An Experimental Study of Forbidden Patterns in Geometric Permutations by Combinatorial Lifting

Xavier Goaoc
Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
xavier.goaoc@loria.fr

Andreas Holmsen
Department of Mathematical Sciences, KAIST, Daejeon, South Korea
andreash@kaist.edu

Cyril Nicaud
Université Paris-Est, LIGM (UMR 8049), CNRS, ENPC, ESIEE, UPEM,
F-77454, Marne-la-Vallée, France
cyril.nicaud@u-pem.fr

Abstract

We study the problem of deciding if a given triple of permutations can be realized as geometric permutations of disjoint convex sets in $\mathbb{R}^3$. We show that this question, which is equivalent to deciding the emptiness of certain semi-algebraic sets bounded by cubic polynomials, can be "lifted" to a purely combinatorial problem. We propose an effective algorithm for that problem, and use it to gain new insights into the structure of geometric permutations.

2012 ACM Subject Classification Theory of computation → Computational geometry; Computing methodologies → Combinatorial algorithms

Keywords and phrases Geometric permutation, Emptiness testing of semi-algebraic sets, Computer-aided proof

Digital Object Identifier 10.4230/LIPIcs.SoCG.2019.40

Related Version A full version of this paper, available at `https://arxiv.org/abs/1903.03014`, contains some details omitted from this version due to space constraints.

Funding *Xavier Goaoc*: Supported by Institut Universitaire de France.
Andreas Holmsen: Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03930998).

1 Introduction

Consider pairwise disjoint convex sets $C_1, C_2, \ldots, C_n$ and lines $\ell_1, \ell_2, \ldots, \ell_k$ in $\mathbb{R}^d$, where every line intersects every set. Each line ℓ_i defines two orders on the sets, namely the orders in which the two orientations of ℓ_i meet the sets; this pair of orders, one the reverse of the other, are identified to form the *geometric permutation* realized by ℓ_i on $C_1, C_2, \ldots, C_n$. Going in the other direction, one may ask if a given family of permutations can occur as geometric permutations of a family of pairwise disjoint convex sets in $\mathbb{R}^d$, *i.e.* whether it is *geometrically realizable* in $\mathbb{R}^d$.

In $\mathbb{R}^2$ there exist pairs of permutations that are unrealizable, while in $\mathbb{R}^3$, every pair of permutations is realizable by a family of segments with endpoints on two skew lines. The simplest non-trivial question is therefore to understand which triples of permutations are geometrically realizable. This question is equivalent to testing the non-emptiness of certain semi-algebraic sets bounded by cubic polynomials. We show that the structure of these polynomials allow to “lift” this algebraic question to a purely combinatorial one, then propose an algorithm for that combinatorial problem, and present some new results on geometric permutations obtained with its assistance.

Conventions. To simplify the discussion, we work with *oriented* lines and thus with *permutations*, in place of the non-oriented lines and geometric permutations customary in this line of inquiry. We represent permutations by words such as 1423 or *badc*, to be interpreted as follows. The letters of the word are the elements being permuted and they come with a natural order, namely $<$ for integer and the alphabetical order for letters. The word gives the sequence of images of the elements by increasing order; for example, 312 codes the permutation mapping 1 to 3, 2 to 1 and 3 to 2, and *badc* codes the permutation exchanging a with b and c with d. The *size* of a permutation is the number of elements being permuted, that is the length of this word. We say that a triple of permutations is *realizable* (resp. *forbidden*) to mean that it is realizable (resp. not realizable) in $\mathbb{R}^3$.

1.1 Contributions

Our results are of two types, methodological and geometrical.

Combinatorial lifting. Our first contribution is a new approach for deciding the emptiness of a semi-algebraic set with a special structure. We describe it for the geometric realizability problem, here and in Section 3, but stress that it applies more broadly.

As spelled out in Section 2, deciding if a triple of permutations is realizable amounts to testing the emptiness of a semi-algebraic set $R \subseteq \mathbb{R}^n$. Let $u_1, u_2, \ldots, u_n$ denote the variables and $P_1, P_2, \ldots, P_m$ the polynomials used in a Boolean formula defining R. The structure we take advantage of is that here, each P_k can be written as a product of terms, each of which is of the form $u_i - u_j$, $u_i - 1$, or $u_i - f(u_j)$, with $f(t) = \frac{1}{1-t}$. If only terms of the form $u_i - u_j$ or $u_i - 1$ occur, then we can test the emptiness of R by examining the possible orders on $(1, u_1, u_2, \ldots, u_n)$. We propose to handle the terms of the form $u_i - f(u_j)$ in the same way, by exploring the orders that can arise on $(1, u_1, f(u_1), u_2, f(u_2), \ldots, u_n, f(u_n))$.

The main difficulty in this approach is to restrict the exploration to the orders that can be realized by a sequence of the form $(1, u_1, f(u_1), u_2, f(u_2), \ldots, u_n, f(u_n))$. This turns out to be easier if we extend the lifting to

$$\Lambda : \left\{ \begin{array}{ccc} (\mathbb{R} \setminus \{0,1\})^n & \to & \mathbb{R}^{3n} \\ (u_1, u_2, \ldots, u_n) & \mapsto & (u_1, f(u_1), f^{(2)}(u_1), \ldots, u_n, f(u_n), f^{(2)}(u_n)). \end{array} \right.$$

This extended lifting allows to take advantage of the facts that $f^{(3)} = f \circ f \circ f$ is the identity, that f permutes circularly the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$, and that f is increasing on each of them. In Proposition 7, we essentially show that *an order on the* $3n$ *lifted variables can be realized by a point of* $\Lambda(\mathbb{R}^{3n})$ *if and only if it is compatible with the action of* f*, as captured by these properties.*

Algorithm. Our second contribution is an algorithm that puts the combinatorial lifting in practice, and decides if a triple of permutations is realizable in $O\left(6^n n^{10}\right)$ time and $O(n^2)$ space in the worst-case. We provide an implementation in Python, see the full version of this paper.

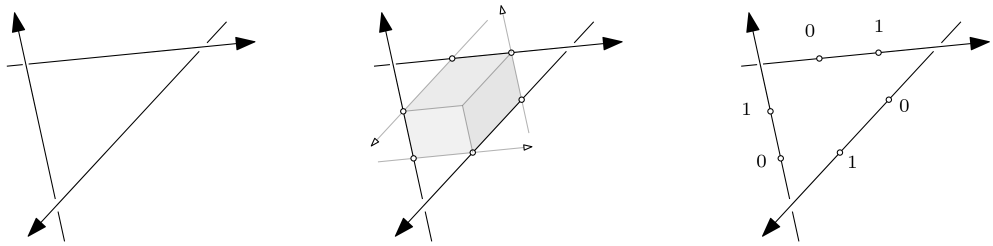


Figure 1 Three skew lines (left), the parallelotope (middle) and the marked points (right).

New geometric results. Our remaining contributions are new geometric results obtained with the aid of our implementation. A first systematic exploration reveals:

▶ **Theorem 1.** *Every triple of permutations of size* 5 *is geometrically realizable in* $\mathbb{R}^3$*.*

The smallest known triple of geometric permutations forbidden in $\mathbb{R}^3$ has size 6 (see Section 1.2), so Theorem 1 proves that it is minimal. We also obtained the complete list of forbidden triples of size 6 (see the full version). Interestingly, although everything is realizable up to size 5, something can be said on geometric permutations of size 4. Recall that the *side operator* $(pq) \odot (rs)$ of the two lines (pq) and (rs), oriented respectively from p to q and from r to s, is the orientation of the tetrahedron $pqrs$; it captures the mutual disposition of the two lines. We prove:

▶ **Theorem 2.** *Let* ℓ_1 *and* ℓ_2 *be two oriented lines intersecting four pairwise disjoint convex sets in the order* 1234*. Any oriented line* ℓ_3 *that intersects those four sets in the order* 2143 *satisfies* $\ell_1 \odot \ell_3 = \ell_2 \odot \ell_3$*.*

The pattern $(1234, 2143)$ is known to be forbidden in some cases (see Section 1.2), but this is the first condition valid for arbitrary disjoint convex sets. We could prove Theorem 2 because our algorithm solves a more constrained problem than just realizability of permutations. Given three lines in general position in $\mathbb{R}^3$, there is a unique parallelotope with three disjoint edges supported on these three lines (see Figure 1). Combinatorial lifting, and therefore our algorithm, can decide whether three permutations can be realized *with the vertices of that parallelotope in prescribed positions in the permutations.*

We label the vertices of the parallelotope with $^{\mathbf{0}}$ and $^{\mathbf{1}}$ as in Figure 1 and work with permutations where two extra elements, $^{\mathbf{0}}$ and $^{\mathbf{1}}$, are inserted; we call them *tagged permutations*. We examine triples of tagged permutations realizable on a canonical system of lines (see Equation (1)), and characterize those minimally unrealizable up to size 4 (Proposition 10); for size 2 and 3, we provide independent, direct, geometric proofs of unrealizability (Section 7). We conjecture that no other minimally unrealizable triples of tagged permutations exist, and verified this experimentally up to size 6 (not counting $^{\mathbf{0}}$ and $^{\mathbf{1}}$). A weaker conjecture is:

▶ **Conjecture 3.** *There exists a polynomial time algorithm that decides the geometric realizability of a triple of permutations of size* n *in* $\mathbb{R}^3$*.*

1.2 Discussion and related work

We now put our contribution in context, starting with motivations for studying geometric permutations.

Geometric transversals. In the 1950's, Grünbaum [7] conjectured that, given a family of disjoint translates of a convex figure in the plane, if every five members of the family can be met by a line, then there exists a line that meets the entire family. (Such a statement, if true, is an example of *Helly-type theorem.*) Progress on Grünbaum's conjecture was slow until the 1980's, when the notion of geometric permutations of families of convex sets was introduced [13, 14]. Their systematic study in the plane was refined by Katchalski [12] in order to prove a weak version of Grünbaum's conjecture (with 128 in place of 5). Tverberg [24] soon followed up with a proof of the conjecture, again using a careful analysis of planar geometric permutations. This initial success and further conjectures about Helly-type theorems stimulated a more systematic study of geometric permutations realizable under various geometric restrictions; cf. [10] and the references therein.

Another motivation to study geometric permutations comes from computational geometry, more precisely the study of geometric structures such as *arrangements.* There, geometric permutations appear as a coarse measure of complexity of the space of line transversals to families of sets, and relates to various algorithmic problems such as ray-shooting or smallest enclosing cylinder computation [18, §7.6]. From this point of view, the main question is to estimate the maximum number $g_d(n)$ of distinct geometric permutations of n pairwise disjoint convex sets in $\mathbb{R}^d$. Broadly speaking, while $g_2(n)$ is known to equal $2n-2$ [6], even the order of magnitude of $g_d(n)$ as $n \to \infty$ is open for every $d \geq 3$; the gap is between $\Omega(n^{d-1})$ and $O(n^{2d-3}\log n)$. Bridging this gap has been identified as an important problem in discrete geometry [18, §7.6], yet, over the last fifteen years, the only progress has been an improvement of the upper bound from $O(n^{2d-2})$ down to $O(n^{2d-3}\log n)$; moreover, while the former bound follows from a fairly direct argument, the latter is a technical tour de force [20]. We hope that a better understanding of small forbidden configurations will suggest new approaches to this question.

Geometric realizability problems. Combinatorial structures that arise from geometric configurations such as arrangements, polytopes, or intersection graphs are classical objects of enquiry in discrete and computational geometry (see *e.g.* [23, § 1, 5, 6, 10, 15, 17, 28]). We are concerned here with the *membership testing problem*: given an instance of a combinatorial structure, decide if there exists a geometric configuration that induces it. Such problems can be difficult: for instance, deciding whether a given graph can be obtained as intersection graph of segments in the plane is NP-hard [15].

A natural approach to membership testing is to parameterize the candidate geometric configuration and express the combinatorial structure as conditions on these parameters. This often results in a semi-algebraic set. In the real-RAM model[1], the emptiness of a semi-algebraic set in $\mathbb{R}^d$ with real coefficients can be tested in time $(nD)^{O(d)}$ [19, Prop. 4.1], where n is the number of polynomials and D their maximum degree. (Other approaches exist but have worse complexity bounds, see [5, 16, 11, 4]). Given three permutations of size n, we describe their realizations as a semi-algebraic set defined by $O(n^2)$ cubic polynomials in n variables; the above method thus has complexity $n^{O(n)}$, making our $O\left(6^n n^{10}\right)$ solution competitive in theory. Practical effectiveness is usually difficult to predict as it depends on the geometry of the underlying algebraic surfaces; for example, deciding if two geometric permutations of *size four* are realizable by disjoint unit balls in $\mathbb{R}^3$ was recently checked to be out of reach [9].

[1] See the same reference for a similar bound in the bit model.

Some geometric realizability problems, for example the recognition of unit disk graphs, are $\exists\mathbb{R}$-hard [21] and therefore as difficult from a complexity point of view as deciding the emptiness of a general semi-algebraic set. We do not know whether deciding the emptiness of semi-algebraic sets amenable to our combinatorial lifting remains $\exists\mathbb{R}$-hard; we believe, however, that deciding if a triple of permutations is realizable is not (cf. Conjecture 3).

Forbidden patterns. A dimension count shows that any k permutations are realizable in $\mathbb{R}^{2k-1}$. Our most direct predecessor is the work of Asinowski and Katchalski [2] who proved that this argument is sharp by constructing, for every k, a set of k permutations that are not realizable in $\mathbb{R}^{2k-2}$. They also showed that the triple $(123456, 321654, 246135)$ is not realizable in $\mathbb{R}^3$, a fact that follows easily from our list of obstructions.

In a sense, our work tries to generalize some arguments previously used to analyze geometric permutations in the plane. For example, the (standard) proof that the pair $(1234, 2143)$ is non-realizable as geometric permutations in $\mathbb{R}^2$ essentially analyzes tagged permutations. Indeed, if we augment the permutations by an additional label $\mathbf{^0}$ marking the intersection of the lines realizing the two orders, we get that $(^{\mathbf{0}}ab, ^{\mathbf{0}}ba)$, $(^{\mathbf{0}}ab, ab^{\mathbf{0}})$, $(ba^{\mathbf{0}}, ^{\mathbf{0}}ba)$ and $(ba^{\mathbf{0}}, ab^{\mathbf{0}})$ are forbidden, and there is nowhere to place $^{\mathbf{0}}$ in $(1234, 2143)$. We should, however, emphasize that already in $\mathbb{R}^3$ the geometry is much more subtle.

Forbidden patterns were used to bound the number of geometric permutations for certain restricted families of convex sets. For pairwise disjoint translates of a convex planar figure [12, 24, 25], it is known that a given family can have at most three geometric permutations, and the possible sets of realizable geometric permutations have been characterized. The situation is similar for families of pairwise disjoint unit balls in $\mathbb{R}^d$. Here, an analysis of forbidden patterns in geometric permutations showed that a given family can have at most a constant number of geometric permutations (in fact only two if the family is sufficiently large) [22, 3, 9]. Another example is [1], where it is shown that the maximum number of geometric permutations for convex objects in $\mathbb{R}^d$ induced by lines that pass through the origin, is in $\Theta(n^{d-1})$. The restriction that the lines pass through the origin, allows them to deal with permutations augmented by one additional label, and their argument relies on the forbidden tagged pattern $(^{\mathbf{0}}ab, ^{\mathbf{0}}ba)$ [1, Lemma 2.1].

In these examples, the bounds use highly structured sets of forbidden patterns. In general, one cannot expect polynomial bounds on the sole basis of excluding a handful of patterns; for instance it is not hard to construct an exponential size family of permutations of $[n]$ which avoids the pattern $(1234, 2143)$. Such questions are well-studied in the area of "pattern-avoidance" and usually the best one could hope for is an exponential upper bound on the size of the family [17].

2 Semi-algebraic parameterization

Let $P = (\pi_1, \pi_2, \pi_3)$ denote a triple of permutations of $\{1, 2, \ldots, n\}$. We now describe a semi-algebraic set that is nonempty if and only if P has a geometric realization in $\mathbb{R}^3$.

Canonical realizations. We say that a geometric realization of P is *canonical* if the oriented line transversals are

$$\ell_x = \begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \ell_y = \begin{pmatrix}0\\0\\1\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \quad \ell_z = \begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad (1)$$

and if the convex sets are triangles with vertices on ℓ_x, ℓ_y and ℓ_z.

▶ **Lemma 4.** *If P is geometrically realizable in $\mathbb{R}^3$, then it has a canonical realization, possibly after reversing some of the permutations π_i.*

Proof. Consider a realization of P by three lines and n pairwise disjoint sets. For each convex set we select a point from the intersection with each of the lines and replace it by the (possibly degenerate) triangle spanned by these points. This realizes P by *compact* convex sets. By taking the Minkowski sum of each set with a sufficiently small ball, the sets remain disjoint and the line intersect the sets in their interior. We may now perturb the lines into three lines L_x, L_y and L_z that are pairwise skew and not all parallel to a common plane. We then, again, crop each set to a triangle with vertices on L_x, L_y and L_z.

We now use an affine map to send our three lines to ℓ_x, ℓ_y and ℓ_z. An affine transform is defined by 12 parameters and fixing the image of one line amounts to four linear conditions on these parameters; these constraints determine a unique transform because the lines are in general position. Note, however, that the oriented line L_x is mapped to either ℓ_x or $-\ell_x$, so π_1 may have to be reversed; the same applies to the permutations π_2 and π_3. ◀

We equip the line ℓ_x (resp. ℓ_y, ℓ_z) with the coordinate system obtained by projecting the x-coordinate (resp. y-coordinate, z-coordinate) of $\mathbb{R}^3$. This parameterizes the space of canonical realizations by $\mathbb{R}^{3n}$. Specifically, we equip $\mathbb{R}^{3n}$ with a coordinate system $(O, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n)$ and for any point $c \in \mathbb{R}^{3n}$ we put

$$\mathcal{T}(c) = \{\operatorname{conv}\{X_i, Y_i, Z_i\}\}_{1 \le i \le n}, \quad \text{where} \quad X_i = \begin{pmatrix} x_i \\ 1 \\ 0 \end{pmatrix}, \quad Y_i = \begin{pmatrix} 0 \\ y_i \\ 1 \end{pmatrix}, \text{ and } \quad Z_i = \begin{pmatrix} 1 \\ 0 \\ z_i \end{pmatrix}.$$

Each element of $\mathcal{T}(c)$ is thus a triangle with a vertex on each of ℓ_x, ℓ_y and ℓ_z. We define:

$$R = \{c \in \mathbb{R}^{3n} \colon \mathcal{T}(c) \text{ consists of disjoint triangles and realizes } P\}.$$

The triple P is realizable if and only if R is non-empty.

Triangle disjointedness. We now review an algorithm of Guigue and Devillers [8] to decide if two triangles are disjoint, and use it to formulate the condition that two triangles $X_iY_iZ_i$ and $X_jY_jZ_j$ be disjoint as a semi-algebraic condition on $x_i, \ldots, z_j$.

The algorithm and our description are expressed in terms of orientations, where the *orientation* of four points $p, q, r, s \in \mathbb{R}^3$ is

$$[p, q, r, s] \stackrel{\text{def}}{=} \operatorname{sign} \det \begin{pmatrix} x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ z_p & z_q & z_r & z_s \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Intuitively, the orientation indicates whether point s is "above" $(+1)$, on (0), or "below" (-1) the plane spanned by p, q, r, where above and below refer to the orientation of the plane that makes the directed triangle pqr positively oriented. We only consider orientations of non-coplanar quadruples of points, so orientations take values in $\{\pm 1\}$.

If one triangle is on one side of the plane spanned by the other, then the triangles are disjoint. We check this by computing

$$v(i,j) \stackrel{\text{def}}{=} \begin{pmatrix} [X_i, Y_i, Z_i, X_j] \\ [X_i, Y_i, Z_i, Y_j] \\ [X_i, Y_i, Z_i, Z_j] \\ [X_j, Y_j, Z_j, X_i] \\ [X_j, Y_j, Z_j, Y_i] \\ [X_j, Y_j, Z_j, Z_i] \end{pmatrix} \in \{-1,1\}^6$$

and testing if $v(i,j)_1 = v(i,j)_2 = v(i,j)_3$ or $v(i,j)_4 = v(i,j)_5 = v(i,j)_6$. If this fails, then we rename $\{X_i, Y_i, Z_i\}$ into $\{A_i, B_i, C_i\}$ and $\{X_j, Y_j, Z_j\}$ into $\{A_j, B_j, C_j\}$ so that

$$\begin{pmatrix} [A_i, B_i, C_i, A_j] \\ [A_i, B_i, C_i, B_j] \\ [A_i, B_i, C_i, C_j] \\ [A_j, B_j, C_j, A_i] \\ [A_j, B_j, C_j, B_i] \\ [A_j, B_j, C_j, C_i] \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then, the triangles are disjoint if and only if $[A_i, B_i, A_j, B_j] = 1$ or $[A_i, C_i, C_j, A_j] = 1$ [8]. The renaming is done as follows. Since the first test is inconclusive, the plane spanned by a triple of points separates the other triple of points. We let (A_i, B_i, C_i) be the circular permutation of (X_i, Y_i, Z_i) such that A_i is separated from B_i and C_i by the plane spanned by X_j, Y_j, and Z_j. We let (A_j, B_j, C_j) be the circular permutation of (X_j, Y_j, Z_j) such that A_j is separated from B_j and C_j by the plane spanned by A_i, B_i, and C_i. If $[A_i, B_i, C_i, A_j] = -1$ then we exchange B_i and C_i. If $[A_j, B_j, C_j, A_i] = -1$ then we exchange B_j and C_j.

Semi-algebraicity. Every step in the Guigue-Devillers algorithm can be expressed as a logical proposition in terms of orientation predicates which are, when specialized to our parameterization, conditions on the sign of polynomials in the coordinates of c. Checking that each of ℓ_x, ℓ_y and ℓ_z intersects the triangles in the prescribed order amounts to comparing coordinates of c. Altogether, the set R is a semi-algebraic subset of $\mathbb{R}^{3n}$.

3 Combinatorial lifting

We now explain how to test combinatorially the emptiness of our semi-algebraic set R.

Definitions. We start by decomposing each orientation predicate used in the definition of R as indicated in Table 1. For the last three rows, this is not a factorization since one of the factors is of the form $u - f(v)$ where $f : t \mapsto \frac{1}{1-t}$.

In light of the third column of Table 1, it may seem natural to "linearize" the problem by considering the map $(x_1, x_2, \ldots, z_n) \mapsto (x_1, f(x_1), x_2, f(x_2), \ldots, z_n, f(z_n))$ from $\mathbb{R}^{3n}$ to $\mathbb{R}^{6n}$. Indeed, the order on the lifted coordinates and 1 determines the sign of all polynomials defining R. We must, however, identify the orders on the coordinates in $\mathbb{R}^{6n}$ that can be realized by lifts of points from $\mathbb{R}^{3n}$. Perhaps surprisingly, the task gets easier if we lift to even higher dimension. For convenience we let $\mathbb{R}_* \stackrel{\text{def}}{=} \mathbb{R} \setminus \{0,1\}$. The lifting map we use is:

$$\Lambda : \begin{cases} \mathbb{R}_*^{3n} & \to \quad \mathbb{R}^{9n} \\ (x_1, x_2, \ldots, z_n) & \mapsto \quad (x_1, f(x_1), f^{(2)}(x_1), x_2, \ldots, z_n, f(z_n), f^{(2)}(z_n)) \end{cases}$$

To determine the image of $\Lambda\left(\mathbb{R}_*^{3n}\right)$, we will use the following properties of f:

Table 1 Orientation predicates used in the Guigue-Devillers algorithm when specialized to points from ℓ_x, ℓ_y and ℓ_z.

Orientation	Determinant	Decomposition
$[X_a, X_b, Y_c, Y_d]$	$(x_a - x_b)(y_c - y_d)$	$(x_a - x_b)(y_c - y_d)$
$[X_a, X_b, Z_c, Z_d]$	$(x_a - x_b)(z_c - z_d)$	$(x_a - x_b)(z_c - z_d)$
$[Y_a, Y_b, Z_c, Z_d]$	$(y_a - y_b)(z_c - z_d)$	$(y_a - y_b)(z_c - z_d)$
$[X_a, X_b, Y_c, Z_d]$	$(x_a - x_b)(y_c z_d - z_d + 1)$	$(x_a - x_b)(y_c - 1)\left(z_d - \frac{1}{1-y_c}\right)$
$[X_a, Y_b, Y_c, Z_d]$	$(y_b - y_c)(x_a - x_a z_d - 1)$	$-(y_b - y_c)(z_d - 1)\left(x_a - \frac{1}{1-z_d}\right)$
$[X_a, Y_b, Z_c, Z_d]$	$(z_c - z_d)(x_a y_b + 1 - y_b)$	$(z_c - z_d)(x_a - 1)\left(y_b - \frac{1}{1-x_a}\right)$

▷ Claim 5. $f^{(3)} = f \circ f \circ f$ is the identity on $\mathbb{R}_*$, f permutes the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$ circularly, and f is monotone on each of these intervals.

Let us denote the points of $\mathbb{R}^{9n}$ by vectors $(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{9n})$. We next "lift" the semi-algebraic description of R:

1. We pick a Boolean formula ϕ describing R in terms of orientations (for the triangle disjointedness) and comparisons of coordinates (for the geometric permutations).
2. We decompose every orientation predicate ocurring in ϕ as in the third row of Table 1.
3. We then construct another Boolean formula ψ by substituting[2] in ϕ every $f(x_1)$ by the variable $\mathbf{t}_2$ (to which it is mapped under Λ). We similarly substitute every $f(x_i)$, $f(y_i)$ and $f(z_i)$, then every remaining x_i, y_i and z_i by the corresponding variable $\mathbf{t}_*$.
4. We let $S \subset \mathbb{R}^{9n}$ be the (semi-algebraic) set of points that satisfy ψ.

We finally let $\mathcal{H}$ denote the arrangement in $\mathbb{R}^{9n}$ of the set of hyperplanes:

$$\{\mathbf{t}_i = \mathbf{t}_j\}_{1 \le i < j \le 9n} \cup \{\mathbf{t}_i = 0\}_{1 \le i \le 9n} \cup \{\mathbf{t}_i = 1\}_{1 \le i \le 9n}.$$

Note that the full-dimensional (open) cells in $\mathcal{H}$ are in bijection with the total orders on $\{0, 1, \mathbf{t}_1, \ldots, \mathbf{t}_{9n}\}$ in which 0 comes before 1. We write $\prec_A$ for the order associated with a full-dimensional cell A of $\mathcal{H}$.

▶ **Lemma 6.** *Every full-dimensional cell of $\mathcal{H}$ is disjoint from or contained in S. Moreover, R is nonempty if and only if there exists a full-dimensional cell of $\mathcal{H}$ that is contained in S and intersects $\Lambda\left(\mathbb{R}_*^{3n}\right)$.*

Proof. The set S is defined by the positivity or negativity of polynomials, each of which is a product of terms of the form $(\mathbf{t}_i - \mathbf{t}_j)$ or $(\mathbf{t}_i - 1)$. The first statement thus follows from the fact the coordinates of all points in a full-dimensional cell realize the same order on $\{0, 1, \mathbf{t}_1, \ldots, \mathbf{t}_{9n}\}$. By the perturbation argument used in the proof of Lemma 4, if R is non-empty, then it contains a point with no coordinate in $\{0, 1\}$. Thus, R is non-empty if and only if $\Lambda(R)$ is non-empty. The construction of S ensures that $\Lambda(R) = S \cap \Lambda\left(\mathbb{R}_*^{3n}\right)$. Again, a perturbation argument ensures that if $\Lambda(R)$ is nonempty, it contains a point outside of the union of the hyperplanes of $\mathcal{H}$. The second statement follows. ◀

[2] For example, with $n = 3$, the product $(x_1 - x_2)(y_2 - 1)\left(z_3 - \frac{1}{1-y_2}\right) = (x_1 - x_2)(y_2 - 1)(z_3 - f(y_2))$ appearing in ϕ is translated in ψ as $(\mathbf{t}_1 - \mathbf{t}_4)(\mathbf{t}_{13} - 1)(\mathbf{t}_{25} - \mathbf{t}_{14})$.

Zone characterization. Inspired by Lemma 6, we now characterize the orders $\prec_A$ such that A intersects $\Lambda\left(\mathbb{R}_*^{3n}\right)$. We split the $9n$ variables $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{9n}$ into $3n$ blocks of three consecutive variables $\mathbf{t}_{3i+1}, \mathbf{t}_{3i+2}$, $\mathbf{t}_{3i+3}$ (representing $x_i, f(x_i), f^{(2)}(x_i)$ for $0 \le i < n$, $y_i, f(y_i), f^{(2)}(y_i)$ for $n \le i < 2n$, and $z_i, f(z_i), f^{(2)}(z_i)$ for $2n \le i < 3n$). We also define an operator $\mathbf{f}$ that shifts the variables cyclically within each individual block:

$$\mathbf{f}(\mathbf{t}_{3i+1}) = \mathbf{t}_{3i+2}, \quad \mathbf{f}(\mathbf{t}_{3i+2}) = \mathbf{t}_{3i+3} \quad \text{and} \quad \mathbf{f}(\mathbf{t}_{3i+3}) = \mathbf{t}_{3i+1}.$$

By convention, $\mathbf{f}^0$ means the identity. The fact that $\mathbf{f}$ mimicks, symbolically, the action of f yields the following characterization.

▶ **Proposition 7.** *A full-dimensional cell A of $\mathcal{H}$ intersects $\Lambda\left(\mathbb{R}_*^{3n}\right)$ if and only if*

(i) *For any $0 \le i < 3n$, there exists $j \in \{0,1,2\}$ s. t.*

$$\mathbf{f}^{(j)}(\mathbf{t}_{3i+1}) \prec_A 0 \prec_A \mathbf{f}^{(j+1)}(\mathbf{t}_{3i+1}) \prec_A 1 \prec_A \mathbf{f}^{(j+2)}(\mathbf{t}_{3i+1}).$$

(ii) *For any $1 \le i,j \le 9n$,* $\quad \{\mathbf{t}_i \prec_A \mathbf{t}_j \quad \textit{and} \quad \mathbf{f}(\mathbf{t}_j) \prec_A \mathbf{f}(\mathbf{t}_i)\} \quad \Rightarrow \quad \mathbf{t}_i \prec_A 1 \prec_A \mathbf{t}_j.$

Proof. Let us first see why the conditions are necessary. Let $c = (x_1, x_2, \ldots, z_n) \in \mathbb{R}_*^{3n}$ such that $\Lambda(c) \in A$. Fix some $0 \le i < n$. As j ranges over $\{0,1,2\}$, the coordinate $\mathbf{f}^{(j)}(\mathbf{t}_{3i+i})$ of $\Lambda(c)$ ranges over $\{x_i, f(x_i), f^{(2)}(x_i)\}$, and Condition (i) holds because f permutes the intervals $(-\infty, 0)$, $(0,1)$ and $(1,+\infty)$ circularly. The cases $n \le i < 3n$ are similar. Condition (ii) follows in a similar manner from the fact that f permutes the intervals $(-\infty,0)$, $(0,1)$ and $(1,+\infty)$ circularly and is increasing on each of them.

To examine sufficiency we need some notations. We let $\mathcal{V} = \{0, 1, \mathbf{t}_1, \ldots, \mathbf{t}_{9n}\}$. Given an order $\prec$ on $\mathcal{V}$ and two elements $a, b \in \mathcal{V}$ we write $(a,b)_\prec \stackrel{\text{def}}{=} \{c \in \mathcal{V} : a \prec c \prec b\}$. We also write $(\cdot, a)_\prec$ for the set of elements smaller than a, and $(a, \cdot)_\prec$ for the set of elements larger than a, and $[a,b)_\prec$, $[a,b)_\prec$ or $[a,b]_\prec$ to include one or both bounds in the interval.

Let $\prec_*$ be an order on $\{0, 1, \mathbf{t}_1, \ldots, \mathbf{t}_{9n}\}$ such that $0 \prec_* 1$. By Condition (i), $(1,\cdot)_{\prec_*}$ has size $3n$, so let us write $(1,\cdot)_{\prec_*} = \{b_1, b_2, \ldots, b_{3n}\}$ with $1 \prec_* b_1 \prec_* b_2 \prec_* \ldots \prec_* b_{3n}$. Condition (i) also ensures that for every $0 \le i < 3n$, exactly one of $\{\mathbf{t}_{3i+1}, \mathbf{f}(\mathbf{t}_{3i+1}), \mathbf{f}^{(2)}(\mathbf{t}_{3i+1})\}$ belongs to $(1,\cdot)_{\prec_*}$. Hence, for every $0 \le i < 3n$ there are uniquely defined integers $0 \le \alpha(i) \le 2$ and $1 \le \beta(i) \le 3n$ such that $b_{\beta(i)} = \mathbf{f}^{\alpha(i)}(\mathbf{t}_{3i+1})$.

We next pick $3n$ real numbers $1 < r_1 < r_2 < \ldots < r_{3n}$, put

$$\begin{aligned} x_i &= f^{3-\alpha(i)}\left(r_{\beta(i)}\right), && \text{for } 0 \le i < n \\ y_i &= f^{3-\alpha(i)}\left(r_{\beta(i)}\right), && \text{for } n \le i < 2n \\ z_i &= f^{3-\alpha(i)}\left(r_{\beta(i)}\right), && \text{for } 2n \le i < 3n, \end{aligned}$$

and let $p = (x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) \in \mathbb{R}^{3n}$. Note that $\Lambda(p)$ lies in a full-dimensional cell of the arrangement $\mathcal{H}$; let us denote it by A.

Now, 0 precedes 1 in both $\prec_*$ and $\prec_A$. Also, $[1,\cdot)_{\prec_*} = [1,\cdot)_{\prec_A}$ and the two orders coincide on that interval by construction of p. Remark that $\mathbf{f}$ acts similarly for both orders:

- $\mathbf{f}$ maps $[1,\cdot)_{\prec_*}$ to $(\cdot,0]_{\prec_*}$ increasingly for $\prec_*$ by Conditions (i) and (ii).
- $\mathbf{f}$ maps $[1,\cdot)_{\prec_A}$ to $(\cdot,0]_{\prec_A}$ increasingly for $\prec_A$ by definition of p and Claim 5.

We therefore also have $(\cdot,0]_{\prec_*} = (\cdot,0]_{\prec_A}$ and the orders coincide on that interval as well. The same argument applied to $\mathbf{f}^2$ shows that $[0,1]_{\prec_*} = [0,1]_{\prec_A}$ and that the two orders coincide on that interval as well. Altogether, $\prec_*$ and $\prec_A$ coincide. ◀

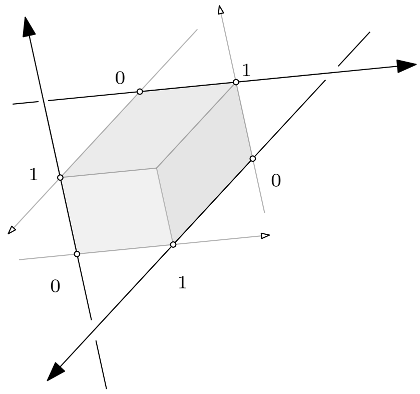


Figure 2 Tagging permutations by adding the $^{\mathbf{0}}$ and $^{\mathbf{1}}$.

4 Tagged patterns and their canonical realization

Let us clarify the geometric problem that is really captured by our combinatorial lifting.

As it appears from Table 1, the combinatorial lift searches only for a *canonical* realization. By Lemma 4, however, this is not a restriction. More importantly, the lift compares the lifted variable to the constants 0 and 1. In the coordinate systems of ℓ_x, ℓ_y and ℓ_z, the 0 and 1 are at certain corners of the unique parallelotope with three disjoint edges supported by these lines. Let us label these points $^{\mathbf{0}}$ and $^{\mathbf{1}}$ as Figure 2. Note that fixing the comparisons of 0 and each x_i is equivalent to specifying the position of the point $^{\mathbf{0}}$ of ℓ_x in the first permutation. Other coordinates behave similarly.

Formally, we define a *tagged permutation* as a permutation of $\{^{\mathbf{0}}, ^{\mathbf{1}}, 1, 2, \dots, n\}$ in which $^{\mathbf{0}}$ precedes $^{\mathbf{1}}$. We call a triple of tagged permutations a *tagged pattern*. A *canonical realization* of a tagged pattern is a set of triangles, with vertices on ℓ_x, ℓ_y and ℓ_z, such that ℓ_x (resp. ℓ_y, ℓ_z) intersects the triangles in the first (resp. second, third) permutation and such that the tagged corners of the parallelotope appear in the right position on each line.

Our experiments will use two more notions. Two tagged patterns are *equivalent* for canonical realizability if one can be transformed into the other by (i) relabeling the symbols other than $^{\mathbf{0}}$ and $^{\mathbf{1}}$ bijectively, and (ii) applying a circular permutation to the triple. A tagged pattern is *minimally forbidden* if it has no canonical realization, and deleting any symbol other than $^{\mathbf{0}}$ and $^{\mathbf{1}}$ from the three tagged permutations produces a tagged pattern which has a canonical realization.

5 Algorithm

We now present an algorithm that takes a tagged pattern as input and decides if it admits a canonical realization. Our initial problem of testing the geometric realizability of a triple of permutations of size n reduces to $8\binom{n+2}{2}^3$ instances of that problem.

5.1 Outline

Following Sections 2 and 3, we search for an order on $\{0, 1, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{9n}\}$ satisfying the conditions of Proposition 7 and the formula ψ (which defines S). To save breath, we call such an order *good*. We say that *triangles i and j are disjoint in a partial order $\mathcal{P}$* if for every $c \in \mathbb{R}^{3n}$ such that the order on $\Lambda(c)$ is a linear extension of $\mathcal{P}$, the triangles i and j of $\mathcal{T}(c)$ are disjoint.

Our algorithm gradually refines a set of partial orders on $\{0, 1, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{9n}\}$ with the constraint that, at any time, every good order is a linear extension of at least one of these partial orders. (Note that we do not need to make ψ explicit.) Every partial order is refined until all or none of its extensions are good, so that we can report success or discard that partial order. Refinements are done in two ways:

- *branching* over an uncomparable pair, meaning duplicating the partial order and adding the comparison in one copy, and its reverse in the other copy,
- *forcing* a comparison when it is required for the formula S to be satisfiable.

We keep our algorithm as simple as possible to facilitate the verification of the algorithm, its implementation, and the geometric results proven with their aid. This comes at the cost of some efficiency, but we discuss some possible improvements in Section 5.3.

5.2 Description

Our poset representation stores (i) for each lifted variable the interval $(\cdot, 0)$, $(0, 1)$ or $(1, \cdot)$ that contains it, and (ii) a directed graph over the variables contained in the interval $(1, \cdot)$. The graph has $3n$ vertices, by Lemma 6. To compare two variables, we first retrieve the intervals containing them. If they differ, we can return the comparison readily. If they agree, then up to composing by f or $f^{(2)}$ we can assume that both variables are in $(1, \cdot)$ and we use the graph to reply. We ensure throughout that the graph is saturated, *i.e.* is its own transitive closure. In our implementation, initialization takes $O(n^3)$ time, elements comparison takes $O(1)$ time, and edge addition to the graph takes $O(n^2)$ time.

We start with the poset of the comparisons forced by the tagged pattern: all pairs (x_i, x_j), $(f(x_i), f(x_j))$, ..., $\big(f^{(2)}(z_i), f^{(2)}(z_j)\big)$ as well as pairs separated by 0 or 1. We next collect in a set U the comparisons missing to compute the vectors $v(i, j)$.

▶ **Lemma 8.** *U contains only pairs of the form $z_k - f(y_k)$, $x_k - f(z_k)$, or $y_k - f(x_k)$.*

Proof. Every orientation predicate considered involves three points of the same index. Consider for instance $[X_i, Y_i, Z_i, X_j]$. Following Table 1, this decomposes into $(x_i - x_j)(y_i - 1)(z_i - f(y_i))$ and only the sign of the last term may be undecided. Other cases are similar and show that U can only contain terms the form $z_k - f(y_k)$, $x_k - f(z_k)$, or $y_k - f(x_k)$. ◀

Every pair in U corresponds to two variables *with same index*, so $|U| \le 3n$. If U contains the three pairs with a given index, then two of the eight choices for these three comparisons are cyclic, and can thus be ignored. We thus have at most 6^n ways to decide the order of the undeterminate pairs of U; call them *candidates*. For each candidate, we make a separate copy of our current graph and perform the following operations on that copy:

1. We add the $|U|$ edges ordering the undecided pairs as fixed by the candidate and compute its transitive closure. We check that the result is acyclic; if not, we discard that candidate (as it makes contradictory choices) and move to the next candidate.
2. Let $\mathcal{P}$ denote the resulting partial order. We consider every $1 \le i < j \le n$ in turn. (Note that $v(i, j)$ is determined and equal for all linear extensions of $\mathcal{P}$.)

   2a If $v(i,j)_1 = v(i,j)_2 = v(i,j)_3$ or $v(i,j)_4 = v(i,j)_5 = v(i,j)_6$ then triangles i and j are disjoint in $\mathcal{P}$. We move on to the next pair (i, j).

2b Otherwise, the extensions of $\mathcal{P}$ in which the triangles i and j are disjoint are those in which $[A_i, B_i, A_j, B_j] = 1$ or $[A_i, C_i, C_j, A_j] = 1$ (in the notations of Section 2). Lemma 9 asserts that $\mathcal{P}$ already determines at least one of these two predicates.

2b1 If both tests are determined to false, then triangles i and j intersect in $\mathcal{P}$. We then discard $\mathcal{P}$ and move on to the next candidate.

2b2 If one test is determined to false and the other is undetermined, then that second test must evaluate to true in every good extension of $\mathcal{P}$. Again, by Table 1 we are missing exactly one comparison to decide that test. We add it to our graph.

2b3 In the remaining cases, at least one test is determined to true, so triangles i and j are disjoint in $\mathcal{P}$. We move on to the next pair (i, j).

3. If we exhaust all (i, j) for a candidate, then we report “realizable”.

4. If we exhaust the candidates without reaching step 3, then we report “unrealizable”.

This algorithm relies on property whose computer-aided proof is discussed in Section 6:

▶ **Lemma 9.** *At step 2b, at least one of* $[A_i, B_i, A_j, B_j]$ *or* $[A_i, C_i, C_j, A_j]$ *is determined.*

5.3 Discussion

Let us make a few comments on our algorithm.

Correctness. Let $\mathcal{P}_0$ denote the initial poset. First, remark that we explore the candidates exhaustively, so every good extension of $\mathcal{P}_0$ is a good extension of $\mathcal{P}_0$ augmented by (at least) one of the candidates. Next, consider the poset $\mathcal{P}$ obtained in step 2. When processing a pair (i, j), we either discard $\mathcal{P}$ if we detect that i and j intersects in it (2b1) or we move on to the next pair (i, j) after having checked (2a, 2b3) or ensured (2b2) that triangles i and j are disjoint in $\mathcal{P}$. If we reach step 3, then all extensions of the current partial order are good and we correctly report feasibility. If a candidate is discarded then no linear extension of $\mathcal{P}$ augmented by that candidate is a good order. If we reach Step 4, then every candidate has been discarded, so no linear extension of $\mathcal{P}_0$ was a good order to begin with, and we correctly report unfeasibility.

Complexity. Initializing the poset and computing U take $O(n^3)$ time. We have at most 6^n candidates to consider. Step 1 takes $O(n^3)$ time. The steps 2a-2b3 are executed $O\left(n^2\right)$ times, and the bottleneck among them is 2b2, which takes $O(n^2)$ time. Altogether, our algorithm decides if a tagged pattern is realizable in $O\left(6^n n^4\right)$ time.

Improvements. In practice, the algorithm we presented can be sped up in several ways. For example, it is much better to branch over the pairs of U one by one. Once a branching is done, we can update U by removing the pairs that have become comparable, and thus avoid examining candidates that would get discarded at Step 1. Also, it pays off to record the forbidden tagged patterns of small size, and, given a larger tagged pattern to test, check first that it does not contain a small forbidden pattern.

One-sided certificate. If the algorithm reaches Step 3, we actually know a poset for which every linear extension is good. This means that we can compute an arbitrary linear extension to obtain an order on the variables in $(1, \cdot)$. We can then assign to these variables any values that satisfy this order, say by choosing the integers from 2 to $3n+1$, and then propagate these values via f and $f^{(2)}$ to all lifted variables. From there, we can extract the values of $x_1, x_2, \ldots, z_n$ of a concrete realization of our tagged pattern. In this way, all computations are done on (relatively small) rationals and are therefore easy to do exactly.

6 Experimental results

We now discuss our implementation of the above algorithm as well as its experimental use. Remember that we call a tagged pattern forbidden if it admits no *canonical* realization. We make the raw data available (see the full paper).

Implementation. We implemented the algorithm of Section 5 in Python 3. For simplicity, our implementation makes one adjustment to the algorithm: we branch over all $2^{|U|}$ choices for the pairs of undecided variables; so, we take 8 choice per k, rather than 6. Altogether, the implementation amounts to ~ 470 lines of (commented) code and is sufficiently effective for our experiments: on a standard desktop computer, finding all realizable triples of size 6 (and a realization when it exists) takes about 40 minutes, whereas verifying that no minimally forbidden tagged pattern of size 6 exists took up about a month of computer time; the difference of course is that in the former, for realizable triples we do not have to look at all positions of tags.

Proof of Lemma 9. The statement concerns only two triangles and can be shown by a simple case analysis. Our code sets up an exception that is raised if the statement of the lemma fails (cf line 94 in the code in Appendix B1 in the full version). Checking the realizability of all tagged patterns on two elements exhausts the case analysis, and the exception is not raised.

Minimally forbidden patterns. To state the minimally forbidden tagged patterns of size 3 we compress the notation as follows. We use $\{uv\}$ to mean "uv or vu". Symbols that are omitted may be placed anywhere (this may include $^{\mathbf{0}}$ and $^{\mathbf{1}}$). We use $x_i = y_j$ to mean "any pattern in which the ith symbol on the 1st tagged permutation equals the jth symbol of the 2nd tagged permutation".

▶ **Proposition 10.** *The equivalence classes of minimally forbidden tagged patterns are:*

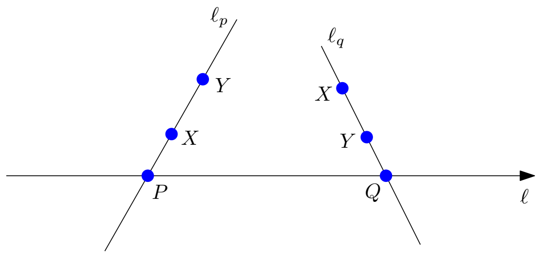
(i) *For size* 2, $(ab^{\mathbf{0}}, {}^{\mathbf{1}}ab, ab)$, $({}^{\mathbf{0}}ab, ab^{\mathbf{1}}, ba)$, $(ab^{\mathbf{0}}, ba^{\mathbf{1}}, ab)$, *and* $({}^{\mathbf{0}}ab, {}^{\mathbf{1}}ba, ba)$.

(ii) *For size* 3, $(\{ab\}^{\mathbf{0}}, \{ab\}c^{\mathbf{0}}, z_2 = y_1)$, $(\{ab\}c^{\mathbf{0}}, {}^{\mathbf{1}}\{ab\}, z_2 = x_1)$, $(\{ab\}^{\mathbf{0}}, {}^{\mathbf{1}}c\{ab\}, z_2 = y_3)$, $({}^{\mathbf{1}}c\{ab\}, {}^{\mathbf{1}}\{ab\}, z_2 = x_3)$, $(abc^{\mathbf{0}}, b^{\mathbf{1}}ac, ca^{\mathbf{0}}b)$, *and* $({}^{\mathbf{1}}abc, b^{\mathbf{1}}ca, ac^{\mathbf{0}}b)$.

(iii) *For size* 4, *the taggings of* $(abcd, badc, cdab)$ *that contains* $({}^{\mathbf{0}}b^{\mathbf{1}}, {}^{\mathbf{1}}d, a^{\mathbf{0}})$ *or* $(b^{\mathbf{0}}c, {}^{\mathbf{1}}a, a^{\mathbf{0}})$, *and the taggings of* $(abcd, badc, dcba)$ *that contains* $(b^{\mathbf{0}}c, {}^{\mathbf{0}}d^{\mathbf{1}}, {}^{\mathbf{1}}c)$ *or* $(c^{\mathbf{0}}, \{{}^{\mathbf{0}}ba\}\{{}^{\mathbf{1}}dc\}, {}^{\mathbf{1}}c)$.

(iv) *None for size* 5 *and* 6.

Realization database. For every tagged pattern that our algorithm declared realizable, we computed a realization (as explained in Section 5.3) and checked it independently.



Geometric permutations. It remains to prove our statements on geometric permutations:

Proof of Theorem 1. For every triple of permutations, we checked that it is realizable by trying all 8 reversals and all $\binom{7}{2}^3$ possible positions of $^{\mathbf{0}}$ and $^{\mathbf{1}}$, until we find a choice that does not contain any minimally forbidden tagged pattern of Proposition 10. ◀

Proof of Theorem 2. We argue by contradiction. Consider four disjoint convex sets met by lines ℓ_1, ℓ_2 in the order $abcd$ and ℓ_3 in the order $badc$; assume that $\ell_1 \odot \ell_3 = -\ell_2 \odot \ell_3$. By the perturbation argument of Lemma 4, we can assume that the three lines are pairwise skew and that the convex sets are triangles with vertices on these lines. Moreover, there exists a nonsingular affine transform A that maps the unoriented lines ℓ_1 to ℓ_x, ℓ_2 to ℓ_y and ℓ_3 to ℓ_z. Remark that A either preserves or reverses all side operators. Since $\ell_x \odot \ell_z = \ell_y \odot \ell_z$ and $\ell_1 \odot \ell_3 = -\ell_2 \odot \ell_3$, the map A sends the oriented lines (ℓ_1, ℓ_2) to either $(\ell_x, -\ell_y)$ or $(-\ell_x, \ell_y)$. We used our program to check that none of $(abcd, dcba, badc)$, $(dcba, abcd, badc)$, $(abcd, dcba, cdab)$, $(dcba, abcd, cdab)$ admits a canonical realization. The statement follows. ◀

7 Geometric analysis (size two)

We present here an independent proof that the tagged patterns of size 2 listed in Proposition 10 do not have a (canonical) realization. We have a similar proof for tagged patterns of size 3 but defer it to the full paper for lack of space. We do not prove the patterns are minimal, nor do we prove that the list is exhaustive; these facts come from the completeness of our computer-aided enumeration.

The following observation was used by Asinowski and Katchalski [2]:

▶ **Observation 11.** *Let X and Y be compact convex sets and let P and Q be points in $\mathbb{R}^3$. Assume that X, Y, P, Q are pairwise disjoint and that there exist lines inducing the geometric permutation (PXY) and (QYX). Any oriented line with direction $\overrightarrow{PQ}$ that intersects X and Y, must intersect X before Y.*

Proof. Refer to the figure. Let h be a plane that separates X and Y. The existence of the geometric permutations (PXY) and (QYX) ensure that h also separates P and Q. Moreover, the halfspace bounded by h that contains X also contains P, so any line with direction $\overrightarrow{PQ}$ traverses h from the side of X to the side of Y. ◀

Observation 11 implies that $(ab^{\mathbf{0}},{}^{\mathbf{1}}ab, ab)$, $({}^{\mathbf{0}}ab, ab^{\mathbf{1}}, ba)$, $(ab^{\mathbf{0}}, ba^{\mathbf{1}}, ab)$, and $({}^{\mathbf{0}}ab,{}^{\mathbf{1}}ba, ba)$, are forbidden. Indeed, consider, by contradiction, a realization of one of these tagged patterns. Let P be the point $^{\mathbf{0}}$ on ℓ_x and Q the point $^{\mathbf{1}}$ on ℓ_y. In each case, we can map X and Y to a and b so that some line $\ell_P \in \{\ell_x, -\ell_x\}$ realizes PXY and $\ell_Q \in \{\ell_y, -\ell_y\}$ realizes

QYX. Then, Observation 11 implies that any line with same direction as the line from P to Q must intersect X before Y; this applies to the line ℓ_z and contradicts the fact that the configuration realizes the chosen tagged pattern.

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