can be effectively decomposed into a finite number of non-degenerate LRS; moreover this reduction can be carried out in polynomial time for rational LRS of bounded order [9, 14].

This is a highly non-trivial result making use of deep number-theoretic tools concerning bounds on the sum of S-units. Unfortunately, the proof is not effective, in the sense that given $\varepsilon > 0$, it does not provide estimates for the corresponding value of N. This effectiveness issue is described as "an important open problem" in [9], where it is furthermore suggested that any progress on the matter would likely hinge upon substantial improvements of deep number-theoretic results, such as Roth's theorem, the prospects of which currently appear to be remote.

Nevertheless – and in particular for algorithmic applications in computer science – effectiveness is of central importance. The sharpest known results in this vein are due to Mignotte [12] as well as Shorey and Stewart [19], capping a long line of work in this area:

▶ **Theorem 2.** *For any homogeneous non-degenerate integral LRS* $\mathbf{u}$ *of order at most* 3 *with dominant modulus* ρ*, there are effective constants* a *and* d *such that, for all* $n \geq 1$*, we have* $|u_n| \geq \frac{a\rho^n}{n^d}$.

For most problems in computer science and automated verification, such as the analysis of the long-run behaviour of dynamical systems or the termination of linear while loops, the primary notion of *divergence* is clearly much more relevant than that of 'divergence in absolute value'. In view of the above results, however, one might expect that little could be said about effective rates of divergence. Somewhat surprisingly, divergence does turn out to be significantly more tractable than absolute divergence. At a high level, the main results of this paper can now be summarised as follows:

> Given a rational LRS $\mathbf{u}$, homogeneous or inhomogeneous, either of order at most 5, or, if the LRS is simple, of order at most 8, we can carry out the following tasks in polynomial time:
> - decide if $\mathbf{u}$ diverges to ∞ or not; and
> - in divergent instances, provide effective fine-grained lower bounds on the rate of divergence of $\mathbf{u}$.

The precise statements can be found in Theorems 12 and 13. The most obvious contrast in comparison with Theorem 2 is the higher order of LRS that can be handled effectively (5 and 8 versus 3). It is also worth noting, however, that our results apply more generally to rational (as opposed to integral) LRS, and that we can handle inhomogeneous sequences at no cost – this is remarkable in that the folklore wisdom usually broadly equates inhomogeneous LRS of order k with homogeneous LRS of order $k+1$ (this assertion, as well as the manner in which we circumvent it, are made precise in the main body of the paper).

Finally, let us point out that our analysis of divergence rates relies, among others, on improvements to results concerning the positivity and ultimate positivity of LRS, which were originally developed in [15, 14, 16]. As a by-product, therefore, stronger results on the Positivity and Ultimate Positivity Problems – notably dealing with inhomogeneous LRS – can be found in the present paper, in particular in the form of Theorems 19 and 20.

2 Preliminaries

2.1 Linear Recurrence Sequences

Let us start by reformulating the notion of linear recurrence more abstractly as follows. Define the *shift operator* $E : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $E(f)(n) = f(n+1)$ for a sequence $f \in \mathbb{R}^{\mathbb{N}}$. The polynomial ring $\mathbb{R}[E]$ acts on the set of sequences $\mathbb{R}^{\mathbb{N}}$ on the left in a natural way, turning

$\mathbb{R}^{\mathbb{N}}$ into a left $\mathbb{R}[E]$ module. Then a sequence $\mathbf{u} = \langle u \rangle_{n=1}^{\infty}$ satisfies the recurrence equation (1) if and only if $p(E) \cdot \mathbf{u} = a_{k+1} \cdot \mathbf{1}$, where p is the characteristic polynomial of the recurrence and $\mathbf{1}$ is the all-ones sequence.

The following homogenization construction is well known.

▶ **Proposition 3.** *Let $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ satisfy an inhomogeneous linear recurrence of order k. Then $\mathbf{u}$ satisfies a homogeneous recurrence of order $k+1$.*

Proof. By assumption we have that $p(E) \cdot \mathbf{u} = \mathbf{c}$ for some monic polynomial $p(x)$ of degree k and constant sequence $\mathbf{c}$. Writing $q(x) = (x-1)p(x)$, we have $q(E) \cdot \mathbf{u} = (E-1) \cdot \boldsymbol{c} = 0$. ◀

We have the following partial converse to Proposition 3.

▶ **Proposition 4.** *Let $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ satisfy a homogeneous linear recurrence of order $k+1$ with a positive real characteristic root ρ. Then the sequence $\mathbf{v} = \langle v_n \rangle_{n=1}^{\infty}$ defined by $v_n = \frac{u_n}{\rho^n}$ satisfies an inhomogeneous linear recurrence of order k.*

Proof. By assumption, $\mathbf{u}$ satisfies the recurrence $f(E) \cdot \mathbf{u} = 0$ for some monic polynomial $f(x) \in \mathbb{R}[x]$ of degree $k+1$ that has a positive real root ρ. Define a sequence $\mathbf{v} = \langle v_n \rangle_{n=1}^{\infty}$ by $v_n := \frac{u_n}{\rho^n}$ for all $n \in \mathbb{N}$. Then $\mathbf{v}$ satisfies the recurrence $g(E) \cdot \mathbf{v} = 0$ where g is the monic polynomial $g(x) = \rho^{-(k+1)} f(\rho x)$.

But $g(1) = 0$ and hence we have the factorization $g(x) = (x-1)h(x)$ for some monic polynomial $h(x) \in \mathbb{R}[x]$. It follows that $(E-1)h(E) \cdot \mathbf{v} = 0$ and hence $h(E) \cdot \mathbf{v}$ is constant, i.e., $\mathbf{v}$ satisfies an inhomogeneous recurrence of order k. ◀

Let $\|\mathbf{u}\|$ denote the binary representation length[4] of $\mathbf{u}$. We remark that the transformations back and forth between homogeneous and inhomogeneous LRS can be carried out in polynomial time in $\|u\|$ if the given LRS have real algebraic coefficients. For an inhomogeneous LRS $\mathbf{u}$ of order k, we refer to the corresponding homogeneous LRS obtained as per Proposition 3 as the *homogenization* of $\mathbf{u}$, denoted $\textsc{hom}(\mathbf{u})$. The proof of Proposition 3 gives us the following useful property.

▶ **Property 5.** *The characteristic roots of $\textsc{hom}(\mathbf{v})$ are the same as those of $\mathbf{v}$, with the same multiplicities, except for the characteristic root 1, which always occurs in $\textsc{hom}(\mathbf{v})$, and whose multiplicity is $m+1$, where m is the multiplicity of 1 in $\mathbf{v}$.*

Consider an LRS $\mathbf{u}$ with integer coefficients. Then, since the characteristic polynomial p of an LRS $\mathbf{u}$ has integer coefficients, the characteristic roots of $\mathbf{u}$ comprise real-algebraic roots $\{\rho_1, \ldots, \rho_d\}$, and conjugate pairs of complex-algebraic roots $\{\gamma_1, \overline{\gamma_1}, \ldots, \gamma_m, \overline{\gamma_m}\}$. There are now univariate polynomials $A_1, \ldots, A_d \in$ with real-algebraic coefficients and $C_1, \ldots, C_m$ with complex-algebraic coefficients such that, for every $n \geq 0$,

$$u_n = \sum_{i=1}^{d} A_i(n)\rho_i^n + \sum_{j=1}^{m} (C_j(n)\gamma^n + \overline{C_j(n)}\overline{\gamma}^n)$$

This expression is referred to as the *exponential polynomial* solution of $\mathbf{u}$. The degree of each of the polynomials is strictly smaller than the multiplicity of the corresponding root. For a fixed order k, the coefficients appearing in the polynomials can be computed in time polynomial in $\|\mathbf{u}\|$.

[4] In general, we denote by $\|\cdot\|$ the binary-representation length of objects.

We now turn to present two results regarding the asymptotic analysis of LRS.

The following result due to Braverman [5] enables us to reason about the complex part of the exponential polynomial above.

▶ **Lemma 6** (Complex Units Lemma). *Let $\zeta_1, \zeta_2, \ldots, \zeta_m \in \mathbf{S}_1 \setminus \{1\}$ be distinct complex numbers (where $\mathbf{S}_1 = \{z \in \mathbb{C} : |z| = 1\}$), and let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C}\setminus\{0\}$. Set $z_n := \sum_{k=1}^{m} \alpha_k \zeta_k^n$. Then there exists $c < 0$ such that for infinitely many n, $Re\,(z_n) < c$.*

In particular, Lemma 6 immediately implies that an LRS without a real dominant characteristic root, is neither positive, ultimately positive, nor divergent.

Finally, the following proposition from [14] allows us to bound the growth rate of the low-order terms in the exponential polynomial of an LRS.

▶ **Proposition 7.** *Consider an LRS $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ of bounded order, with dominant modulus ρ, and write*

$$\frac{u_n}{\rho^n} = A(n) + \sum_{i=1}^{m} \left(C_i(n)\lambda_i^n + \overline{C_i(n)}\overline{\lambda_i}^n \right) + r(n)$$

where A is a real polynomial, C_i are non-zero complex polynomials, $\rho\lambda_i$ and $\rho\overline{\lambda_i}$ are conjugate pairs of non-real dominant roots of $\mathbf{u}$, and r is an exponentially decaying function.

We can compute in polynomial time $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$\frac{1}{\epsilon} = 2^{\|\mathbf{u}\|^{O(1)}},$$
$$N = 2^{\|\mathbf{u}\|^{O(1)}},$$
for all $n > N, |r(n)| < (1 - \epsilon)^n$.

2.2 Mathematical Tools

In this section we introduce several tools that will be used throughout the paper.

Algebraic Numbers. A complex number α is *algebraic* if it is a root of a polynomial $p \in \mathbb{Z}[x]$. The *defining polynomial* of α is the unique monic polynomial of minimal degree that has α as a root, and is denoted p_α. The *degree* and the *height* of α are the degree and the height (i.e., maximum absolute value of the coefficients) of p_α, respectively. An algebraic number α can be represented by a polynomial that has α as a root, along with an approximation of α by a complex number with rational real and imaginary parts. We denote by $\|\alpha\|$ the representation length of α. Basic arithmetic operations as well as equality testing and comparisons for algebraic numbers can be carried out in polynomial time (see [4, 7] for efficient algorithms).

The following lemma from [15] is a consequence of the celebrated lower bound for linear forms in logarithms due to Baker and Wüstholz [2].

▶ **Lemma 8.** *There exists $D \in \mathbb{N}$ such that, for all algebraic numbers $\lambda, \zeta \in \mathbb{C}$ of modulus 1, and for all $n \geq 2$, if $\lambda^n \neq \zeta$, then $|\lambda^n - \zeta| > \frac{1}{n^{(\|\lambda\|+\|\zeta\|)^D}}$.*

Multiplicative Relations. Multiplicative relations between characteristic roots of an LRS play a key role in our analysis. The following result, due to Masser [11] enables us to efficiently elicit these relationships.

▶ **Theorem 9.** *Let m be fixed, and let $\lambda_1, \ldots, \lambda_m$ be complex algebraic numbers of modulus 1. Let $L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda_1^{v_1} \cdots \lambda_m^{v_m} = 1\}$ be the group of multiplicative relations between the λ_i. L has a basis $\{\boldsymbol{\ell_1}, \ldots, \boldsymbol{\ell_p}\} \subseteq \mathbb{Z}^m$ (with $p \leq m$), where the entries of each of the $\boldsymbol{\ell_j}$ are all polynomially bounded in $\|\lambda_1\| + \ldots + \|\lambda_m\|$. Moreover, such a basis can be computed in time polynomial in $\|\lambda_1\| + \ldots + \|\lambda_m\|$.*

The First-Order Theory of the Reals. A sentence in the first-order theory of the reals is of the form $Q_1 x_1 \cdots Q_m x_m \varphi(x_1, \ldots, x_m)$ where each Q_i is a quantifier ($\exists$ or $\forall$), each x_i is a real valued variable, and φ is a boolean combination of atomic predicates of the form $p(x_1, \ldots, x_m) \sim 0$ for some $p \in \mathbb{Z}[x_1, \ldots, x_m]$ and $\sim \in \{>, =\}$. The first-order theory of the reals admits quantifier elimination, a famous result due to Tarski [20], whose procedure unfortunately has non-elementary complexity. In this paper we consider only the case where the number of variables is uniformly bounded. Then, we can invoke the following result due to Renegar [18].

▶ **Theorem 10** (Renegar). *Let $M \in \mathbb{N}$ be fixed. Let $\tau(\mathbf{y})$ be a formula of the first order theory of the reals. Assume that the number of (free and bound) variables in $\tau(\mathbf{y})$ is bounded by M. Denote the degree of $\tau(\mathbf{y})$ by d and the number of atomic predicates in $\tau(\mathbf{y})$ by n.*

There is a polynomial time (polynomial in $\|\tau(\mathbf{y})\|$) procedure which computes an equivalent quantifier-free formula

$$\chi(\mathbf{y}) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j}(y) \sim_{i,j} 0$$

where each $\sim_{i,j}$ is either $>$ or $=$, with the following properties:

1. *Each of I and J_i (for $1 \leq i \leq I$) is bounded by $(n+d)^{O(1)}$.*
2. *The degree of $\chi(\mathbf{y})$ is bounded by $(n+d)^{O(1)}$.*
3. *The height of $\chi(\mathbf{y})$ is bounded by $2^{\|\tau(\mathbf{y})\|(n+d)^{O(1)}}$.*

Asymptotic Analysis. We conclude this section with the following simple lemma from [15].

▶ **Proposition 11.** *Let $a \geq 2$ and $\epsilon \in (0,1)$ be real numbers. Let $B \in \mathbb{Z}[x]$ have degree at most a^{D_1} and height at most $2^{a^{D_2}}$, and assume that $1/\epsilon \leq 2^{a^{D_3}}$ for some $D_1, D_2, D_3 \in \mathbb{N}$. Then there is $D_4 \in \mathbb{N}$ depending only on D_1, D_2, D_3 such that for all $n \geq 2^{a^{D_4}}$, $\frac{1}{B(n)} > (1-\epsilon)^n$.*

3 Divergence

Recall from Theorem 1 that an LRS $\mathbf{u}$ with dominant modulus ρ necessarily diverges in absolute value if $\rho > 1$. More precisely, if $\rho > 1$ then given $\varepsilon > 0$ there exists a threshold N such that $|u_n| > \rho^{(1-\varepsilon)n}$ for all $n > N$. However this result is *ineffective* – it is not known how to compute N given $\mathbf{u}$ and ε.

In this section we derive *effective* divergence bounds for sequences that diverge to ∞ (i.e., sequences that both diverge in absolute value and that are ultimately positive). The bounds on divergence have the following form: for a divergent sequence $\mathbf{u}$ with dominant modulus $\rho = 1$ we aim to show that for every $n > N$, $u_n > an^d$ for effective constants $a > 0, d \in \mathbb{N}$, and $N \in \mathbb{N}$. In case of a dominant modulus $\rho > 1$ we aim to show that for every $n > N$,

$u_n > \frac{a\rho^n}{n^d}$ for effective constants $a > 0, d \in \mathbb{N}$, and $N \in \mathbb{N}$. Henceforth we refer to bounds of these respective forms as *divergence bounds*.[5]

In Section 3.1, we show how to compute effective divergence bounds of LRS up to certain orders. Then, in Section 3.2, we provide hardness results for the decidability of divergence.

3.1 Effective Divergence is Solvable

In this section we prove the following theorems:

▶ **Theorem 12.** *There is a polynomial-time procedure that given a rational LRS of order at most 5 decides whether it diverges and, in case of divergence, outputs divergence bounds.*

▶ **Theorem 13.** *There is a polynomial-time procedure that, given a simple rational LRS of order at most 8, decides whether it diverges and, in case of divergence, outputs divergence bounds.*

The proofs of Theorems 12 and 13 build on techniques developed in [15, 14, 16], using a fine-grained analysis in the results thereof, along with some new ideas. To avoid unnecessary repetition, we sketch the main ideas of the proofs simultaneously.

Consider an LRS $\mathbf{u}$ of order k. For uniformity, if $\mathbf{u}$ is inhomogeneous, we homogenize it as per Proposition 3. Thus, either $k \leq 6$ or $\mathbf{u}$ is simple and $k \leq 9$, where if $k = 6$ or if $\mathbf{u}$ is simple and $k = 9$, then $\mathbf{u}$ has a special structure according to Property 5.

As mentioned in Section 1, we can assume without loss of generality that $\mathbf{u}$ is non-degenerate. Let ρ be the dominant modulus of $\mathbf{u}$, we also note that if $\rho < 1$, then $|u_n| \to 0$ as $n \to \infty$, and in particular the sequence does not diverge. Thus, we may assume $\rho \geq 1$. In addition, by Lemma 6, if $\mathbf{u}$ does not have a real positive dominant root, then $u_n \not\to \infty$. Thus, we may assume a real positive dominant characteristic root ρ. Note that all other dominant roots must be complex, and come in conjugate pairs, since if $-\rho$ were a root, then $\mathbf{u}$ would be degenerate.

Writing u_n as an exponential polynomial and dividing by ρ^n, we have

$$\frac{u_n}{\rho^n} = A(n) + \sum_{i=1}^{m} \left(C_i(n)\lambda_i^n + \overline{C_i(n)}\overline{\lambda}_i^n \right) + r(n) \tag{2}$$

where A is a real polynomial, C_i are non-zero complex polynomials, $\rho\lambda_i$ and $\rho\overline{\lambda}_i$ are conjugate pairs of non-real dominant characteristic roots of $\mathbf{u}$ (so $|\lambda_i| = 1$), and $r(n)$ is an exponentially decaying function (possibly identically zero). More precisely, the degree of each of $A(n), C_1(n), \ldots, C_m(n)$ is strictly smaller than the multiplicity of the corresponding characteristic root. We can assume that either $A(n) \not\equiv 0$ or $m \neq 0$. Indeed, otherwise we can consider the LRS $\langle \rho^n r(n) \rangle_{n=1}^{\infty}$, which is of lower order than $\mathbf{u}$.

In the following, if $A(n)$ (resp. $C_i(n)$ for some $1 \leq i \leq m$) is a constant, we denote it by A (resp. C_i).

We proceed to decide divergence by a case analysis of Equation (2).

[5] Note that not only do we seek effective divergence bounds, but also that these bounds are asymptotically tighter than the bounds from Theorem 1 since for any fixed $d > 0$, it is clear that $a\rho^n/n^d$ eventually dominates $\rho^{(1-\varepsilon)n}$ for any $\varepsilon > 0$.

Case 1: $\rho = 1$ and $A(n) = A$ is a constant

Note that in this case, $\frac{u_n}{\rho^n} = u_n$. Since A is a constant, then it does not affect the divergence of $\mathbf{u}$. We claim that $u_n \not\to \infty$. Indeed, by Lemma 6, the expression $\sum_{i=1}^{m} \left(C_i(n)\lambda_i^n + \overline{C_i(n)\lambda_i^n} \right)$ becomes negative infinitely often (regardless of whether $C_i(n)$ are constants or polynomials), whereas the effect of $r(n)$ is exponentially decreasing. Thus, $\mathbf{u}$ does not diverge.

Case 2: $\rho = 1$, $A(n)$ is not a constant, and every C_i is a constant

In this case we can rewrite Equation (2) as

$$u_n = A(n) + \sum_{i=1}^{m} \left(C_i\lambda_i^n + \overline{C_i\lambda_i^n} \right) + r(n) \tag{3}$$

Since $|\lambda_i| = 1$ for all i, and since $r(n)$ is exponentially decreasing, then clearly $u_n \to \infty$ iff the leading coefficient of $A(n)$ is positive.

Recall that since $\rho = 1$, then if $\mathbf{u}$ diverges, there exist $N, d \in \mathbb{N}$ and $a > 0$ such that $u_n \geq an^d$ for all $n > N$. We now show how to effectively compute N, d, and a.

From Proposition 7, we can compute in polynomial time $\epsilon \in (0, 1)$ and $N_1 \in \mathbb{N}$ such that $r(n) < (1 - \epsilon)^n < 1$ for all $n > N_1$. We thus have that $u_n \geq A(n) - 2\sum_{i=1}^{m} |C_i| - 1$, and we can easily compute $N_2 \in \mathbb{N}$ and $a \in \mathbb{Q}$ (depending on the coefficients of $A(n)$) such that for all $n > N_2$ we have $A(n) - 2\sum_{i=1}^{m} |C_i| - 1 \geq an^d$, where d is the degree of $A(n)$. Taking $N = \max\{N_1, N_2\}$, we conclude this case.

Case 3: $\rho = 1$, $A(n)$ is not a constant, and there exists a non-constant $C_i(n)$

We notice that if there exists a non-constant $C_i(n)$, it follows by Property 5 that $\mathbf{u}$ is not obtained by homogenizing a simple LRS. That is, we are in the case where $k \leq 6$. In the notations of Equation (2), we then have that $m = 1$, $A(n)$ is linear, $C_1(n)$ is linear, and $r(n) \equiv 0$. Indeed, this corresponds to the case where the characteristic roots of u_n are $1, \lambda, \overline{\lambda}$, each with multiplicity 2. Let $A(n) = a_1 n + b_1$ and $C_1(n) = a_2 n + b_2$, then we can write

$$u_n = a_1 n + b_1 + (a_2 n + b_2)\lambda^n + (\overline{a_2} n + \overline{b_2})\overline{\lambda}^n = n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n}) + (b_1 + b_2\lambda^n + \overline{b_2\lambda^n})$$

Since $|(b_1 + b_2\lambda^n + \overline{b_2\lambda^n})|$ is bounded, then u_n diverges iff $n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n})$ diverges. Let $\theta = \arg\lambda$ and $\varphi = \arg a_2$. We have $n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n}) = n(a_1 + 2|a_2|\cos(n\theta + \varphi))$.

Observe that since $\mathbf{u}$ is non-degenerate, then θ is not a rational multiple of π. It follows that $\{[n\theta + \varphi]_{2\pi} : n \in \mathbb{N}\}$ (where $[x]_{2\pi} = x - 2\pi j$ where j is the unique integer such that $0 \leq x - 2\pi j < 2\pi$) is dense in $[0, 2\pi)$, so $\{\cos(n\theta + \varphi) : n \in \mathbb{N}\}$ is dense in $[-1, 1]$. Again, we split into cases.

- If $a_1 > 2|a_2|$, we have that u_n diverges. Then, we can compute in polynomial time a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that $a_1 - 2|a_2| > \epsilon$ and $n(a_1 + 2|a_2|) - (b_1 - 2|b_2|) > \epsilon n$ for all $n > N$. We then have that $u_n > \epsilon n$ for all $n > N$, thus concluding effective decidability of divergence in this case.
- If $a_1 < 2|a_2|$, then u_n does not diverge, as it becomes negative infinitely often, by the density argument above.
- The remaining case is when $a_1 = 2|a_2|$, and the expression above becomes $na_1(1+\cos(n\theta+\varphi))$. We show that in this case, u_n does not diverge.

By Taylor approximation, for every $x \in (-\pi, \pi]$ it holds that $1 - \cos(x) \leq \frac{x^2}{2}$. For $n \in \mathbb{N}$, write $\Lambda(n) = n\theta + \varphi - (2j+1)\pi$, where $j \in \mathbb{Z}$ is the unique integer such that $-\pi < \Lambda(n) \leq \pi$. We now have that

$$na_1(1 + \cos(n\theta + \varphi)) = na_1(1 - \cos(n\theta + \varphi + \pi)) = na_1(1 - \cos(\Lambda(n))) < na_1 \frac{\Lambda(n)^2}{2} .$$

By Dirichlet's Approximation Theorem, we have that $|\Lambda(n)| < \frac{t}{n}$ for infinitely many values of n, where t is a constant depending on φ. Thus, we have $na_1 \frac{\Lambda(n)^2}{2} < \frac{a_1 t^2}{2n}$ for infinitely many values of n. It follows that u_n is infinitely often bounded by a constant, so it does not diverge.

Case 4: $\rho > 1$ and there exists a non-constant $C_i(n)$

As in Case 3, it holds that $k \leq 6$. Moreover, since $\rho > 1$, then whether or not $\mathbf{u}$ was obtained by homogenization, the characteristic root 1 (if it exists) is captured in $r(n)$. Therefore, we have that $m = 1$, C_1 is linear, and $A(n) = A$ is constant. Let C_1 have leading coefficient $b \neq 0$. By Lemma 6, there exists $\epsilon > 0$ such that $b\lambda^n + \overline{b\lambda}^n < -\epsilon$ infinitely often. Then $C_1(n)\lambda_1^n + \overline{C_1(n)\lambda_1}^n$ (and hence u_n) is unbounded below, so u_n does not diverge.

Case 5: $\rho > 1$, $A(n)$ is not a constant, and every C_i is a constant

Since $A(n)$ is not a constant and $\rho > 1$, this case may only arise for $k \leq 6$ and $m \leq 1$. We write

$$\frac{u_n}{\rho^n} = A(n) + C_1\lambda_1^n + \overline{C_1\lambda_1}^n + r(n).$$

where if $m = 0$ then take $C_1 = 0$.

If $A(n)$ has a negative leading coefficient, then u_n is unbounded from below, and in particular u_n does not diverge.

If $A(n)$ has a positive leading coefficient, we can compute in polynomial time $N_0 \in \mathbb{N}$ and a rational $\epsilon_0 > 0$ such that $A(n) - 2|C_1| > 2\epsilon_0$ for all $n > N_0$. By Proposition 7, we can also compute in polynomial time $N_1 \in \mathbb{N}$ and $\epsilon_1 \in (0,1)$ such that $|r(n)| < (1-\epsilon_1)^n$ for all $n > N_1$. Taking $N_2 \geq \log_{1-\epsilon_1} \epsilon_0$, we have that for all $n > \max\{N_0, N_1, N_2\}$, $|r(n)| < \epsilon_0$, and thus

$$\frac{u_n}{\rho^n} \geq A(n) - 2|C_1| + r(n) \geq A(n) - 2|C_1| - \epsilon_0 > 2\epsilon_0 - \epsilon_0 = \epsilon_0 .$$

Thus we have $u_n \geq \epsilon_0 \rho^n$ for all $n > \max\{N_0, N_1, N_2\}$, which immediately yields effective divergence bounds in this case.

Case 6: $\rho > 1$, $A(n) = A$ is a constant, and every C_i is a constant

This case is the most involved, and utilizes deep mathematical results. Our proof works along the lines of [16]. For completeness, the full proof can be found in the full version.

We rewrite Equation (2) as

$$\frac{u_n}{\rho^n} = A + \sum_{i=1}^{m} \left(C_i\lambda_i^n + \overline{C_i\lambda_i}^n \right) + r(n) \tag{4}$$

Observe that $m \leq 3$. Indeed, if $k \leq 8$ this is trivial, and if $k = 9$ then by Property 5, 1 must be a non-dominant characteristic root of $\mathbf{u}$, so $r(n) \not\equiv 0$ and thus $m \leq 3$.

In the following, we handle the case $m = 3$. The cases where $m < 3$ are similar and slightly simpler.

Let $L = \{(v_1, \ldots, v_3) \in \mathbb{Z}^3 : \lambda^{v_1} \cdots \lambda^{v_3} = 1\}$, and let $\{\boldsymbol{\ell_1}, \ldots, \boldsymbol{\ell_p}\}$ be a basis for L of cardinality p. Write $\boldsymbol{\ell_q} = (\ell_{q,1}, \ldots, \ell_{q,3})$ for $1 \le q \le p$. From Theorem 9, such a basis can be computed in polynomial time, and moreover – each $\ell_{q,j}$ may be assumed to have magnitude polynomial in $\|u\|$.

Consider the set $\mathbb{T} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3| = 1$ and for each $1 \le q \le p$, $z_1^{\ell_{q,1}} z_2^{\ell_{q,2}} z_3^{\ell_{q,3}} = 1\}$.

Define $h : \mathbb{T} \to \mathbb{R}$ by setting $h(z_1, z_2, z_3) = \sum_{i=1}^{3}(C_i z_i + \overline{C_i}\overline{z_i})$, so that for every $n \in \mathbb{N}$, $\frac{u_n}{\rho^n} = A + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) + r(n)$. By Kronecker's theorem on inhomogeneous Diophantine approximation [6], the set $\{\lambda_1^n, \lambda_2^n, \lambda_3^n : n \in \mathbb{N}\}$ is a dense subset of $\mathbb{T}$. Since h is continuous, it follows that $\inf\{h(\lambda_1^n, \lambda_2^n, \lambda_3^n) : n \in \mathbb{N}\} = \min h|_{\mathbb{T}} = \mu$ for some $\mu \in \mathbb{R}$.

In the full proof, we show that μ is algebraic, computable in polynomial time, with $\|\mu\| = \|\mathbf{u}\|^{O(1)}$.

We now split to cases according to the sign of $A + \mu$.

- If $A + \mu < 0$, then $\mathbf{u}$ is infinitely often negative, and does not diverge.
- If $A + \mu > 0$, then $\mathbf{u}$ diverges, and we obtain an effective bound similarly to Case 5.
- It remains to analyze the case where $A+\mu = 0$. To this end, let $\lambda_j = e^{i\theta_j}$ and $C_j = |C_j|e^{i\varphi_j}$ for $1 \le j \le 3$. From Equation (4) we have

$$\frac{u_n}{\rho^n} = A + \sum_{j=1}^{3} 2|C_j|\cos(n\theta_j + \varphi_j) + r(n).$$

We further assume that all the C_j are non-zero. Indeed, if this does not hold, we can recast our analysis in lower dimension.

In the full proof, we use zero-dimensionality results to show that h achieves its minimum μ over $\mathbb{T}$ only at a finite set of points $Z = \{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{T} : h(\zeta_1, \zeta_2, \zeta_3) = \mu\}$.

We concentrate on the set Z_1 of first coordinates of Z. Write

$$\tau_1(x) = \exists z_1(\mathrm{Re}(z_1) = x \wedge z_1 \in Z_1)$$
$$\tau_2(y) = \exists z_1(\mathrm{Im}(z_1) = y \wedge z_1 \in Z_1)$$

By rewriting these formulas in the first order theory of the reals, we are able to show, using Theorem 10, that any $\zeta_1 = \hat{x} + i\hat{y} \in Z_1$ is algebraic, and moreover satisfies $\|\zeta_1\| = \|\mathbf{u}\|^{O(1)}$. In addition, we show that the cardinality of Z_1 is at most polynomial in $\|\mathbf{u}\|$.

Since λ_1 is not a root of unity, for each $\zeta_1 \in Z_1$ there is at most one value of n such that $\lambda_1^n = \zeta_1$. Theorem 9 then entails that this value (if it exists) is at most $M = \|\mathbf{u}\|^{O(1)}$, which we can take to be uniform across all $\zeta_1 \in Z_1$. We can now invoke Corollary 8 to conclude that, for $n > M$, and for all $\zeta_1 \in Z_1$, we have

$$|\lambda_1^n - \zeta_1| > \frac{1}{n^{\|\mathbf{u}\|^D}} \tag{5}$$

where $D \in \mathbb{N}$ is some absolute constant.

Let $b > 0$ be minimal such that the set

$$\left\{z_1 \in \mathbb{C} : |z_1| = 1 \text{ and, for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \ge \frac{1}{b}\right\}$$

is non empty. Thanks to our bounds on the cardinality of Z_1, we can use the first-order theory of the reals, together with Theorem 10, to conclude that b is algebraic and $\|b\| = \|\mathbf{u}\|^{O(1)}$.

Define the function $g : [b, \infty) \to \mathbb{R}$ as follows:

$$g(x) = \min \left\{ h(z_1, z_2, z_3) - \mu : (z_1, z_2, z_3) \in \mathbb{T} \text{ and for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \geq \frac{1}{x} \right\}.$$

In the full proof we show that we can compute in polynomial time a polynomial $P \in \mathbb{Z}[x]$ such that, for all $x \in [b, \infty)$,

$$g(x) \geq \frac{1}{P(x)} \tag{6}$$

with $\|P\| = \|\mathbf{u}\|^{O(1)}$

By Proposition 7 we can find $\epsilon \in (0, 1)$ and $N = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for all $n > N$, we have $|r(n)| < (1 - \epsilon)^n$, and moreover $1/\epsilon = 2^{\|\mathbf{u}\|^{O(1)}}$. In addition, by Proposition 11, there is $N' = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for every $n \geq N'$

$$\frac{1}{2P(n^{\|\mathbf{u}\|^D})} > (1 - \epsilon)^n. \tag{7}$$

Combining Equations (4)–(7), we get

$$\begin{aligned} \frac{u_n}{\rho^n} &= A + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) + r(n) \geq -\mu + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) - (1 - \epsilon)^n \geq g(n^{\|\mathbf{u}\|^D}) - (1 - \epsilon)^n \\ &\geq \frac{1}{P(n^{\|\mathbf{u}\|^D})} - (1 - \epsilon)^n = \frac{1}{2P(n^{\|\mathbf{u}\|^D})} + \frac{1}{2P(n^{\|\mathbf{u}\|^D})} - (1 - \epsilon)^n \geq \frac{1}{2P(n^{\|\mathbf{u}\|^D})} \end{aligned}$$

provided $n > \max\{M, N, N'\}$. We thus have that $\frac{u_n}{\rho^n}$ is eventually lower bounded by an inverse polynomial and hence we have effective divergence bounds in this case.

Finally, Cases 1–6 allow us to conclude both Theorem 12 and Theorem 13.

3.2 Hardness of Divergence

We now turn to show lower bounds for the divergence problem. Surprisingly, our lower bounds hold already for homogeneous LRS, and for the divergence decision problem, even without requiring effectively computable bounds.

In [14], it is shown that the Ultimate Positivity problem for homogeneous LRS of order at least 6 is hard, in the sense that if Ultimate Positivity is decidable for such LRS, then certain hard open problems in Diophantine approximation become solvable. We show hardness of divergence for homogeneous LRS of order at least 6 by reducing from Ultimate Positivity.

▶ **Theorem 14.** *Ultimate Positivity is reducible to Divergence.*

Proof. We show a reduction from the Ultimate Positivity problem for non-degenerate LRS of order 6, shown to be hard in [14]. The key ingredient in the reduction is Theorem 1.

Consider a non-degenerate homogeneous LRS $\langle u_n \rangle$ of order 6 with dominant modulus ρ, and let $\mu = \max\left\{2, \frac{2}{\rho}\right\}$, then the sequence $v_n = \mu^n u_n$ is a non-degenerate homogeneous LRS of order 6 with dominant modulus $\mu\rho \geq 2$. By Theorem 1, taking $\epsilon = \frac{1}{2}$, it follows that there exists $N \in \mathbb{N}$ such that $|v_n| \geq 2^{n/2}$ for every $n > N$. It immediately follows that v_n is ultimately positive iff $v_n \to \infty$. Clearly, however, v_n and u_n have the same sign, and therefore u_n is ultimately positive iff v_n diverges, and we are done. ◀

4 Positivity and Ultimate Positivity

In this section we study the Positivity and Ultimate Positivity problems for inhomogeneous LRS. These problems were studied in [14, 15, 16] for homogeneous LRS. Using Proposition 3 and some careful analysis, we extend the decidability results to the inhomogeneous case.

We start by citing some results from [14, 15, 16], split to upper and lower bounds.

▶ **Theorem 15** (Upper Bounds from [14, 15, 16]).

1. *Positivity and Ultimate Positivity are decidable for homogeneous LRS of order 5 or less with complexities in* $\mathbf{coNP}^{\mathbf{PosSLP}}$ *and* **PTIME***, respectively.*
2. *Positivity is decidable for simple homogeneous LRS or order 9 or less with complexity in* $\mathbf{coNP}^{\mathbf{PosSLP}}$*.*
3. *Ultimate Positivity is decidable for simple homogeneous LRS of any order with complexity in* **PTIME***.*
4. *Effective Ultimate Positivity is solvable for simple homogeneous LRS of order 9 or less with complexity in* **PTIME***.*

The following notion of hardness will be made precise in Section 4.2.

▶ **Theorem 16** (Lower Bounds from [14, 15, 16]). *Positivity and Ultimate Positivity for LRS of order at least 6 are hard with respect to certain hard open problems in Diophantine approximation.*

4.1 Upper Bounds

We proceed to prove analogous results to Theorem 15 for inhomogeneous LRS.

Theorem 15(1.) along with Proposition 3 readily give us the following:

▶ **Theorem 17.** *Positivity and Ultimate Positivity are decidable for inhomogeneous LRS of order* 4 *or less, with complexity in* $\mathbf{coNP}^{\mathbf{PosSLP}}$*.*

For simple LRS, things become more involved, as Proposition 3 does not preserve simplicity. However, Property 5 shows that simplicity is almost preserved, up to the multiplicity of the characteristic root 1. As we now show, this is sufficient to obtain upper bounds for inhomogeneous simple LRS.

We start by addressing effective Ultimate Positivity, which we then use for addressing Positivity.

▶ **Theorem 18.** *Effective Ultimate Positivity is solvable in polynomial time for simple inhomogeneous LRS of order* 8 *or less.*

Proof. Let $\mathbf{v}$ be a simple, non-degenerate, inhomogeneous LRS or order 8 or less, and consider the homogeneous LRS $\mathbf{u} = \textsc{hom}(\mathbf{v})$. By Proposition 3, $\mathbf{u}$ is of order at most 9. If $\mathbf{u}$ is a simple LRS, then by [15] we can effectively decide its Ultimate Positivity. We hence assume that $\mathbf{u}$ is not simple.

By Property 5, it follows that the characteristic roots of $\mathbf{u}$ all have multiplicity 1, apart from the characteristic root 1 which has multiplicity 2. Consider the dominant modulus ρ of $\mathbf{u}$. If $\rho > 1$, then by writing the exponential polynomial of $\mathbf{u}$, we have

$$\frac{u_n}{\rho^n} = a + \sum_{i=1}^{m}(c_i\lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{8}$$

with $a \in \mathbb{R}$, $c_i \in \mathbb{C} \setminus \mathbb{R}$ and $|\lambda_i| = 1$ for every $1 \leq i \leq m$, and $|r(n)|$ exponentially decaying. Crucially, since 1 is not a dominant characteristic root, its effect is enveloped in $r(n)$. Specifically, we observe that the analysis of effective Ultimate Positivity in [15] only relies on Proposition 7. Since this holds in the case at hand, we can effectively decide Ultimate Positivity when 1 is not a dominant characteristic root.

Finally, if 1 is a dominant characteristic root, the exponential polynomial of $\mathbf{u}$ can be written as

$$u_n = A(n) + \sum_{i=1}^{m} (c_i \lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n). \tag{9}$$

We observe that in this case, u_n is ultimately positive iff it diverges (indeed, clearly $|u_n| \to \infty$). Thus, we can reduce the problem to divergence, and proceed with the analysis as in Section 3 Case 2.

This concludes the proof that Ultimate Positivity is effectively decidable for simple inhomogeneous LRS of order at most 8. ◀

Similarly to Theorem 18, we are able to conclude the following result, whose proof can be found in the full version.

▶ **Theorem 19.** *Ultimate Positivity is decidable in polynomial time for simple inhomogeneous LRS of any order.*

Finally, using Theorem 18, we can solve the Positivity problem (see the full version for the proof).

▶ **Theorem 20.** *Positivity is decidable for simple inhomogeneous LRS of order* 8 *or less, with complexity in* $\mathbf{coNP}^{\mathbf{PosSLP}}$.

4.2 Lower Bounds

We now turn to study lower bounds, proving analogous results to Theorem 16 for inhomogeneous LRS. Similarly to [15], the hardness results we present are with respect to long standing open problems in Diophantine approximation. Before stating our results, we require some definitions from Diophantine approximation. We refer the reader to [13, 15] for comprehensive references.

For any $x \in \mathbb{R}$, we define the *Lagrange constant* of x as

$$L_\infty(x) = \inf\{c \in \mathbb{R} : |x - \frac{n}{m}| \leq \frac{c}{m^2} \text{ for infinitely many } m, n \in \mathbb{Z}\}$$

and the *approximation type* of x as

$$L(x) = \inf\{c \in \mathbb{R} : |x - \frac{n}{m}| \leq \frac{c}{m^2} \text{ for some } m, n \in \mathbb{Z}\}$$

For the vast majority of transcendental numbers, the Lagrange constant and the approximation type are unknown, despite significant work [8, 15], and the problem of computing them is a major open problem. In the following, we show that the decidability of Ultimate Positivity (resp. Positivity) for inhomogeneous LRS of order 5 or more would imply a major breakthrough in computing the Lagrange constant (resp. approximation type) for a large class of transcendental numbers.

▶ **Theorem 21.** *If Ultimate Positivity is decidable for inhomogeneous rational LRS of order at least 5 then there is an algorithm that computes the Lagrange constant of any number* $\theta/2\pi$ *such that* $e^{i\theta}$ *has rational real and imaginary parts.*

Proof. In [15], it is shown that deciding Ultimate Positivity of the homogeneous LRS of order 6 given by

$$u_n = r \sin n\theta - n(1 - \cos n\theta) \text{ and } v_n = -r \sin n\theta - n(1 - \cos n\theta)$$

for every $r \in \mathbb{Q}$ such that $r > 0$ and $\theta \in (0, 2\pi)$ such that $e^{i\theta}$ has rational real and imaginary parts would allow one to compute $L_\infty(\theta/2\pi)$.

We observe that both sequences u_n and v_n fall under the premise of Proposition 4. Thus, by applying Proposition 4, we obtain an equivalent inhomogeneous LRS of order 5, concluding the proof. ◀

A similar proof, using the results of [15], gives us also the following theorem.

▶ **Theorem 22.** *If Positivity is decidable for inhomogeneous rational LRS of order at least 5 then there is an algorithm that computes the approximation type of any number $\theta/2\pi$ such that $e^{i\theta}$ has rational real and imaginary parts.*

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