

MacNeille Completion and Buchholz' Omega Rule for Parameter-Free Second Order Logics

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Abstract

Buchholz' Ω -rule is a way to give a syntactic, possibly ordinal-free proof of cut elimination for various subsystems of second order arithmetic. Our goal is to understand it from an algebraic point of view. Among many proofs of cut elimination for higher order logics, Maehara and Okada's algebraic proofs are of particular interest, since the essence of their arguments can be algebraically described as the (Dedekind-) *MacNeille completion* together with Girard's reducibility candidates. Interestingly, it turns out that the Ω -rule, formulated as a rule of logical inference, finds its algebraic foundation in the MacNeille completion.

In this paper, we consider a family of sequent calculi $\mathbf{LIP} = \bigcup_{n \geq -1} \mathbf{LIP}_n$ for the *parameter-free* fragments of second order intuitionistic logic, that corresponds to the family $\mathbf{ID}_{<\omega} = \bigcup_{n < \omega} \mathbf{ID}_n$ of arithmetical theories of inductive definitions up to ω . In this setting, we observe a formal connection between the Ω -rule and the MacNeille completion, that leads to a way of interpreting second order quantifiers in a first order way in Heyting-valued semantics, called the Ω -*interpretation*. Based on this, we give a (partly) algebraic proof of cut elimination for \mathbf{LIP}_n , in which quantification over reducibility candidates, that are genuinely second order, is replaced by the Ω -interpretation, that is essentially first order. As a consequence, our proof is locally formalizable in \mathbf{ID} -theories.

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1 Introduction

This paper is concerned with cut elimination for subsystems of second order logics. It is of course very well known that the full second order classical/intuitionistic logics admit cut elimination. Then why are we interested in their subsystems? A primary reason is that proving cut elimination for a subsystem is often very hard if one is sensitive to the metatheory within which (s)he works. This is witnessed by the vast literature in the traditional proof theory. In fact, proof theorists are not just interested in proving cut elimination itself, but in identifying a *characteristic principle* P (e.g. ordinals, ordinal diagrams, combinatorial principles and inductive definitions) for each system of logic, arithmetic and set theory, by proving cut elimination within a weak metatheory (e.g. \mathbf{PRA} , \mathbf{IS}_1 and \mathbf{RCA}_0) extended by P . Our motivation is to understand those hard proofs and results from an algebraic perspective.



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One can distinguish several types of cut elimination proofs for higher order logics/arithmetic: (i) syntactic proofs by ordinal assignment (e.g. Gentzen's consistency proof for **PA**), (ii) syntactic but ordinal-free proofs, (iii) semantic proofs based on Schütte's semivaluation or its variants (e.g. [30]), (iv) algebraic proofs based on completions (the list is not intended to be exhaustive). Historically (i) and (iii) precede (ii) and (iv), but understanding (i) takes years just to catch up with the expanding universe of ordinal notations, while (iii) is slightly unsatisfactory for the truly constructive logician since it involves *reductio ad absurdum* and weak König's lemma. Hence we address (ii) and (iv) in this paper.

For (ii), a very useful and versatile technique is Buchholz' Ω -rule. Introduced in the context of ordinal analysis of **ID**-theories [11] and further developed in, e.g., [14], it later yielded an ordinal-free proof of cut elimination for fragments/extensions of Π_1^1 -**CA**₀ [12, 4, 3]. However, the Ω -rule is notoriously complicated, and is hard to grasp its meaning at a glance. Even its semantic soundness is not clear at all. While Buchholz gives an account based on the BHK interpretation [11], we will try to give an algebraic account in this paper.

For (iv), there is a very conspicuous algebraic proof of cut elimination for higher order logics which may be primarily ascribed to Maehara [24] and Okada [26, 28]. In contrast to (iii), these algebraic proofs are fully constructive; no use of *reductio ad absurdum* or any nondeterministic principle. More importantly, it extends to proofs of normalization for proof nets and typed lambda calculi [27]. While their arguments can be described in various dialects (e.g. phase semantics in linear logic), apparently most neutral and most widely accepted would be to speak in terms of algebraic completions: the essence of their arguments can be described as the (*Dedekind-*)*MacNeille completion* together with Girard's reducibility candidates, as we will explain in Section 6.

Having a syntactic technique on one hand and an algebraic methodology on the other, it is natural to ask the relationship between them. To make things concrete, we consider, in addition to the standard sequent calculus **LI2** for second order intuitionistic logic, a family of subcalculi $\mathbf{LIP} = \bigcup_{n \geq -1} \mathbf{LIP}_n$ for the *parameter-free* fragments of **LI2**. **LIP** is the intuitionistic counterpart of the classical sequent calculus studied in [32]. Although we primarily work on intuitionistic logic, all results in this paper (except Proposition 11) carry over to classical logic too.

As we will see, cut elimination based on the Ω -rule technique works for **LIP**. Moreover, it turns out to be intimately related to the MacNeille completion in that the Ω -rule in our setting is not sound in Heyting-valued semantics in general, but is sound when the underlying algebra is the MacNeille completion of the Lindenbaum algebra. This observation leads to a curious way of interpreting second order formulas in a first order way, that we call the *Ω -interpretation*. The basic idea already appears in Altenkirch and Coquand [6], but ours is better founded and accommodates the existential quantifier too.

The Ω -rule and Ω -interpretation are two sides of the same coin. Combining them together, we obtain a (partly) algebraic proof of cut elimination for \mathbf{LIP}_n ($n \geq 0$), that is comparable with Aehlig's result [1] for the parameter-free, negative fragments of second order Heyting arithmetic. As with [1], our proof does not rely on (second order quantification over) reducibility candidates, and is formalizable in theories of finitely iterated inductive definitions.

The rest of this paper is organized as follows. In Section 2 we recall some basics of the MacNeille completion. In Section 3 we give some background on iterated inductive definitions and then introduce a family of sequent calculi $\mathbf{LIP} = \bigcup \mathbf{LIP}_n$. In Section 4 we transform the arithmetical Ω -rule into a logical one and explain how it works for **LIP**. In Section 5, we turn to the algebraic side of the Ω -rule, establish a connection with the MacNeille completion, and

motivate the Ω -interpretation. In Section 6, we review an algebraic proof of cut elimination for **LI2**, and then gives an algebraic proof for **LIP_n** based on the Ω -interpretation. Appendix A fully describes the sequent calculi studied in this paper. Omitted proofs are found in the full version of this paper available at <https://arxiv.org/abs/1804.11066>.

2 MacNeille completion

Let $\mathbf{A} = \langle A, \wedge, \vee \rangle$ be a lattice. A *completion* of \mathbf{A} is an embedding $e : \mathbf{A} \rightarrow \mathbf{B}$ into a complete lattice $\mathbf{B} = \langle B, \wedge, \vee \rangle$. We often assume that e is an inclusion map so that $\mathbf{A} \subseteq \mathbf{B}$.

For example, let $[0, 1]_{\mathbb{Q}} := [0, 1] \cap \mathbb{Q}$ be the chain of rational numbers in the unit interval (seen as a lattice). Then it admits an obvious completion $[0, 1]_{\mathbb{Q}} \subseteq [0, 1]$. For another example, let \mathbf{A} be a Boolean algebra. Then it also admits a completion $e : \mathbf{A} \rightarrow \mathbf{A}^{\sigma}$, where $\mathbf{A}^{\sigma} := \langle \wp(\text{uf}(\mathbf{A})), \cap, \cup, -, A, \emptyset \rangle$, the powerset algebra on the set of ultrafilters of \mathbf{A} , and $e(a) := \{u \in \text{uf}(\mathbf{A}) : a \in u\}$.

A completion $\mathbf{A} \subseteq \mathbf{B}$ is \vee -dense if $x = \vee\{a \in A : a \leq x\}$ holds for every $x \in B$. It is \wedge -dense if $x = \wedge\{a \in A : x \leq a\}$. A \vee -dense and \wedge -dense completion is called a *MacNeille completion*.

► **Theorem 1.** *Every lattice \mathbf{A} has a MacNeille completion unique up to isomorphism [8, 29]. A MacNeille completion is regular, i.e., preserves all joins and meets that already exist in \mathbf{A} .*

Coming back to the previous examples:

- $[0, 1]_{\mathbb{Q}} \subseteq [0, 1]$ is MacNeille, since $x = \inf\{a \in \mathbb{Q} : x \leq a\} = \sup\{a \in \mathbb{Q} : a \leq x\}$ for any $x \in [0, 1]$. It is regular since if $q = \lim_{n \rightarrow \infty} q_n$ holds in \mathbb{Q} , then it holds in \mathbb{R} too.
- $e : \mathbf{A} \rightarrow \mathbf{A}^{\sigma}$ is not regular when \mathbf{A} is an infinite Boolean algebra. In fact, the Stone space $\text{uf}(\mathbf{A})$ is compact, so collapses any infinite union of open sets into a finite one. It is actually a *canonical extension*, that has been extensively studied in ordered algebra and modal logic [23, 21, 20].

MacNeille completions behave better than canonical extensions in preservation of existing limits, but the price to pay is loss of generality. Let $\mathcal{DL}(\mathcal{HA}, \mathcal{BA}, \text{resp.})$ be the variety of distributive lattices (Heyting algebras, Boolean algebras, resp.).

► **Theorem 2.** ■ *\mathcal{DL} is not closed under MacNeille completions [18].*

- \mathcal{HA} and \mathcal{BA} are closed under MacNeille completions.
- \mathcal{HA} and \mathcal{BA} are the only nontrivial subvarieties of \mathcal{HA} closed under MacNeille completions [9].

As is well known, completion is a standard algebraic way to prove conservativity of extending first order logics to higher order ones. The above result indicates that MacNeille completions work for classical and intuitionistic logics, but not for proper intermediate logics. See [33] for more on MacNeille completions.

Now an easy but crucial observation follows.

► **Proposition 3.** *A completion $\mathbf{A} \subseteq \mathbf{B}$ is MacNeille iff the rules below are valid:*

$$\frac{\{a \leq y\}_{a \leq x}}{x \leq y} \quad \frac{\{x \leq a\}_{y \leq a}}{x \leq y}$$

where x, y range over B and a over A .

The left rule has infinitely many premises indexed by the set $\{a \in A : a \leq x\}$. It states that if $a \leq x$ implies $a \leq y$ for every $a \in A$, then $x \leq y$. This is valid just in case $x = \bigvee\{a \in A : a \leq x\}$. Likewise, the right rule states that if $y \leq a$ implies $x \leq a$ for every $a \in A$, then $x \leq y$. This is valid just in case $y = \bigwedge\{a \in A : y \leq a\}$.

As we will see, the above looks very similar to the Ω -rule. This provides a link between lattice theory and proof theory.

3 Parameter-free second order intuitionistic logic

3.1 Arithmetic

We here recall theories of inductive definitions. Let \mathbf{IS}_1 , \mathbf{PA} and $\mathbf{PA2}$ be the first order arithmetic with Σ_1^0 induction, that with full induction, and the second order arithmetic with full induction and comprehension, respectively. Given a theory T of arithmetic, $T[X]$ denotes the extension of T with a single set variable X and atomic formulas of the form $X(t)$.

A great many subsystems of $\mathbf{PA2}$ are considered in the literature. For instance, the system $\Pi_1^1\text{-CA}_0$ is obtained by restricting the induction and comprehension axiom schemata to Π_1^1 formulas. Even weaker are theories of iterated inductive definitions \mathbf{ID}_n with $n < \omega$, that are obtained as follows.

\mathbf{ID}_0 is just \mathbf{PA} . To obtain \mathbf{ID}_{n+1} , consider a formula $\varphi(X, x)$ in $\mathbf{ID}_n[X]$ which contains no first order free variables other than x and no negative occurrences of X . It can be seen as a monotone map $\varphi^{\mathbb{N}} : \wp(\mathbb{N}) \rightarrow \wp(\mathbb{N})$ sending a set $X \subseteq \mathbb{N}$ to $\{n \in \mathbb{N} : \mathbb{N} \models \varphi(X, n)\}$, so it has a least fixed point $I_\varphi^{\mathbb{N}}$. Based on this intuition, one adds a unary predicate symbol I_φ for each such φ to the language of \mathbf{ID}_n and axioms

$$\varphi(I_\varphi) \subseteq I_\varphi, \quad \varphi(\tau) \subseteq \tau \rightarrow I_\varphi \subseteq \tau$$

for every abstract $\tau = \lambda x.\xi(x)$ in the new language. Here $\varphi(I_\varphi)$ is a shorthand for the abstract $\lambda x.\varphi(I_\varphi, x)$ and $\tau_1 \subseteq \tau_2$ is for $\forall x.\tau_1(x) \rightarrow \tau_2(x)$. The induction schema is extended to the new language. This defines the system \mathbf{ID}_{n+1} . Notice that \mathbf{ID}_{n+1} does not involve any set variable. Finally, let $\mathbf{ID}_{<\omega}$ be the union of all \mathbf{ID}_n with $n < \omega$.

Clearly $\mathbf{ID}_{<\omega}$ can be seen as a subsystem of $\Pi_1^1\text{-CA}_0$. In fact, any fixed point atom $I_\varphi(t)$ can be replaced by second order formula

$$I_\varphi(t) := \forall X.\forall x(\varphi(X, x) \rightarrow X(x)) \rightarrow X(t).$$

Given a formula ψ of $\mathbf{ID}_{<\omega}$, we write ψ^I for the formula of $\mathbf{PA2}$ obtained by repeating the above replacement. This makes the axioms of $\mathbf{ID}_{<\omega}$ all provable in $\Pi_1^1\text{-CA}_0$.

The converse is not strictly true, but it is known that $\mathbf{ID}_{<\omega}$ has the same proof theoretic strength and the same arithmetical consequences with $\Pi_1^1\text{-CA}_0$.

Let us point out that a typical use of inductive definition is to define a provability predicate. Let T be a sequent calculus system, and suppose that we are given a formula $\varphi(X, x)$ saying that there is a rule in T with conclusion sequent x (coded by a natural number) and premises $Y \subseteq X$. Then $I_\varphi^{\mathbb{N}}$ gives the set of all provable sequents in T . Notice that the premise set Y can be infinite. It is for this reason that \mathbf{ID} -theories are suitable metatheories for infinitary proof systems. See [13] for more on inductive definitions.

3.2 Second order intuitionistic logic

In this subsection, we formally introduce sequent calculus $\mathbf{LI2}$ for the second order intuitionistic logic with full comprehension, that is an intuitionistic counterpart of Takeuti's classical calculus $\mathbf{G}^1\mathbf{LC}$ [31].

Consider a language L that consists of (first order) function symbols and predicate symbols. A typical example is the language $L_{\mathbf{PA}}$ of Peano arithmetic, which contains a predicate symbol for equality and function symbols for all primitive recursive functions. Let

- **Var**: a countable set of term variables x, y, z, \dots ,
- **Tm**(L): the set of first order terms t, u, v, \dots over L ,
- **VAR**: the set of set variables X, Y, Z, \dots

The set $\mathbf{FM}(L)$ of second order formulas is defined by:

$$\varphi, \psi ::= p(\vec{t}) \mid X(t) \mid \perp \mid \varphi \star \psi \mid Qx.\varphi \mid QX.\varphi,$$

where $p \in L$, $\star \in \{\wedge, \vee, \rightarrow\}$ and $Q \in \{\forall, \exists\}$. We define $\top := \perp \rightarrow \perp$. When the language L is irrelevant, we write $\mathbf{Tm} := \mathbf{Tm}(L)$ and $\mathbf{FM} := \mathbf{FM}(L)$. Given φ , let $\mathbf{FV}(\varphi)$ and $\mathbf{Fv}(\varphi)$ be the set of free set variables and that of free term variables in φ , respectively.

Typical formulas in $\mathbf{FM}(L_{\mathbf{PA}})$ are

$$\begin{aligned} \mathbf{N}(t) &:= \forall X. [\forall x (X(x) \rightarrow X(x+1)) \wedge X(0) \rightarrow X(t)], \\ \mathbf{E}(t) &:= \forall X. \forall x. [t = x \wedge X(x) \rightarrow X(t)]. \end{aligned}$$

We assume the standard variable convention that α -equivalent formulas are syntactically identical, so that substitutions can be applied without variable clash. A *term substitution* is a function $\circ : \mathbf{Var} \rightarrow \mathbf{Tm}$. Given $\varphi \in \mathbf{FM}$, the substitution instance φ° is defined as usual. Likewise, a *set substitution* is a function $\bullet : \mathbf{VAR} \rightarrow \mathbf{ABS}$, where $\mathbf{ABS} := \{\lambda x. \xi : \xi \in \mathbf{FM}\}$ is the set of *abstracts*. Instance φ^\bullet is obtained by replacing each atomic formula $X(t)$ with $X^\bullet(t)$ and applying β -reduction.

Let $\mathbf{SEQ} := \{\Gamma \Rightarrow \Pi : \Gamma, \Pi \subseteq_{\text{fin}} \mathbf{FM}, |\Pi| \leq 1\}$ be the set of *sequents of LI2*. We write Γ, Δ to denote $\Gamma \cup \Delta$. Rules of **LI2** include:

$$\begin{array}{c} \frac{}{\Gamma, \varphi \Rightarrow \varphi} \text{ (id)} \quad \frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X. \varphi(X), \Gamma \Rightarrow \Pi} \text{ (\forall X left)} \quad \frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X. \varphi(X)} \text{ (\forall X right)} \\ \frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \text{ (cut)} \quad \frac{\varphi(Y), \Gamma \Rightarrow \Pi}{\exists X. \varphi(X), \Gamma \Rightarrow \Pi} \text{ (\exists X left)} \quad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X. \varphi(X)} \text{ (\exists X right)} \end{array}$$

where $\tau \in \mathbf{ABS}$ and rules $(\forall X \text{ right})$ and $(\exists X \text{ left})$ are subject to the eigenvariable condition $Y \notin \mathbf{FV}(\Gamma, \Pi)$. The inference rules for other connectives can be found in Appendix A. The indicated occurrence of $\forall X. \varphi(X)$ in $(\forall X \text{ left})$ is the *main formula* and $\varphi(\tau)$ is the *minor formula* of rule $(\forall X \text{ left})$. The same terminology applies to other inference rules too.

A well known fact essentially due to [31] is that if a Π_2^0 sentence φ is provable in **PA2**, then $\forall y. \mathbf{E}(y), \Gamma^{\mathbf{N}} \Rightarrow \varphi^{\mathbf{N}}$ is provable in **LI2**, where Γ is a finite set of true Π_1^0 sentences (equality axioms, basic axioms of Peano arithmetic and defining axioms of primitive recursive functions), and $\varphi^{\mathbf{N}}$ is obtained from φ by relativizing each first order quantifier Qx to $Qx \in \mathbf{N}$. In particular if φ is Σ_1^0 , we obtain $\forall y. \mathbf{E}(y), \Gamma \Rightarrow \varphi$, and the assumption $\forall y. \mathbf{E}(y)$ can be eliminated by another relativization with respect to \mathbf{E} , so that we eventually obtain $\Gamma \Rightarrow \varphi$ in **LI2**. A consequence is that

$$\mathbf{IS}_1 \vdash \mathbf{CE}(\mathbf{LI2}) \rightarrow \mathbf{1CON}(\mathbf{PA2}),$$

where $\mathbf{CE}(\mathbf{LI2})$ is a Π_2^0 sentence stating that **LI2** admits cut elimination, and $\mathbf{1CON}(\mathbf{PA2})$ is that **PA2** is *1-consistent*, that is, all provable Σ_1^0 sentences are true.

Thus 1-consistency of **PA2** is reduced to cut elimination for **LI2**. We also have the converse, also provably in \mathbf{IS}_1 . The reason is that cut elimination for **LI2** is “locally” provable in **PA2**, that is, whenever $\mathbf{LI2} \vdash \Gamma \Rightarrow \Pi$, **PA2** proves a Σ_1^0 statement “ $\mathbf{LI2} \vdash^{cf} \Gamma \Rightarrow \Pi$ ” (that

is, “ $\Gamma \Rightarrow \Pi$ is cut-free provable in **LI2**”), and moreover, a derivation of the latter statement (in **PA2**) can be primitive recursively obtained from any derivation of the former (in **LI2**). Hence 1-consistency of **PA2** implies cut elimination for **LI2** (in **IS₁**). See [7] for a concise explanation.

The equivalence holds because **PA2** and **LI2** have a “matching” proof theoretic strength. We are going to introduce subsystems of **LI2** that match $\mathbf{ID}_{<\omega} = \bigcup_{n \in \omega} \mathbf{ID}_n$ in this sense.

3.3 Parameter-free fragments

Now let us introduce parameter-free subsystems of **LI2**. We first define the set $\mathbf{FMP}_n \subseteq \mathbf{FM}$ of *parameter-free formulas at level n* for every $n \geq -1$.

\mathbf{FMP}_{-1} is just the set of formulas in **FM** without second order quantifiers. It is also denoted by **Fm**. For $n \geq 0$, \mathbf{FMP}_n is defined by:

$$\varphi, \psi ::= p(\vec{t}) \mid t \in X \mid \perp \mid \varphi \star \psi \mid Qx.\varphi \mid QX.\xi,$$

where $\star \in \{\wedge, \vee, \rightarrow\}$, $Q \in \{\forall, \exists\}$ and ξ is any formula in \mathbf{FMP}_{n-1} such that $\mathbf{FV}(\xi) \subseteq \{X\}$. Thus $QX.\xi$ is free of set parameters, though may contain first order free variables. Finally, **FMP** is the union of all \mathbf{FMP}_n .

For instance, both $\mathbf{N}(t)$ and $\mathbf{E}(t)$ belong to \mathbf{FMP}_0 so that relativizations $\varphi^{\mathbf{N}}$, $\varphi^{\mathbf{E}}$ belong to \mathbf{FMP}_0 too, whenever φ is an arithmetical formula. Furthermore, each fixed point atom I_φ with φ arithmetical translates to

$$I_\varphi^{\mathbf{N}}(t) := \forall X.\forall x \in \mathbf{N}(\varphi^{\mathbf{N}}(X, x) \rightarrow X(x)) \rightarrow X(t),$$

that belongs to \mathbf{FMP}_1 . We write $\varphi^{I\mathbf{N}}$ to denote the translation of \mathbf{ID}_1 -formula φ in \mathbf{FMP}_1 . Likewise, any formula φ of \mathbf{ID}_n translates to a formula $\varphi^{I\mathbf{N}}$ in \mathbf{FMP}_n . On the other hand, second order definitions of positive connectives $\{\exists, \vee\}$:

$$\begin{aligned} \exists X.\varphi(X) &:= \forall Y.\forall X(\varphi(X) \rightarrow Y(*)) \rightarrow Y(*), \\ \varphi \vee \psi &:= \forall Y.(\varphi \rightarrow Y(*)) \wedge (\psi \rightarrow Y(*)) \rightarrow Y(*) \end{aligned}$$

with $Y \notin \mathbf{FV}(\varphi, \psi)$ and $*$ a constant, are no longer available. They do not belong to **FMP**, so restricting to the negative fragment $\{\forall, \wedge, \rightarrow\}$ causes a serious loss of expressivity in the parameter-free setting.

Sequent calculus **LIP** (resp. **LIP_n**) is obtained from **LI2** by restricting the formulas to **FMP** (resp. \mathbf{FMP}_n). Most importantly, when one applies rules ($\forall X$ left) and ($\exists X$ right) to introduce $QX.\varphi$, the minor formula $\varphi(\tau)$ must belong to **FMP** (resp. \mathbf{FMP}_n).

LIP is an intuitionistic counterpart of the classical calculus studied in [32], and **LIP₋₁** is just the ordinary sequent calculus for first order intuitionistic logic, that is also denoted by **LI**.

As before, arithmetical systems \mathbf{ID}_n reduce to logical systems **LIP_n**. For every Π_2^0 sentence φ of \mathbf{ID}_n , $\mathbf{ID}_n \vdash \varphi$ implies $\mathbf{LIP}_n \vdash \forall y.\mathbf{E}(y), \Gamma^{\mathbf{N}} \Rightarrow \varphi^{I\mathbf{N}}$, where Γ is a finite set of true Π_1^0 sentences. In particular, if φ is a Σ_1^0 sentence of **PA**, we obtain $\mathbf{LIP}_n \vdash \Gamma \Rightarrow \varphi$. As a consequence,

$$\mathbf{IS}_1 \vdash \mathbf{CE}(\mathbf{LIP}_n) \rightarrow \mathbf{1CON}(\mathbf{ID}_n), \quad \mathbf{IS}_1 \vdash \mathbf{CE}(\mathbf{LIP}) \rightarrow \mathbf{1CON}(\mathbf{ID}_{<\omega}).$$

The converse is obtained by proving cut elimination for **LIP_n** locally within \mathbf{ID}_n .

4 Ω -rule

4.1 Introduction to Ω -rule

Cut elimination in a higher order setting is tricky, since a principal reduction step

$$\frac{\frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)} (\forall X \text{ right}) \quad \frac{\varphi(\tau) \Rightarrow \Pi}{\forall X.\varphi(X) \Rightarrow \Pi} (\forall X \text{ left})}{\Gamma \Rightarrow \Pi} (\text{cut}) \quad \Longrightarrow \quad \frac{\Gamma \Rightarrow \varphi(\tau) \quad \varphi(\tau) \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} (\text{cut})$$

may yield a bigger cut formula so that one cannot simply argue by induction on the complexity of the cut formula. The Ω -rule, introduced by [11], is an alternative of rule ($\forall X$ left) that allows us to circumvent this difficulty. Buchholz [12] includes an ordinal-free proof of (partial) cut elimination for a parameter-free subsystem \mathbf{BI}_1^- of analysis. It was later extended to complete cut elimination for the same system [4], and to complete cut elimination for $\Pi_1^1\text{-CA}_0 + \mathbf{BI}$ (bar induction) [3]. The Ω -rule further finds applications in modal fixed point logics [22, 25]. It is used to show strong normalization for the parameter-free fragments of System F, provably in \mathbf{ID} -theories [5].

As a starter, let us consider the most direct translation of the arithmetical Ω -rule [12] into our setting¹. We extend \mathbf{LI} by enlarging the formulas to \mathbf{FMP}_0 and adding rules ($\forall X$ right) and

$$\frac{\{ \Delta, \Gamma \Rightarrow \Pi \}_{\Delta \in |\forall X.\varphi|^b}}{\forall X.\varphi, \Gamma \Rightarrow \Pi} (\Omega^b)$$

where $|\forall X.\varphi|^b$ consists of $\Delta \subseteq_{\text{fin}} \mathbf{Fm}$ such that $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ for some $Y \notin \mathbf{FV}(\Delta)$ (recall that “ cf ” indicates cut-free provability).

Rule (Ω^b) has infinitely many premises indexed by $|\forall X.\varphi|^b$. Observe a similarity with the characteristic rules of MacNeille completion (Proposition 3). In Section 5, we will provide a further link between them.

(Ω^b) is intended to be an alternative of ($\forall X$ left). Indeed, we can prove $\forall X.\varphi \Rightarrow \varphi(\tau)$ for an arbitrary abstract τ as follows. Let $\Delta \in |\forall X.\varphi|^b$, that is, $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ for some $Y \notin \mathbf{FV}(\Delta)$. We then have $\Delta \Rightarrow \varphi(\tau)$ in the extended system by substituting τ for Y . Hence rule (Ω^b) yields $\forall X.\varphi \Rightarrow \varphi(\tau)$.

Moreover, rule (Ω^b) suggests a natural step of cut elimination. Consider a cut:

$$\frac{\frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi(X)} (\forall X \text{ right}) \quad \frac{\{ \Delta \Rightarrow \Pi \}_{\Delta \in |\forall X.\varphi|^b}}{\forall X.\varphi \Rightarrow \Pi} (\Omega^b)}{\Gamma \Rightarrow \Pi} (\text{cut})$$

If $\Gamma \subseteq_{\text{fin}} \mathbf{Fm}$ and $\Gamma \Rightarrow \varphi(Y)$ is cut-free provable, then Γ belongs to $|\forall X.\varphi|^b$, so the conclusion $\Gamma \Rightarrow \Pi$ is just one of the infinitely many premises.

However, rule (Ω^b) cannot be combined with the standard rules for first order quantifiers.

► **Proposition 4.** *System $\mathbf{LI} + (\forall X \text{ right}) + (\Omega^b)$ is inconsistent.*

¹ Actually the original rule has assumptions indexed by *derivations* of $\Delta \Rightarrow \varphi(Y)$, not by Δ 's themselves. As an advantage, one obtains a concrete operator for cut elimination and reduces the complexity of inductive definition: the original semiformal system can be defined by inductive definition on a bounded formula, while ours requires a Π_1^0 formula. However, this point is irrelevant for the subsequent argument.

Proof. Consider formula $\varphi := X(c) \rightarrow X(x)$ with c a constant. We claim that $\forall X.\varphi \Rightarrow \perp$ is provable. Let $\Delta \in |\forall X.\varphi|^b$, that is, $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow Y(c) \rightarrow Y(x)$ for some $Y \notin \mathbf{FV}(\Delta)$. Since the sequent is first order and $Y(c) \rightarrow Y(x)$ is not provable, Craig's interpolation theorem yields $\Delta \Rightarrow \perp$. Hence $\forall X.\varphi \Rightarrow \perp$ follows by (Ω^b) . Since both $\exists x.\forall X.\varphi \Rightarrow \perp$ and $\Rightarrow \exists x.\forall X.\varphi$ are provable, we obtain \perp . \blacktriangleleft

The primary reason for inconsistency is that (Ω^b) is not closed under term substitutions, while the standard treatment of first order quantifiers assumes that all rules are closed under term substitutions. Hence we have to weaken first order quantifier rules to obtain a consistent system. A reasonable way is to replace $(\forall x \text{ right})$ and $(\exists x \text{ left})$ with Schütte's ω -rules:

$$\frac{\{\Gamma \Rightarrow \varphi(t)\}_{t \in \mathbf{Tm}}}{\Gamma \Rightarrow \forall x.\varphi(x)} \quad (\omega \text{ right}) \qquad \frac{\{\varphi(t), \Gamma \Rightarrow \Pi\}_{t \in \mathbf{Tm}}}{\exists x.\varphi(x), \Gamma \Rightarrow \Pi} \quad (\omega \text{ left})$$

This allows us to prove *partial* cut elimination: if a sequent $\Gamma \Rightarrow \Pi$ is provable, then it is cut-free provable, *provided that* $\Gamma \cup \Pi \subseteq \mathbf{Fm}$. To prove *complete* cut elimination, we need to work with more sophisticated calculi.

4.2 Cut elimination by Ω -rule

We now introduce a family of infinitary sequent calculi and use them to prove complete cut elimination for \mathbf{LIP} . The proof idea is entirely due to [3].

We first prepare an isomorphic copy of each \mathbf{FMP}_n , denoted by $\overline{\mathbf{FMP}}_n$. $\overline{\mathbf{FMP}}_{-1}$ is just $\mathbf{FMP}_{-1} = \mathbf{Fm}$. For $n \geq 0$, $\overline{\mathbf{FMP}}_n$ is defined by:

$$\vartheta, \vartheta' ::= p(\vec{t}) \mid t \in X \mid \perp \mid \vartheta \star \vartheta' \mid Qx.\vartheta \mid \overline{\forall}X.\chi \mid \overline{\exists}X.\chi,$$

where $\star \in \{\wedge, \vee, \rightarrow\}$, $Q \in \{\forall, \exists\}$ and χ is any formula in $\overline{\mathbf{FMP}}_{n-1}$ such that $\mathbf{FV}(\chi) \subseteq \{X\}$. Given $\vartheta \in \overline{\mathbf{FMP}} := \bigcup \overline{\mathbf{FMP}}_n$, its *level* is defined by $\text{level}(\vartheta) := \min\{k : \vartheta \in \overline{\mathbf{FMP}}_k\}$. Given a formula $\varphi \in \mathbf{FMP}$, $\overline{\varphi} \in \overline{\mathbf{FMP}}$ is obtained by overlining all the second order quantifiers in it.

We are going to introduce a hybrid calculus $\mathbf{LI}\Omega_n$ for each $n \geq -1$ in which sequents are made of formulas in $\mathbf{FMP} \cup \overline{\mathbf{FMP}}_n$. Those in $\overline{\mathbf{FMP}}_n$ are intended to be potential cut formulas, i.e., ancestors of cut formulas in a derivation (called *implicit* in [32]), and are treated by using Ω -rules. Those in \mathbf{FMP} are remaining formulas, that are treated as in \mathbf{LIP} .

Calculus $\mathbf{LI}\Omega_{-1}$ is just \mathbf{LIP} where sequents consist of formulas in $\mathbf{FMP} = \mathbf{FMP} \cup \overline{\mathbf{FMP}}_{-1}$ and cut formulas are restricted to $\mathbf{Fm} = \overline{\mathbf{FMP}}_{-1}$.

Suppose that $\mathbf{LI}\Omega_{k-1}$ has been defined for every $0 \leq k \leq n$. For each $\overline{\forall}X.\vartheta$ and $\overline{\exists}X.\vartheta$ of level k , let

$$\begin{aligned} |\overline{\forall}X.\vartheta(X)| &:= \{\Delta : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y) \text{ for some } Y \notin \mathbf{FV}(\Delta)\} \\ |\overline{\exists}X.\vartheta(X)| &:= \{(\Delta \Rightarrow \Lambda) : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \vartheta(Y), \Delta \Rightarrow \Lambda \text{ for some } Y \notin \mathbf{FV}(\Delta, \Lambda)\}. \end{aligned}$$

Note that $\Delta \cup \Lambda \subseteq \mathbf{FMP} \cup \overline{\mathbf{FMP}}_{k-1}$. Calculus $\mathbf{LI}\Omega_n$ is defined as follows:

- Sequents consist of formulas in $\mathbf{FMP} \cup \overline{\mathbf{FMP}}_n$.
- Cut formulas are restricted to $\overline{\mathbf{FMP}}_n$.
- First order quantifiers are treated by rules $(\forall x \text{ left})$, $(\exists x \text{ right})$, $(\omega \text{ right})$ and $(\omega \text{ left})$.
- Second order quantifiers in \mathbf{FMP} are treated by rules $(\forall X \text{ left})$, $(\forall X \text{ right})$, $(\exists X \text{ left})$ and $(\exists X \text{ right})$ as in \mathbf{LIP} .

- Second order quantifiers in $\overline{\text{FMP}}_n$ are treated by the following rules ($k = 0, \dots, n$):

$$\frac{\vartheta(Y), \Gamma \Rightarrow \Pi}{\overline{\exists}X.\vartheta(X), \Gamma \Rightarrow \Pi} (\overline{\exists}X \text{ left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y)}{\Gamma \Rightarrow \overline{\forall}X.\vartheta(X)} (\overline{\forall}X \text{ right})$$

$$\frac{\{\Delta, \Gamma \Rightarrow \Pi\}_{\Delta \in |\overline{\forall}X.\vartheta|}}{\overline{\forall}X.\vartheta, \Gamma \Rightarrow \Pi} (\Omega_k \text{ left}) \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad \{\Delta, \Gamma \Rightarrow \Pi\}_{\Delta \in |\overline{\forall}X.\vartheta|}}{\Gamma \Rightarrow \Pi} (\tilde{\Omega}_k \text{ left})$$

$$\frac{\{\Gamma, \Delta \Rightarrow \Lambda\}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists}X.\vartheta|}}{\Gamma \Rightarrow \overline{\exists}X.\vartheta} (\Omega_k \text{ right}) \qquad \frac{\{\Gamma, \Delta \Rightarrow \Lambda\}_{(\Delta \Rightarrow \Lambda) \in |\overline{\exists}X.\vartheta|} \quad \vartheta(Y), \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} (\tilde{\Omega}_k \text{ right})$$

where k is the level of $\overline{\forall}X.\vartheta$, $\overline{\exists}X.\vartheta$ and rules $(\overline{\exists}X \text{ left})$, $(\overline{\forall}X \text{ right})$, $(\tilde{\Omega}_k \text{ left})$ and $(\tilde{\Omega}_k \text{ right})$ are subject to the eigenvariable condition ($Y \notin \text{FV}(\Gamma, \Pi)$).

- Other connectives are treated as in **LIP**. See Appendix A for a complete list of inference rules.

It is admittedly complicated. First of all, notice that the rule $(\tilde{\Omega}_k \text{ left})$ is derivable by combining $(\overline{\forall}X \text{ right})$, $(\Omega_k \text{ left})$ and (cut) . It is nevertheless included for a technical reason. The same applies to rule $(\tilde{\Omega}_k \text{ right})$.

On the other hand, rules $(\Omega_k \text{ left})$ and $(\Omega_k \text{ right})$ are our real concern. The former should be read as follows: whenever $\mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y)$ implies $\mathbf{LI}\Omega_n \vdash \Delta, \Gamma \Rightarrow \Pi$ for every Δ with $Y \notin \text{FV}(\Delta)$, one can conclude $\mathbf{LI}\Omega_n \vdash \overline{\forall}X.\vartheta, \Gamma \Rightarrow \Pi$.

Now let us list some key lemmas for cut elimination. The proofs are found in the full version.

► **Lemma 5** (Embedding). $\mathbf{LIP}_n \vdash \Gamma \Rightarrow \Pi$ implies $\mathbf{LI}\Omega_n \vdash \Gamma \Rightarrow \Pi$.

► **Lemma 6**. $\mathbf{LI}\Omega_n \vdash \Gamma \Rightarrow \Pi$ implies $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$.

► **Lemma 7** (Collapsing). $\mathbf{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$ implies $\mathbf{LI}\Omega_{n-1} \vdash^{cf} \Gamma \Rightarrow \Pi$, provided that $\Gamma \cup \Pi \subseteq \text{FMP} \cup \overline{\text{FMP}}_{n-1}$.

Proof. By induction on the length of the cut-free derivation of $\Gamma \Rightarrow \Pi$ in $\mathbf{LI}\Omega_n$. If it ends with $(\tilde{\Omega}_n \text{ left})$ (see above), we have $\mathbf{LI}\Omega_{n-1} \vdash^{cf} \Gamma \Rightarrow \vartheta(Y)$ by the induction hypothesis, noting that $\vartheta(Y) \in \overline{\text{FMP}}_{n-1}$. Hence $\Gamma \in |\overline{\forall}X.\vartheta|$, so $\Gamma, \Gamma \Rightarrow \Pi$ is among the premises. Therefore $\mathbf{LI}\Omega_{n-1} \vdash^{cf} \Gamma \Rightarrow \Pi$ by the induction hypothesis again.

Rule $(\tilde{\Omega}_n \text{ left})$ is treated similarly. When $n = 0$, one has to replace $(\omega \text{ right})$ and $(\omega \text{ left})$ by $(\forall x \text{ right})$ and $(\exists x \text{ left})$ respectively, that is easy. ◀

► **Theorem 8** (Cut elimination). $\mathbf{LIP} \vdash \Gamma \Rightarrow \Pi$ implies $\mathbf{LIP} \vdash^{cf} \Gamma \Rightarrow \Pi$.

Proof. The sequent is provable in \mathbf{LIP}_n for some $n < \omega$, so in $\mathbf{LI}\Omega_n$ by Lemma 5. Noting that $\Gamma \cup \Pi \subseteq \text{FMP}$, we obtain a cut-free derivation in $\mathbf{LI}\Omega_{-1}$ by Lemmas 6 and 7, that is also a cut-free derivation in **LIP**. ◀

Of course the above argument can be restricted to a proof of cut elimination for \mathbf{LIP}_n . From a metatheoretical point of view, the most significant part is to define provability predicates $\mathbf{LI}\Omega_{-1}, \dots, \mathbf{LI}\Omega_n$. $\mathbf{LI}\Omega_{-1}$ is finitary, so is definable in $\mathbf{PA} = \mathbf{ID}_0$. $\mathbf{LI}\Omega_0$ is obtained by an inductive definition relying on $\mathbf{LI}\Omega_{-1}$, so is definable in \mathbf{ID}_1 . By repetition, we observe that $\mathbf{LI}\Omega_n$ is definable in \mathbf{ID}_{n+1} . Moreover, $\mathbf{LI}\Omega$ is definable with a uniform inductive definition in \mathbf{ID}_ω . Once a suitable provability predicate has been defined, the rest of argument can be smoothly formalized. Hence we obtain a folklore:

$$\mathbf{ID}_{n+1} \vdash \text{CE}(\mathbf{LIP}_n), \quad \mathbf{ID}_\omega \vdash \text{CE}(\mathbf{LIP}).$$

5 Ω -rule and MacNeille completion

In this section, we establish a formal connection between the Ω -rule and the MacNeille completion. Let us start by introducing algebraic semantics for full second order calculus **LI2**.

Let L be a language. A (complete) *Heyting-valued prestructure* for L is $\mathcal{M} = \langle \mathbf{A}, M, \mathcal{D}, \mathcal{L} \rangle$ where $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \top, \perp \rangle$ is a complete Heyting algebra, M is a nonempty set (*term domain*), $\emptyset \neq \mathcal{D} \subseteq A^M$ (*abstract domain*) and \mathcal{L} consists of a function $f^{\mathcal{M}} : M^n \rightarrow M$ for each n -ary function symbol $f \in L$ and $p^{\mathcal{M}} : M^n \rightarrow A$ for each n -ary predicate symbol $p \in L$. Thus $p^{\mathcal{M}}$ is an \mathbf{A} -valued subset of M^n .

It is not our purpose to systematically develop a model theory for intuitionistic logic. We will use prestructures only for proving conservative extension and cut elimination. Hence we assume $M = \mathsf{Tm}$ and $f^{\mathcal{M}}(\vec{t}) = f(\vec{t})$ below, that simplifies the interpretation of formulas a lot.

A *valuation* on \mathcal{M} is a function $\mathcal{V} : \mathsf{VAR} \rightarrow \mathcal{D}$. The *interpretation* of formulas $\mathcal{V} : \mathsf{FM} \rightarrow \mathbf{A}$ is inductively defined as follows:

$$\begin{array}{llll} \mathcal{V}(p(\vec{t})) & := & p^{\mathcal{M}}(\vec{t}) & \mathcal{V}(X(t)) & := & \mathcal{V}(X)(t) \\ \mathcal{V}(\perp) & := & \perp & \mathcal{V}(\varphi \star \psi) & := & \mathcal{V}(\varphi) \star \mathcal{V}(\psi) \\ \mathcal{V}(\forall x.\varphi(x)) & := & \bigwedge_{t \in \mathsf{Tm}} \mathcal{V}(\varphi(t)) & \mathcal{V}(\exists x.\varphi(x)) & := & \bigvee_{t \in \mathsf{Tm}} \mathcal{V}(\varphi(t)) \\ \mathcal{V}(\forall X.\varphi) & := & \bigwedge_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi) & \mathcal{V}(\exists X.\varphi) & := & \bigvee_{F \in \mathcal{D}} \mathcal{V}[F/X](\varphi) \end{array}$$

where $\star \in \{\wedge, \vee, \rightarrow\}$ and $\mathcal{V}[F/X]$ is an update of \mathcal{V} that maps X to F . \mathcal{V} can also be extended to a function $\mathcal{V} : \mathsf{ABS} \rightarrow \mathbf{A}^{\mathsf{Tm}}$ by $\mathcal{V}(\lambda x.\varphi)(t) := \mathcal{V}(\varphi[t/x])$. \mathcal{M} is called a *Heyting-valued structure* if $\mathcal{V}(\tau) \in \mathcal{D}$ holds for every valuation \mathcal{V} and every $\tau \in \mathsf{ABS}$. Clearly \mathcal{M} is a Heyting-valued structure if $\mathcal{D} = \mathbf{A}^{\mathsf{Tm}}$. Such a structure is called *full*.

Given a sequent $\Gamma \Rightarrow \Pi$, let $\mathcal{V}(\Gamma) := \bigwedge \{\mathcal{V}(\varphi) : \varphi \in \Gamma\}$ ($:= \top$ if Γ is empty). $\mathcal{V}(\Pi) := \mathcal{V}(\psi)$ if $\Pi = \{\psi\}$, and $\mathcal{V}(\Pi) := \perp$ if Π is empty. It is routine to verify:

► **Lemma 9** (Soundness). *If $\mathbf{LI2} \vdash \Gamma \Rightarrow \Pi$, then $\Gamma \Rightarrow \Pi$ is valid, that is, $\mathcal{V}(\Gamma^\circ) \leq \mathcal{V}(\Pi^\circ)$ holds for every valuation \mathcal{V} on every Heyting structure \mathcal{M} and every term substitution \circ .*

To illustrate use of algebraic semantics, we prove an elementary fact that **LI2** is a conservative extension of **LI**.

Let \mathbf{L} be the Lindenbaum algebra for **LI**, that is, $\mathbf{L} := \langle \mathsf{Fm}/\sim, \wedge, \vee, \rightarrow, \top, \perp \rangle$ where $\varphi \sim \psi$ iff $\mathbf{LI} \vdash \varphi \leftrightarrow \psi$. The equivalence class of φ with respect to \sim is denoted by $[\varphi]$. \mathbf{L} is a Heyting algebra in which

$$(*) \quad [\forall x.\varphi(x)] = \bigwedge_{t \in \mathsf{Tm}} [\varphi(t)], \quad [\exists x.\varphi(x)] = \bigvee_{t \in \mathsf{Tm}} [\varphi(t)]$$

hold. Given a sequent $\Gamma \Rightarrow \Pi$, elements $[\Gamma]$ and $[\Pi]$ in \mathbf{L} are naturally defined.

Let \mathbf{G} be a *regular* completion of \mathbf{L} . Then $\mathcal{M}(\mathbf{G}) := \langle \mathbf{G}, \mathsf{Tm}, \mathbf{G}^{\mathsf{Tm}}, \mathcal{L} \rangle$ is a full Heyting structure, where \mathcal{L} consists of a \mathbf{G} -valued predicate $p^{\mathcal{M}(\mathbf{G})}$ defined by $p^{\mathcal{M}(\mathbf{G})}(\vec{t}) := [p(\vec{t})]$ for each $p \in L$ (in addition to interpretations of function symbols). Define a valuation \mathcal{I} by $\mathcal{I}(X)(t) := [X(t)]$. We then have $\mathcal{I}(\varphi) = [\varphi]$ for every $\varphi \in \mathsf{Fm}$ by regularity (be careful here: $(*)$ may fail in \mathbf{G} if it is not regular).

Now, suppose that **LI2** proves $\Gamma \Rightarrow \Pi$ with $\Gamma \cup \Pi \subseteq \mathsf{Fm}$. Then we have $\mathcal{I}(\Gamma) \leq \mathcal{I}(\Pi)$ by Lemma 9, so $[\Gamma] \leq [\Pi]$, that is, $\mathbf{LI} \vdash \Gamma \Rightarrow \Pi$. This proves that **LI2** is a conservative extension of **LI**.

Although this argument cannot be fully formalized in **PA2** because of Gödel's second incompleteness, it does admit a local formalization in **PA2**. In contrast, the above argument, when applied to **LIP_n**, cannot be locally formalized in **ID_n**. The reason is simply that

\mathbf{ID}_n does not have second order quantifiers, which are needed to write down the definitions of $\mathcal{V}(\forall X.\varphi)$ and $\mathcal{V}(\exists X.\varphi)$. To circumvent this, a crucial observation is that $\mathcal{V}(\forall X.\varphi)$ and $\mathcal{V}(\exists X.\varphi)$ admit alternative first order definitions if the completion is MacNeille. It is here that one finds a connection between the MacNeille completion and the Ω -rule.

► **Theorem 10.** *Let \mathbf{L} be the Lindenbaum algebra for \mathbf{LI} and $\mathbf{L} \subseteq \mathbf{G}$ a regular completion. $\mathcal{M}(\mathbf{G})$ and \mathcal{I} are defined as above. For every sentence $\forall X.\varphi$ in \mathbf{FMP}_0 , the following are equivalent.*

1. $\mathcal{I}(\forall X.\varphi) = \bigvee \{a \in \mathbf{L} : a \leq \mathcal{I}(\forall X.\varphi)\}$.
2. $\mathcal{I}(\forall X.\varphi) = \bigvee \{[\Delta] \in \mathbf{L} : \Delta \in |\forall X.\varphi|^b\}$.
3. *The inference below is sound for every $y \in \mathbf{G}$:*

$$\frac{\{ \mathcal{I}(\Delta) \leq y \}_{\Delta \in |\forall X.\varphi|^b}}{\mathcal{I}(\forall X.\varphi) \leq y}$$

If \mathbf{G} is the MacNeille completion of \mathbf{F} , all the above hold.

Proof. (1. \Leftrightarrow 2.) Let $a = [\Delta]$. It is sufficient to prove that $a \leq \mathcal{I}(\forall X.\varphi)$ iff $\Delta \in |\forall X.\varphi|^b$, i.e., $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ for some $Y \notin \mathbf{FV}(\Delta)$. If $a \leq \mathcal{I}(\forall X.\varphi)$, choose $Y \notin \mathbf{FV}(\Delta)$ and let $F_Y(t) := [Y(t)]$. We then have $[\Delta] \leq \mathcal{I}[F_Y/X](\varphi(X)) = [\varphi(Y)]$, that is, $\mathbf{LI} \vdash \Delta \Rightarrow \varphi(Y)$. By cut elimination for \mathbf{LI} , we obtain $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$. Conversely, suppose that $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ with $Y \notin \mathbf{FV}(\Delta)$. It implies $[\Delta] = \mathcal{I}(\Delta) = \mathcal{I}[F/Y](\Delta) \leq \mathcal{I}[F/Y](\varphi(Y))$ for every $F \in \mathbf{G}^{\text{tm}}$ by Lemma 9. Hence $[\Delta] \leq \mathcal{I}(\forall X.\varphi(X))$.

(2. \Rightarrow 3.) Straightforward by noting that $[\Delta] = \mathcal{I}(\Delta)$.

(3. \Rightarrow 2.) Let $y := \bigvee \{[\Delta] \in \mathbf{L} : \Delta \in |\forall X.\varphi|^b\}$. Then $\mathcal{I}(\Delta) = [\Delta] \leq y$ holds for every $\Delta \in |\forall X.\varphi|^b$, so $\mathcal{I}(\forall X.\varphi) \leq y$ by 3. Since $\Delta \in |\forall X.\varphi|^b$ implies $[\Delta] \leq \mathcal{I}(\forall X.\varphi)$ as proved above, we also have $y \leq \mathcal{I}(\forall X.\varphi)$. ◀

The equivalence in Theorem 10 is quite suggestive, since 3. is an algebraic interpretation of rule (Ω^b), while 1. is a characteristic of the MacNeille completion (Proposition 3). Equation 2. suggests a way of interpreting second order formulas without using second order quantifiers at the meta-level. All these are true if the completion is MacNeille. It should be mentioned that essentially the same as 2. has been already observed by Altenkirch and Coquand [6] in the context of lambda calculus (without making any connection to the Ω -rule and the MacNeille completion). Indeed, they consider a logic which roughly amounts to the negative fragment of our \mathbf{LIP}_0 and employ equation 2. to give a “finitary” proof of (partial) normalization theorem for a parameter-free fragment of System F (see also [2, 5] for extensions). However, their argument is technically based on a downset completion, that is not MacNeille. As is well known, such a naive completion does not work well for the positive connectives $\{\exists, \vee\}$. In contrast, when \mathbf{G} is the MacNeille completion of \mathbf{L} , we also have

$$\mathcal{I}(\exists X.\varphi) = \bigwedge \{[\Delta] \rightarrow [\Lambda] \in \mathbf{L} : (\Delta \Rightarrow \Lambda) \in |\exists X.\varphi|^b\},$$

where $(\Delta \Rightarrow \Lambda) \in |\exists X.\varphi|^b$ iff $\mathbf{LI} \vdash^{cf} \varphi(Y), \Delta \Rightarrow \Lambda$ for some $Y \notin \mathbf{FV}(\Delta, \Lambda)$. We thus claim that the insight by Altenkirch and Coquand is augmented and better understood in terms of the MacNeille completion.

It is interesting to see that (second order) \forall is interpreted by (first order) \bigvee while \exists is by \bigwedge . We call this style of interpretation the Ω -interpretation, that is the algebraic side of the Ω -rule, and that will play a key role in the next section. We conclude our discussion by reporting a counterexample for general soundness.

► **Proposition 11.** *There is a Heyting-valued structure in which (Ω^b) is not sound.*

Proof. Let A be the three-element chain $\{0 < 0.5 < 1\}$ seen as a Heyting algebra. Consider the language that only consists of a term constant $*$. Then a full Heyting-valued structure $\mathcal{A} := \langle \mathbf{A}, \text{Tm}, \mathbf{A}^{\text{Tm}}, \mathcal{L} \rangle$ is naturally obtained. Let $\varphi := (X(*) \rightarrow \perp) \vee X(*)$. It is easy to see that $\mathcal{V}(\forall X.\varphi) = 0.5$ for every valuation \mathcal{V} .

Now consider the following instance:

$$\frac{\{ \Delta \Rightarrow \perp \}_{\Delta \in |\forall X.\varphi|^b}}{\forall X.\varphi \Rightarrow \perp} (\Omega^b)$$

We claim that it is not sound for a valuation \mathcal{V} such that $\mathcal{V}(X(t)) = 0$ for every $X \in \text{VAR}$ and $t \in \text{Tm}$. Suppose that $\Delta \in |\forall X.\varphi|^b$, i.e., $\mathbf{LI} \vdash^{cf} \Delta \Rightarrow \varphi(Y)$ with $Y \notin \text{FV}(\Delta)$. Then $\mathcal{V}(\Delta) \leq \bigwedge_{F \in \mathbf{A}^{\text{Tm}}} \mathcal{V}[F/X](\varphi) = 0.5$ by Lemma 9. But Δ is first order, so only takes value 0 or 1 under our assumption on \mathcal{V} . Hence $\mathcal{V}(\Delta) = 0$, that is, all premises are satisfied. However, $\mathcal{V}(\forall X.\varphi) = 0.5 > 0$, that is, the conclusion is not satisfied. ◀

This invokes a natural question. Is it possible to find a *Boolean-valued* counterexample? In other words, is the Ω -rule classically sound? This question is left open.

6 Algebraic cut elimination

6.1 Polarities and Heyting frames

This section is devoted to algebraic proofs of cut elimination. We begin with a very old concept due to Birkhoff [10], that provides a uniform framework for both MacNeille completion and cut elimination.

A *polarity* $\mathbf{W} = \langle W, W', R \rangle$ consists of two sets W, W' and a binary relation $R \subseteq W \times W'$. Given $X \subseteq W$ and $Z \subseteq W'$, let

$$X^\triangleright := \{z \in W' : x R z \text{ for every } x \in X\}, \quad Z^\triangleleft := \{x \in W : x R z \text{ for every } z \in Z\}.$$

For example, let $\mathbf{Q} := \langle \mathbb{Q}, \mathbb{Q}, \leq \rangle$. Then X^\triangleright is the set of upper bounds of X and Z^\triangleleft is the set of lower bounds of Z . Hence $(X^{\triangleright\triangleleft}, X^\triangleright)$ is a Dedekind cut for every $X \subseteq \mathbb{Q}$ bounded above.

The pair $(\triangleright, \triangleleft)$ forms a *Galois connection*:

$$X \subseteq Z^\triangleleft \iff X^\triangleright \supseteq Z$$

so induces a closure operator $\gamma(X) := X^{\triangleright\triangleleft}$ on $\wp(W)$, that is, $X \subseteq \gamma(Y)$ iff $\gamma(X) \subseteq \gamma(Y)$ for any $X, Y \subseteq W$. Note that $X \subseteq W$ is closed iff there is $Z \subseteq W'$ such that $X = Z^\triangleleft$.

In the following, we write $\gamma(x) := \gamma(\{x\})$, $x^\triangleright := \{x\}^\triangleright$ and $z^\triangleleft := \{z\}^\triangleleft$. Let

$$\mathcal{G}(\mathbf{W}) := \{X \subseteq W : X = \gamma(X)\},$$

$$X \wedge Y := X \cap Y, \quad X \vee Y := \gamma(X \cup Y), \quad \top := W \text{ and } \perp := \gamma(\emptyset).$$

► **Lemma 12.** *If \mathbf{W} is a polarity, then $\mathbf{W}^+ := \langle \mathcal{G}(\mathbf{W}), \wedge, \vee \rangle$ is a complete lattice.*

The lattice \mathbf{W}^+ is not always distributive because of the use of γ in the definition of \vee . To ensure distributivity, we have to impose a further structure on \mathbf{W} .

A *Heyting frame* is $\mathbf{W} = \langle W, W', R, \circ, \varepsilon, \backslash \rangle$, where

- $\langle W, W', R \rangle$ is a polarity,
- $\langle W, \circ, \varepsilon \rangle$ is a monoid,

- $\parallel : W \times W' \longrightarrow W'$ satisfies $x \circ y R z \iff y R x \parallel z$ for every $x, y \in W$ and $z \in W'$,
- the following inferences are valid:

$$\frac{x \circ y R z}{y \circ x R z} (e) \quad \frac{\varepsilon R z}{x R z} (w) \quad \frac{x \circ x R z}{x R z} (c)$$

Clearly $x R z$ is an analogue of a sequent and (e), (w) and (c) correspond to exchange, weakening and contraction rules. By removing some/all of them, one obtains *residuated frames* that work for substructural logics as well [19, 16].

► **Lemma 13.** *If \mathbf{W} is a Heyting frame, $\mathbf{W}^+ := \langle \mathcal{G}(\mathbf{W}), \wedge, \vee, \rightarrow, \top, \perp \rangle$ is a complete Heyting algebra, where $X \rightarrow Y := \{y \in W : x \circ y \in Y \text{ for every } x \in X\}$.*

Polarities and Heyting frames are handy devices to obtain MacNeille completions. Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \top, \perp \rangle$ be a Heyting algebra. Then $\mathbf{W}_{\mathbf{A}} := \langle A, A, \leq, \wedge, \top, \rightarrow \rangle$ is a Heyting frame. Notice that the third condition above amounts to $x \wedge y \leq z$ iff $y \leq x \rightarrow z$.

► **Theorem 14.** *If \mathbf{A} is a Heyting algebra, then $\gamma : \mathbf{A} \longrightarrow \mathbf{W}_{\mathbf{A}}^+$ is a MacNeille completion.*

6.2 Algebraic cut elimination for full second order logic

We here outline an algebraic proof of cut elimination for the full second order calculus **LI2** that we attribute to Maehara [24] and Okada [26, 28]. This will be useful for a comparison with the parameter-free case **LIP**_{n+1}, that is to be discussed in the next subsection.

Let $\wp_{\text{fin}}(\text{FM})$ be the set of finite sets of formulas, so that $\langle \wp_{\text{fin}}(\text{FM}), \cup, \emptyset \rangle$ is a commutative idempotent monoid. Recall that **SEQ** denotes the set of sequents of **LI2**. There is a natural map $\parallel : \wp_{\text{fin}}(\text{FM}) \times \text{SEQ} \longrightarrow \text{SEQ}$ defined by $\Gamma \parallel (\Sigma \Rightarrow \Pi) := (\Gamma, \Sigma \Rightarrow \Pi)$. So

$$\mathbf{CF} := \langle \wp_{\text{fin}}(\text{FM}), \text{SEQ}, \Rightarrow_{\mathbf{LI2}}^{\text{cf}}, \cup, \emptyset, \parallel \rangle$$

is a Heyting frame, where $\Gamma \Rightarrow_{\mathbf{LI2}}^{\text{cf}} (\Sigma \Rightarrow \Pi)$ iff **LI2** $\vdash^{\text{cf}} \Gamma, \Sigma \Rightarrow \Pi$. In the following, we simply write φ for sequent $(\emptyset \Rightarrow \varphi) \in \text{SEQ}$.

CF is a frame in which $\Gamma \in \varphi^{\triangleleft}$ holds iff $\Gamma \Rightarrow \varphi$ is cut-free provable in **LI2**. In particular, $\varphi \in \varphi^{\triangleleft}$ always holds, so $\varphi \in \gamma(\varphi) \subseteq \varphi^{\triangleleft}$. It should also be noted that each $X \in \mathcal{G}(\mathbf{CF})$ is closed under weakening: if $\Delta \in X$ and $\Delta \subseteq \Sigma$, then $\Sigma \in X$.

Define a Heyting prestructure $\mathcal{CF} := \langle \mathbf{CF}^+, \text{Tm}, \mathcal{D}, \mathcal{L} \rangle$ by $p^{\mathcal{CF}}(\vec{t}) := \gamma(p(\vec{t}))$ for each predicate symbol p and

$$\mathcal{D} := \{F \in \mathcal{G}(\mathbf{CF})^{\text{Tm}} : F \text{ matches some } \tau \in \text{ABS}\},$$

where F matches $\lambda x. \xi(x)$ just in case $\xi(t) \in F(t) \subseteq \xi(t)^{\triangleleft}$ holds for every $t \in \text{Tm}$. This choice of $\mathcal{D} \subseteq \mathcal{G}(\mathbf{CF})^{\text{Tm}}$ is a logical analogue of Girard's *reducibility candidates* as noticed by Okada.

Given a set substitution \bullet and a valuation $\mathcal{V} : \text{VAR} \longrightarrow \mathcal{D}$, we say that \mathcal{V} matches \bullet if $\mathcal{V}(X)$ matches $X^{\bullet} \in \text{ABS}$ for every $X \in \text{VAR}$. That is, $X^{\bullet}(t) \in \mathcal{V}(X)(t) \subseteq X^{\bullet}(t)^{\triangleleft}$ holds for every $X \in \text{VAR}$ and $t \in \text{Tm}$. The following is what Okada [28] calls his *main lemma*.

► **Lemma 15.** *Let $\bullet : \text{VAR} \longrightarrow \text{ABS}$ be a substitution and \mathcal{V} be a valuation that matches \bullet . Then for every $\varphi \in \text{FM}$,*

$$\varphi^{\bullet} \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet \triangleleft}.$$

As a consequence, $\mathcal{V}(\tau) \in \mathcal{D}$ for every $\tau \in \text{ABS}$ (recall that $\mathcal{V}(\lambda x.\xi(x))(t) := \mathcal{V}(\xi(t))$). That is, \mathcal{CF} is a Heyting structure. For another consequence, define a valuation \mathcal{I} by $\mathcal{I}(X)(t) := \gamma(X(t))$, that matches the identity substitution. Then we have $\varphi \in \mathcal{I}(\varphi) \subseteq \varphi^\triangleleft$. More generally, for every sequent $\Gamma \Rightarrow \Pi$ we have $\Gamma \in \mathcal{I}(\Gamma)$ (by closure under weakening and $\mathcal{I}(\Gamma) = \bigcap \{\mathcal{I}(\varphi) : \varphi \in \Gamma\}$) and $\mathcal{I}(\Pi) \subseteq \Pi^\triangleleft$.

► **Theorem 16** (Completeness and cut elimination). *For every sequent $\Gamma \Rightarrow \Pi$, the following are equivalent.*

1. $\Gamma \Rightarrow \Pi$ is provable in **LI2**.
2. $\Gamma \Rightarrow \Pi$ is valid in all Heyting structures.
3. $\Gamma \Rightarrow \Pi$ is cut-free provable in **LI2**.

Proof. (1. \Rightarrow 2.) holds by Lemma 9, and (2. \Rightarrow 3.) by $\Gamma \in \mathcal{I}(\Gamma) \subseteq \mathcal{I}(\Pi) \subseteq \Pi^\triangleleft$ in \mathcal{CF} . ◀

Recall that the frame **CF** is defined by referring to cut-free provability in **LI2**. But the above theorem states that it coincides with provability. As a consequence, we have $\gamma(\varphi) = \varphi^\triangleleft$ for every formula φ , so that there is exactly one closed set X such that $\varphi \in X \subseteq \varphi^\triangleleft$. Hence the complete algebra **CF**⁺ can be restricted to a subalgebra **CF**₀⁺ with underlying set $\{\gamma(\varphi) : \varphi \in \text{FM}\}$. It is easy to see that **CF**₀⁺ is isomorphic to the Lindenbaum algebra for **LI2** (defined analogously to **L** in Section 5) and **CF**⁺ is the MacNeille completion of **CF**₀⁺. To sum up:

► **Proposition 17.** **CF**⁺ is the MacNeille completion of the Lindenbaum algebra for **LI2**.

Thus it turns out *a fortiori* that the essence of Maehara and Okada's proof lies in "MacNeille completion + Girard's reducibility candidates."

6.3 Algebraic cut elimination for **LIP**_{*n*+1}

We now proceed to an algebraic proof of cut elimination for **LIP**_{*n*+1} ($n \geq -1$). Although we have already shown cut elimination for **LIP**_{*n*+1} in Section 3, the proof does not formalize in **ID**_{*n*+1} but only in **ID**_{*n*+2}. Our goal here is to give another proof that locally formalizes in **ID**_{*n*+1}. To this end, we combine the algebraic argument in the previous subsection with the Ω -interpretation technique discussed in Section 5. To be more precise, our proof is only partly algebraic, since we employ calculus **LI** Ω _{*n*} and presuppose Lemmas 6 and 7 for **LI** Ω _{*n*} (but *not* for **LI** Ω _{*n*+1} unlike before).

Define a Heyting frame by

$$\mathbf{CF}_n := \langle \wp_{\text{fin}}(\text{FMP}_{n+1} \cup \overline{\text{FMP}}_n), \text{SEQ}_n, \Rightarrow_n^{cf}, \cup, \emptyset, \setminus \rangle,$$

where SEQ_n consists of sequents $\Gamma \Rightarrow \Pi$ with $\Gamma \cup \Pi \subseteq \text{FMP}_{n+1} \cup \overline{\text{FMP}}_n$, and $\Gamma \Rightarrow_n^{cf} (\Sigma \Rightarrow \Pi)$ holds just in case **LI** Ω _{*n*} $\vdash^{cf} \Gamma, \Sigma \Rightarrow \Pi$. This yields a *full* Heyting structure $\mathcal{CF}_n := \langle \mathbf{CF}_n^+, \text{Im}, \mathcal{G}(\mathbf{CF}_n)^{\text{Im}}, \mathcal{L} \rangle$, where $p^{\mathcal{CF}_n}(\vec{t}) := \gamma(p(\vec{t}))$.

Let $\mathcal{I} : \text{VAR} \rightarrow \mathcal{G}(\mathbf{CF}_n)^{\text{Im}}$ be a valuation given by $\mathcal{I}(X)(t) := \gamma(X(t))$. The interpretation $\mathcal{I} : \text{FMP}_{n+1} \rightarrow \mathcal{G}(\mathbf{CF}_n)$ is defined as in Section 5, except that

$$\begin{aligned} \mathcal{I}(\forall X.\varphi) &:= \gamma(\{\Delta : \Delta \Rightarrow_n^{cf} \overline{\varphi}(Y) \text{ for some } Y \notin \text{FV}(\Delta)\}), \\ \mathcal{I}(\exists X.\varphi) &:= \{(\Delta \Rightarrow \Lambda) : \overline{\varphi}(Y), \Delta \Rightarrow_n^{cf} \Lambda \text{ for some } Y \notin \text{FV}(\Delta, \Lambda)\}^\triangleleft. \end{aligned}$$

This interpretation is inspired by Theorem 10. As before, it avoids use of second order quantifiers at the meta-level, that is what we have called the Ω -interpretation in Section 5. Notice the use of overlining. The main lemma nevertheless holds with respect to \mathcal{I} .

► **Lemma 18.** $\bar{\varphi} \in \mathcal{I}(\varphi) \subseteq \bar{\varphi}^\triangleleft$ for every $\varphi \in \text{FMP}_n$. $\varphi \in \mathcal{I}(\varphi) \subseteq \varphi^\triangleleft$ for every $\varphi \in \text{FMP}_{n+1}$.

The following lemma is the hardest part of the proof.

► **Lemma 19.** Suppose that $F \in \mathcal{G}(\mathbf{CF}_n)^{\text{Tm}}$ satisfies $\tau(t) \in F(t) \subseteq \tau(t)^\triangleleft$ for some $\tau(x) \in \text{FMP}_{n+1}$. Then for every $\forall X.\varphi \in \text{FMP}_{n+1}$, we have $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X](\varphi) \subseteq \mathcal{I}(\exists X.\varphi)$.

Once the hardest lemma has been proved, the rest is an easy soundness argument.

► **Lemma 20.** If $\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$, then $\mathcal{I}(\Gamma^\circ) \subseteq \mathcal{I}(\Pi^\circ)$ holds for every substitution \circ .

Proof. We assume $\circ = \text{id}$ for simplicity. The proof proceeds by induction on the length of the derivation.

Suppose that it ends with $(\forall X \text{ left})$ with main formula $\forall X.\varphi$ and minor formula $\varphi(\tau)$. Define $F \in \mathcal{G}(\mathbf{CF}_n)^{\text{Tm}}$ by $F(t) := \mathcal{I}(\tau(t))$. By Lemma 18, this F satisfies the precondition of Lemma 19. Hence $\mathcal{I}(\forall X.\varphi) \subseteq \mathcal{I}[F/X](\varphi) = \mathcal{I}(\varphi(\tau))$, where the last equation can be shown by induction on φ . Soundness of $(\forall X \text{ left})$ follows immediately.

Suppose that the derivation ends with:

$$\frac{\Gamma \Rightarrow \varphi(Y)}{\Gamma \Rightarrow \forall X.\varphi} \text{ (\forall X right)}$$

Let $\Delta \in \mathcal{I}(\Gamma)$. We may assume that $Y \notin \text{FV}(\Delta)$, since otherwise we can rename Y to a new set variable. By the induction hypothesis and Lemma 18, we have $\Delta \in \mathcal{I}(\varphi(Y)) \subseteq \bar{\varphi}(Y)^\triangleleft$. Hence $\Delta \in \mathcal{I}(\forall X.\varphi)$. The other cases are similar. ◀

► **Lemma 21.** If $\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$, then $\text{LI}\Omega_n \vdash^{cf} \Gamma \Rightarrow \Pi$.

Proof. $\Gamma \in \mathcal{I}(\Gamma) \subseteq \mathcal{I}(\Pi) \subseteq \Pi^\triangleleft$ by Lemmas 20 and 18. ◀

Combining it with Lemma 7, we obtain:

► **Theorem 22 (Cut elimination).** Suppose that $\Gamma \cup \Pi \subseteq \text{FMP}_{n+1}$. If $\Gamma \Rightarrow \Pi$ is provable in LIP_{n+1} , then it is cut-free provable in LIP_{n+1} .

As before, the algebra \mathbf{CF}_n^+ coincides with the MacNeille completion of the Lindenbaum algebra for $\text{LI}\Omega_n$. Hence our proof can be described as “MacNeille completion + Ω -interpretation” in contrast to Maehara and Okada’s proof.

What is the gain of an algebraic proof compared with the syntactic one in Section 4? In order to prove Lemma 21, we have only employed provability predicate $\text{LI}\Omega_n$, that is definable in ID_{n+1} . Thus we have saved one inductive definition. Furthermore, the above argument can be locally formalized in ID_{n+1} . Hence by letting $m := n + 1$ we obtain a folklore:

$$\mathbf{I}\Sigma_1 \vdash \text{CE}(\text{LIP}_m) \leftrightarrow \text{1CON}(\text{ID}_m), \quad \mathbf{I}\Sigma_1 \vdash \text{CE}(\text{LIP}) \leftrightarrow \text{1CON}(\text{ID}_{<\omega}).$$

To our knowledge, the idea of combining the Ω -rule with a semantic argument to save one inductive definition is due to Aehlig [1], where Tait’s computability predicate is used instead of the MacNeille completion. He works on the parameter-free, *negative* fragments of second order Heyting arithmetic without induction, and proves a weak form of cut elimination in the matching ID -theories. That is comparable with our result, but ours is concerned with the *full* cut elimination theorem for a logical system with the *full* set of connectives (recall that second order definitions of positive connectives are not available in the parameter-free setting).

Conclusion. In this paper we have brought the Ω -rule into the logical setting, and studied it from an algebraic perspective. We have found an intimate connection with the MacNeille completion (Theorem 10), that is important in two ways. First, it provides a link between syntactic and algebraic approaches to cut elimination. Second, it leads to an algebraic form of the Ω -rule, called the Ω -interpretation, that augments a partial observation by Altenkirch and Coquand [6]. These considerations have led to Theorem 22, the intuitionistic analogue of Takeuti's fundamental cut elimination theorem [32], proved (partly) algebraically.

We prefer the algebraic approach, since it provides a *uniform* perspective to the complicated situation in nonclassical logics. Recall that there is a limitation on MacNeille completions: it does not work for proper intermediate logics (Theorem 2). On the other hand:

- There are infinitely many substructural logics such that the corresponding varieties of algebras are closed under MacNeille completions. As a consequence, these logics, when suitably formalized as sequent calculi, admit an algebraic proof of cut elimination [15, 16].
- There are infinitely many intermediate logics for which *hyper*-MacNeille completions work. As a consequence, these logics, when suitably formalized as *hyper*-sequent calculi, admit an algebraic proof of cut elimination [15, 17].

Thus proving cut elimination amounts to finding a suitable notion of algebraic completion. Although this paper has focused on the easiest case of parameter-free intuitionistic logics, we hope that our approach will eventually lead to an algebraic understanding of *hard* results in proof theory.

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A

 Definitions of sequent calculi

A.1 Sequent calculi LI2, LIP and LIP_n

Sequents of **LI2** consist of formulas in FM. Inference rules are as follows:

$$\begin{array}{c}
 \frac{}{\Gamma, \varphi \Rightarrow \varphi} \text{ (id)} \qquad \frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \text{ (cut)} \\
 \\
 \frac{}{\perp, \Gamma \Rightarrow \Pi} (\perp \text{ left}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \perp} (\perp \text{ right}) \\
 \\
 \frac{\varphi_i, \Gamma \Rightarrow \Pi}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Pi} (\wedge \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi_1 \quad \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2} (\wedge \text{ right}) \\
 \\
 \frac{\varphi_1, \Gamma \Rightarrow \Pi \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Pi} (\vee \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2} (\vee \text{ right}) \\
 \\
 \frac{\Gamma \Rightarrow \varphi_1 \quad \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \rightarrow \varphi_2, \Gamma \Rightarrow \Pi} (\rightarrow \text{ left}) \qquad \frac{\varphi_1, \Gamma \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \rightarrow \varphi_2} (\rightarrow \text{ right}) \\
 \\
 \\
 \\
 \frac{\varphi(t), \Gamma \Rightarrow \Pi}{\forall x. \varphi(x), \Gamma \Rightarrow \Pi} (\forall x \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi(y) \quad y \notin \text{Fv}(\Gamma)}{\Gamma \Rightarrow \forall x. \varphi(x)} (\forall x \text{ right}) \\
 \\
 \frac{\varphi(y), \Gamma \Rightarrow \Pi \quad y \notin \text{Fv}(\Gamma, \Pi)}{\exists x. \varphi(x), \Gamma \Rightarrow \Pi} (\exists x \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x. \varphi(x)} (\exists x \text{ right}) \\
 \\
 \frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X. \varphi(X), \Gamma \Rightarrow \Pi} (\forall X \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi(Y) \quad Y \notin \text{FV}(\Gamma)}{\Gamma \Rightarrow \forall X. \varphi(X)} (\forall X \text{ right}) \\
 \\
 \frac{\varphi(Y), \Gamma \Rightarrow \Pi \quad Y \notin \text{FV}(\Gamma, \Pi)}{\exists X. \varphi(X), \Gamma \Rightarrow \Pi} (\exists X \text{ left}) \qquad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X. \varphi(X)} (\exists X \text{ right})
 \end{array}$$

LIP (resp. **LIP_n** with $n \geq -1$) is obtained by restricting the formulas to FMP (resp. FMP_n).

A.2 Sequent calculi LIΩ_n

LIΩ₋₁ is just **LIP** where cut formulas are restricted to Fm.

For $n \geq 0$, sequents of **LIΩ_n** consist of formulas in $\text{FMP} \cup \overline{\text{FMP}}_n$. Inference rules are (id), (cut), those for propositional connectives and the following rules (where ϑ stands for a

formula in $\overline{\text{FMP}}_{n-1}$):

$$\begin{array}{l}
\frac{\varphi(t), \Gamma \Rightarrow \Pi}{\forall x.\varphi(x), \Gamma \Rightarrow \Pi} \text{ (\forall x left)} \qquad \frac{\{ \Gamma \Rightarrow \varphi(t) \}_{t \in \text{TM}}}{\Gamma \Rightarrow \forall x.\varphi(x)} \text{ (\omega right)} \\
\frac{\{ \varphi(t), \Gamma \Rightarrow \Pi \}_{t \in \text{TM}}}{\exists x.\varphi(x), \Gamma \Rightarrow \Pi} \text{ (\omega left)} \qquad \frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x.\varphi(x)} \text{ (\exists x right)} \\
\frac{\varphi(\tau), \Gamma \Rightarrow \Pi}{\forall X.\varphi(X), \Gamma \Rightarrow \Pi} \text{ (\forall X left)} \qquad \frac{\Gamma \Rightarrow \varphi(Y) \quad Y \notin \text{FV}(\Gamma)}{\Gamma \Rightarrow \forall X.\varphi(X)} \text{ (\forall X right)} \\
\frac{\varphi(Y), \Gamma \Rightarrow \Pi \quad Y \notin \text{FV}(\Gamma, \Pi)}{\exists X.\varphi(X), \Gamma \Rightarrow \Pi} \text{ (\exists X left)} \qquad \frac{\Gamma \Rightarrow \varphi(\tau)}{\Gamma \Rightarrow \exists X.\varphi(X)} \text{ (\exists X right)} \\
\frac{\vartheta(Y), \Gamma \Rightarrow \Pi \quad Y \notin \text{FV}(\Gamma, \Pi)}{\exists X.\vartheta(X), \Gamma \Rightarrow \Pi} \text{ (\exists X left)} \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad Y \notin \text{FV}(\Gamma)}{\Gamma \Rightarrow \forall X.\vartheta(X)} \text{ (\forall X right)} \\
\frac{\{ \Delta, \Gamma \Rightarrow \Pi \}_{\Delta \in |\overline{\forall X}.\vartheta|}}{\overline{\forall X}.\vartheta, \Gamma \Rightarrow \Pi} \text{ (\tilde{\Omega}_k left)} \qquad \frac{\Gamma \Rightarrow \vartheta(Y) \quad \{ \Delta, \Gamma \Rightarrow \Pi \}_{\Delta \in |\overline{\forall X}.\vartheta|}}{\Gamma \Rightarrow \Pi} \text{ (\tilde{\Omega}_k left)} \\
\frac{\{ \Gamma, \Delta \Rightarrow \Lambda \}_{(\Delta \Rightarrow \Lambda) \in |\exists X.\vartheta|}}{\Gamma \Rightarrow \exists X.\vartheta} \text{ (\Omega_k right)} \qquad \frac{\{ \Gamma, \Delta \Rightarrow \Lambda \}_{(\Delta \Rightarrow \Lambda) \in |\exists X.\vartheta|} \quad \vartheta(Y), \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \text{ (\tilde{\Omega}_k right)}
\end{array}$$

where $k = 0, \dots, n$, which is determined by the level of the main formula $\overline{Q}X.\vartheta$. Rules $(\tilde{\Omega}_k \text{ left})$ and $(\tilde{\Omega}_k \text{ right})$ are subject to the eigenvariable condition ($Y \notin \text{FV}(\Gamma, \Pi)$). Index sets are defined by:

$$\begin{aligned}
|\overline{\forall X}.\vartheta(X)| &:= \{ \Delta : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \Delta \Rightarrow \vartheta(Y) \text{ for some } Y \notin \text{FV}(\Delta) \} \\
|\exists X.\vartheta(X)| &:= \{ (\Delta \Rightarrow \Lambda) : \mathbf{LI}\Omega_{k-1} \vdash^{cf} \vartheta(Y), \Delta \Rightarrow \Lambda \text{ for some } Y \notin \text{FV}(\Delta, \Lambda) \}.
\end{aligned}$$