

Sum-Of-Squares Bounds via Boolean Function Analysis

Adam Kurpisz

ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland
adam.kurpisz@ifor.math.ethz.ch

Abstract

We introduce a method for proving bounds on the SoS rank based on Boolean Function Analysis and Approximation Theory. We apply our technique to improve upon existing results, thus making progress towards answering several open questions.

We consider two questions by Laurent. First, finding what is the SoS rank of the linear representation of the set with no integral points. We prove that the SoS rank is between $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} + \sqrt{n \log 2n} \rceil$. Second, proving the bounds on the SoS rank for the instance of the Min Knapsack problem. We show that the SoS rank is at least $\Omega(\sqrt{n})$ and at most $\lceil \frac{n+4\lceil\sqrt{n}\rceil}{2} \rceil$. Finally, we consider the question by Bienstock regarding the instance of the Set Cover problem. For this problem we prove the SoS rank lower bound of $\Omega(\sqrt{n})$.

2012 ACM Subject Classification Theory of computation \rightarrow Semidefinite programming; Theory of computation \rightarrow Convex optimization

Keywords and phrases SoS certificate, SoS rank, hypercube optimization, semidefinite programming

Digital Object Identifier 10.4230/LIPIcs.ICALP.2019.79

Category Track A: Algorithms, Complexity and Games

Funding Supported by SNSF project PZ00P2_174117.

Acknowledgements I would like to express my gratitude to Markus Schweighofer for fruitful discussions.

1 Introduction

A *boolean hypercube optimization problem* is a (constrained) polynomial optimization problem in n -variables, where the set of feasible points is restricted to a subset of an n -dimensional hypercube. This family of problems lies at the heart of theoretical computer science. However, solving general optimization problem over boolean hypercube is NP-hard, since the class contains, e.g., the Independent Set problem.

One of the most successful approaches for constructing theoretically efficient algorithms for this family of problems is the *Sum of Squares (SoS)* algorithm [23, 41, 43, 51]. For a wide variety of combinatorial optimization problems it provides the best available algorithms [1, 20, 5, 24, 36]. Recently, SoS has also been applied to problems in ROBUST ESTIMATION [26], DICTIONARY LEARNING [3, 49] and TENSOR COMPLETION AND DECOMPOSITION [4, 25, 45]. Other applications can be found in [5, 7, 12, 13, 17, 18, 24, 38, 39, 46]; see also the surveys [14, 33, 35].

On the other hand it is known that the SoS algorithm admits certain weaknesses. For example, a $\Omega(n)$ degree SoS certificate is needed to detect a simple integrality argument for some instance of the KNAPSACK problem as showed by Grigoriev in [21], see also [22, 28, 34]. Further weaknesses of SoS for KNAPSACK problems were proved in [11, 31]. Some lower bounds on the effectiveness of the SoS have been shown for CSP problems [27, 52] and for



© Adam Kurpisz;

licensed under Creative Commons License CC-BY

46th International Colloquium on Automata, Languages, and Programming (ICALP 2019).

Editors: Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi;

Article No. 79; pp. 79:1–79:15



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



the planted clique problem [2, 40]. Finally, degree $\Omega(\sqrt{n})$ SoS was proved to have problems scheduling unit size jobs on a single machine to minimize the number of late jobs, see [30]. The problem is solvable in polynomial time using the Moore-Hodgson algorithm.

Proving SoS lower bounds is not an easy task. In most of the results, the lower bound is proved on the *Lasserre relaxation* side, by explicitly giving a *pseudoexpectation* operator that maps polynomials of certain degree to real numbers, such that the corresponding *moment matrix* for variables and *localizing matrices* for constraints are Positive Semi-definite (PSD). Both finding a suitable pseudoexpectation and proving PSDness might be very difficult.

In this paper we follow the less common approach of working on a *dual* side, the *SoS certificate of nonnegativity*. This approach does not require finding a pseudoexpectation operator and proving PSDness of the matrices, which we find very convenient. We prove lower bounds by exploiting the properties of boolean functions and by using the results from approximation theory, especially the theory of approximating boolean functions, see e.g. [44, 50, 53]. This, in our case, directly implies the existence/non-existence of the SoS certificate. To the best of our knowledge there are not that many SoS lower bounds using this approach, see e.g. [37].

On an intuitive level our approach might be explained in the following way. Imagine one wants to write a degree d SoS certificate for some function f that is nonnegative over a subset of the boolean hypercube. Consider a vertex x of the hypercube, where $f(x) < 0$. Note that in order to successfully write a certificate one has to use the constraint g , that made the point x infeasible, and multiply it by some SoS polynomial s of degree at most $2d$. Moreover, the values $f(x)$ and $g(x)$ give some lower bound on the value of $s(x)$. Ideally, we would like s to take value zero on other vertices of the hypercube, this is the case for degree n certificates. However, the lower the degree d , the more other vertices of the hypercube are "affected" by the value of $s(x)$.

This simple observation is strong enough to provide the best known SoS ranks for the following problems:

For $B \geq 2$. The EMPTY INTEGRAL HULL (EIH) problem is a feasibility problem of the form

$$EIH = \{0, 1\}^n \cap \{x \in [0, 1]^n \mid \sum_{i \in [n] \setminus I} x_i + \sum_{i \in I} (1 - x_i) \geq \frac{1}{B} \text{ for all } I \subseteq [n]\} \quad (1.1)$$

For $B = 2$, a long list of results for the performance of the lift and project methods for *EIH* problem is known. In [33], Laurent shows that the Sherali-Adams *rank* is n . They then conjecture the SoS rank of *EIH* is $n - 1$. Moreover, the rank is also equal to n for the Lovász-Schrijver N_+ operator (with positive semidefiniteness) [19], the Lovász-Schrijver N_+ operator strengthen with Chvátal cuts [15], and the N_+ operator combined with Gomory mixed integer cuts (equivalent to disjunctive cuts) [16]. Recently, in [29], the conjecture was disproved, and it was shown that the SoS rank of *EIH* is between $\Omega(\sqrt{n})$ and $n - \Omega(n^{1/3})$. In this paper we prove the following result:

► **Theorem 1.** *For $B = 2$, the SoS rank of EIH problem is between $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} + \sqrt{n \log 2n} \rceil$.*

In this paper we also prove lower and upper bounds on the SoS rank for any $B \geq 2$, see Theorem 12 and Lemma 13.

The second considered problem is the instance of the MIN KNAPSACK (MK) problem. For $P \geq 2$, the problem is defined as:

$$\text{MK:} \quad \min \sum_{i \in [n]} x_i \quad \text{s.t.} \quad \sum_{i \in [n]} x_i \geq \frac{1}{P} \quad (1.2) \\ x \in \{0, 1\}^n$$

For $P = 2$ the problem was previously considered by Cook and Dash [15]. They proved that the Lovasz-Schrijver hierarchy rank is n . For the Sherali-Adams hierarchy Laurent in [33] proved that the rank is also equal to n and raised the open question to find the rank for the SoS hierarchy. For $n = 2$ they also proved that the SoS rank is 2, but whether or not this happens for general n was left open. In [30] the possibility that the Lasserre/SoS rank is n for $n \geq 3$ is ruled out. Finally, in [28], a SoS rank lower bound of $t = \Omega(\log^{1-\epsilon} n)$, for $\epsilon > 0$, was presented. In this paper we prove the following result:

► **Theorem 2.** *For $P = 2$, the SoS rank of the MK problem is between $\Omega(\sqrt{n})$ and $\lceil \frac{n+4\lceil\sqrt{n}\rceil}{2} \rceil$.*

Moreover, we prove a SoS rank lower bound for any $P \geq 2$ of the value $\Omega(\sqrt{n} + \sqrt{n \log P})$, see Lemma 14.

The third problem we consider is the instance of the Set Cover (SC) problem:

$$\text{SC:} \quad \min \sum_{i \in [n]} x_i \quad \text{s.t.} \quad \sum_{i \in [n] \setminus \{j\}} x_i \geq 1 \quad \forall j \in [n] \quad (1.3)$$

$$x \in \{0, 1\}^n$$

This instance was considered in [8], where an open question was raised asking what is the rank of this polytope, conjecturing that the SoS rank is at least $n/4$, based on numerical experiments. In [29], the conjecture was supported by proving that the rank is at least $\log^{1-\delta}(n)$ for any $\delta > 0$. Finally, the instance can be seen as the Min Knapsack instance with Knapsack Cover inequalities, see [29]. In this paper we prove the following result:

► **Theorem 3.** *The SoS rank of the SC problem is at least $\Omega(\sqrt{n})$.*

2 Preliminaries

For any $n \in \mathbb{N}$ we denote $[n] = \{1, \dots, n\}$. Let $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ be the ring of n -variate real polynomials. Furthermore, let \mathcal{G} be the set of polynomials

$$\mathcal{G} := \{g_0 := 1, g_1, \dots, g_m : g_i \in \mathbb{R}[x] \text{ for all } i \in [m]\}.$$

For a given set \mathcal{G} , the corresponding *semialgebraic set* is defined as

$$\mathcal{G}_+ := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \text{ for all } g \in \mathcal{G}\} \subseteq \mathbb{R}^n.$$

Moreover, for any given semialgebraic set $\mathcal{G}_+ \subseteq \mathbb{R}^n$, let $\mathcal{K}(\mathcal{G}_+)$ be the set of *nonnegative polynomials* over the set \mathcal{G}_+

$$\mathcal{K}(\mathcal{G}_+) := \{f \in \mathbb{R}[x] \mid f(x) \geq 0 \text{ for all } x \in \mathcal{G}_+\}.$$

For a given $f \in \mathbb{R}[x]$ and a set \mathcal{G} we define the corresponding *constrained polynomial optimization problem* (CPOP)

$$f^* := \min\{f(x) \mid x \in \mathcal{G}_+\} = \max\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{K}(\mathcal{G}_+)\}. \quad (2.1)$$

and the *constrained polynomial feasibility problem* (CPFP), which consists in testing whether the corresponding set \mathcal{G}_+ is empty or not, see e.g., [9].

Since the problems CPOP and CPFP are NP-hard in general it is desirable to find a proper subset that is a good inner approximation of $\mathcal{K}(\mathcal{G}_+)$ such that the corresponding program is computationally *tractable*.

SoS method

The *SoS method* approximates the cone $\mathcal{K}(\mathcal{G}_+)$ by using the set of *sum of square polynomials*. Let $\mathbb{SOS} := \{s \mid s = \sum_{i=1}^k f_i^2, f_i \in \mathbb{R}[x], k \in \mathbb{N}^*\}$ be the set of (finite) *sum of square polynomials (SoS)*. Let

$$\Sigma_{n,d}^{\mathcal{G}} := \left\{ \sum_{i=1}^m s_i g_i \mid s_i \in \mathbb{SOS}, g_i \in \mathcal{G}, \deg(s_i) \leq 2d, \text{ for all } i \in [m] \text{ and } \deg(s_0) \leq 2 \left\lceil \frac{2d + \deg(\mathcal{G})}{2} \right\rceil \right\},$$

where $\deg(\mathcal{G}) = \max\{\deg(g) \mid g \in \mathcal{G}\}$. The *degree d SoS certificate* for f being nonnegative over \mathcal{G}_+ is $f \in \Sigma_{n,d}^{\mathcal{G}}$. The *degree d SoS program* for CPOP takes the form:

$$f_{\mathbb{SOS}}^d := \max\{\lambda \in \mathbb{R} \mid f - \lambda \in \Sigma_{n,d}^{\mathcal{G}}\}; \quad (2.2)$$

and for CPFP it consists in testing whether or not $-1 \in \Sigma_{n,d}^{\mathcal{G}}$ and, if so, implying that the set \mathcal{G}_+ is empty.

SoS method over the boolean hypercube

In this paper we consider optimization over the boolean hypercube $\mathcal{H} := \{0, 1\}^n$. We assume that \mathcal{G} is such that $\mathcal{G}_+ \subseteq \mathcal{H}$. We fix the following notation. Let \mathcal{H}^+ and \mathcal{H}^- be \mathcal{G}_+ and $\mathcal{H} \setminus \mathcal{G}_+$, respectively. For $I \in [n]$, let $x_I \in \mathbb{R}^n$ be the characteristic vector of set I such that $(x_I)_i = 1$ if $i \in I$ and 0 otherwise. Moreover, for any $f \in \mathbb{R}[x]$, let $\mathcal{H}^-(f) := \{x \in \mathcal{H} \mid f(x) < 0\}$ and $\mathcal{H}^+(f) := \{x \in \mathcal{H} \mid f(x) \geq 0\}$.

Throughout the paper we assume that \mathcal{G} is always of the form

$$\mathcal{G} := \{g_0 := 1, g_1, \dots, g_m, \pm(x_1^2 - x_1), \dots, \pm(x_n^2 - x_n) : g_i \in \mathbb{R}[x] \text{ for all } i \in [m]\}.$$

In this case solving a degree d SoS program can be done via Semi-definite Program (SDP) of size $O(\sum_{k=0}^d \binom{n}{k})$. Moreover, it is known that degree n SoS program is *exact* meaning that for CPOP we have $f_{\mathbb{SOS}}^n = f^*$ and for CPFP we have $-1 \in \Sigma_{n,n}$ if and only if the set \mathcal{G}_+ is empty, see e.g. [6, 32, 33]. A very useful result was recently proved in [48] giving an upper bound on the degree of the SoS certificate for every degree r unconstrained boolean hypercube optimization problem:

► **Theorem 4** ([48]). *Every n -variate polynomial of degree r , nonnegative over the unconstrained boolean hypercube has a degree $\lceil \frac{n+r-1}{2} \rceil$ SoS certificate.*

Now we give the following definition.

► **Definition 5.** *The SoS rank for a CPOP (CPFP) is the smallest degree d such that the degree d SoS program is exact.*

That is why the SoS rank for EIH problem is the smallest degree d such that there exist SoS polynomials s_I , for all $I \subseteq [n]$ of degree at most $2d$ and s_0 of degree at most $2d + 2$ such that: $-1 = s_0 + \sum_{I \subseteq [n]} s_I(x) \left(\sum_{i \in [n] \setminus I} x_i + \sum_{j \in I} (1 - x_j) - 1/B \right)$.

Analogously, the SoS rank for the MK problem is the smallest d such that there exist SoS polynomials s_1 of degree at most $2d$ and s_0 of degree at most $2d + 2$ such that: $\sum_{i \in [n]} x_i - 1 = s_0 + s_1 \left(\sum_{i \in [n]} x_i - 1/P \right)$.

Finally, the SoS rank for the SC problem is the smallest degree d such that there exist SoS polynomials s_1, \dots, s_n of degree at most $2d$ and s_0 of degree at most $2d + 2$ such that: $\sum_{i \in [n]} x_i - 2 = s_0 + \sum_{j \in [n]} s_j(x) \left(\sum_{j \neq i \in [n]} x_j - 1 \right)$.

3 Analysis of boolean functions

In this section we show some properties of boolean functions.

The first one is a very intuitive statement saying that the lower degree SoS boolean function we consider, the less of the total mass of the values is in one particular point.

The second one is the special case of results in [44, 50, 53] giving a lower bound on the degree of a real polynomial that approximates the NOR boolean function in ℓ_∞ -norm.

► **Lemma 6.** *For every function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ of degree at most d and every subset $J \subseteq [n]$ the following holds:*

$$\sum_{S \subseteq [n]} f^2(x_S) \geq \frac{2^n}{\sum_{i=0}^d \binom{n}{i}} f^2(x_J).$$

Moreover, for every $J \subseteq [n]$ there exists a degree d polynomial such that the above inequality is satisfied with equality.

Proof. We will use some elementary Fourier analysis of boolean functions (see e.g. [42, Ch. 1]). For the sake of following an established notation we analyze the function $h : \{\pm 1\}^n \rightarrow \mathbb{R}$ instead of $f : \{0, 1\}^n \rightarrow \mathbb{R}$ using the bijective transformation $f(x) = h(w)$, for $w = 1 - 2x$. Clearly this transformation preserves the degree of the function.

Let $w_I \in \mathbb{R}^n$ be the characteristic vector of set $I \subseteq [n]$ such that $(w_I)_i = -1$ if $i \in I$ and 1 otherwise. The *Fourier expansion* of the function h takes the following form:

$$h(w) = \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \widehat{h}(I) \prod_{i \in I} w_i,$$

where $\widehat{h}(I)$ is the *Fourier coefficient* of the monomial $\prod_{i \in I} w_i$. Let $g : \{\pm 1\}^n \rightarrow \mathbb{R}$ be such that $g := h^2$. Thus $g(w) = \left(\sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \widehat{h}(I) \prod_{i \in I} w_i \right)^2$ and its Fourier expansion is of the form:

$$g(w) = \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \left(\widehat{h}(I) \right)^2 + \sum_{\substack{I \neq K \subseteq [n] \\ |I|, |K| \leq d}} \widehat{h}(I) \widehat{h}(K) \prod_{i \in I \Delta K} w_i, \quad (3.1)$$

since all monomials squared evaluate to 1 over the $\{\pm 1\}^n$ boolean hypercube.

By [42, Fact 1.12] and Equation 3.1 we know that

$$\widehat{g}(\emptyset) = \mathbb{E}_{w \sim \{\pm 1\}^n} g(w) = 2^{-n} \sum_{I \subseteq [n]} g(w_I) = \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \left(\widehat{h}(I) \right)^2 \quad (3.2)$$

Without loss of generality, we assume that the set J in the statement of the Theorem is the empty set. Thus:

$$g(w_\emptyset) = \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \left(\widehat{h}(I) \right)^2 + \sum_{\substack{I \neq K \subseteq [n] \\ |I|, |K| \leq d}} \widehat{h}(I) \widehat{h}(K). \quad (3.3)$$

Now, we show that among all boolean functions normalized to have expected value over $\{\pm 1\}^n$ boolean hypercube equal to $\sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \left(\widehat{h}(I) \right)^2$ the function $g := h^2$, such that all Fourier

coefficients of h are the same, takes the largest value on the point w_\emptyset . Indeed, consider a function g evaluated at point w_\emptyset . The RHS of Equation 3.3 satisfies:

$$\sum_{\substack{I, K \subseteq [n] \\ |I|, |K| \leq d}} \widehat{h}(I)\widehat{h}(K) \leq \frac{1}{2} \sum_{\substack{I, K \subseteq [n] \\ |I|, |K| \leq d}} \left(\widehat{h}^2(I) + \widehat{h}^2(K) \right) = \sum_{i=0}^d \binom{n}{i} \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \widehat{h}^2(I)$$

and the equality holds when all Fourier coefficients of h are the same.

Finally, we show that for the function $g = h^2$, such that all Fourier coefficients of h are the same, the Theorem holds. Assume w.l.o.g. that for every $I \subseteq [n]$, $\widehat{h}(I) = 1$. Then we get

$$h^2(w_\emptyset) = g(w_\emptyset) = \left(\sum_{i=0}^d \binom{n}{i} \right)^2 \quad \text{and} \quad \sum_{I \subseteq [n]} h^2(w_I) = 2^n \sum_{\substack{I \subseteq [n] \\ |I| \leq d}} \left(\widehat{h}(I) \right)^2 = 2^n \sum_{i=0}^d \binom{n}{i}.$$

thus the first part of the statement follows.

Note, that, for $J = \emptyset$, the above proof gives an explicit construction of a degree d polynomial that satisfies the inequality in the statement of Lemma 6 with equality. In an analogous way one can construct a function for other sets $J \subseteq [n]$, thus the second part of the claim follows. ◀

Clearly a similar statement holds for functions being a sum of squares of boolean functions.

► **Corollary 3.1.** *For every function $s = \sum_i h_i^2$ such that for every i , $h_i : \{0, 1\}^n \rightarrow \mathbb{R}$ is of degree at most d and every subset $J \subseteq [n]$ the following holds:*

$$\sum_{S \subseteq [n]} s(x_S) \geq \frac{2^n}{\sum_{i=0}^d \binom{n}{i}} s(x_J).$$

Now we present the second result in [44, 50, 53]. We start with the following definitions.

A boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ is *symmetric* if $f(x) = f_i \in \mathbb{R}$ for every $x \in \{0, 1\}^n$ such that $|x| = i$. A polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$ *approximates* a symmetric boolean function f in ℓ_∞ -norm within an error c , if $|f(x) - p(x)| \leq c$ for every $x \in \{0, 1\}^n$. A symmetric boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is a *NOR* function if it satisfies $f(0, \dots, 0) = 1$ and $f(x) = 0$ for every other $x \in \{0, 1\}^n$.

Now we present the result proved in [44, 50, 53].

► **Theorem 7** ([44, 50, 53]). *For every constant $2^{-n} \leq c < 1/2$ the minimum degree of a real polynomial that approximates a NOR boolean function in ℓ_∞ -norm within an error c is $\Theta(\sqrt{n} + \sqrt{n \log 1/c})$.*

3.1 Application to the SoS certificates

In this section, we apply Theorem 4, Corollary 3.1 and Theorem 7 to provide lower and upper bounds on the SoS rank for some family of problems. We start with the following definition.

► **Definition 8.** *A polynomial f is called a Single Vertex Cutting (SVC) constraint if there exists only one $x \in \{0, 1\}^n$ such that $f(x) < 0$. A set \mathcal{G} is SVC if all functions $g_1, \dots, g_m \in \mathcal{G}$ are SVC and every $x \in \{0, 1\}^n$ is cut by at most one $g \in \mathcal{G}$. A problem is SVC if its corresponding set \mathcal{G} is SVC.*

Note that all three problems considered in this paper are SVC Problems.

► **Theorem 9.** Consider an SVC system \mathcal{G} . For some $f \in \mathbb{R}[x]$, assume that $\mathcal{H}^-(f) = \mathcal{H}^-$, thus $|\mathcal{H}^-(f)| = m$. For every $x_J \in \mathcal{H}^-(f)$ let $g_J \in \mathcal{G}$ be such that $g_J(x_J) < 0$, g_J is unique.

There is no degree d certificate for f over system \mathcal{G} , if for every $c_J \geq 1$, for every J such that $x_J \in \mathcal{H}^-(f)$, the following holds

$$\sum_{x_J \in \mathcal{H}^+(f)} f(x_J) + \sum_{x_J \in \mathcal{H}^-(f)} f(x_J)(1 - c_J) < \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) \sum_{x_I \in \mathcal{H}^-(f)} c_I \frac{f(x_I)}{g_I(x_I)} \cdot \min_{\substack{x_J \in \mathcal{H}^- \\ J \neq I}} g_I(x_J)$$

Proof. Assume, there exists a SoS certificate for f of degree d over \mathcal{G} , then

$$f(x) = s_0 + \sum_{x_J \in \mathcal{H}^-} s_J(x)g_J(x), \quad \text{for every } x \in \{0, 1\}^n,$$

for s_J , being SoS of degree at most $2d$, and s_0 being SoS of degree at most $2 \left\lceil \frac{2d + \deg(\mathcal{G})}{2} \right\rceil$. Consider $x_J \in \mathcal{H}^-(f)$. Since g_J is the only constraint from \mathcal{G} that is negative on this point we get that

$$s_J(x_J) \geq \frac{f(x_J)}{g_J(x_J)}$$

Let $c_J \geq 1$ be such that $s_J(x_J) = c_J \cdot \frac{f(x_J)}{g_J(x_J)}$, we obtain:

$$f(x_J) = \underbrace{s_0(x_J)}_{\geq 0} + \underbrace{s_J(x_J)g_J(x_J)}_{=c_J f(x_J)} + \sum_{\substack{x_I \in \mathcal{H}^- \\ I \neq J}} \underbrace{s_I(x_J)g_I(x_J)}_{\geq 0}.$$

By summing up over all points in $\mathcal{H}^-(f)$ we get:

$$\sum_{x_J \in \mathcal{H}^-(f)} f(x_J) = \underbrace{\sum_{x_J \in \mathcal{H}^-(f)} s_0(x_J)}_{\geq 0} + \underbrace{\sum_{x_J \in \mathcal{H}^-(f)} s_J(x_J)g_J(x_J)}_{=\sum_{x_J \in \mathcal{H}^-(f)} c_J f(x_J)} + \sum_{x_J \in \mathcal{H}^-(f)} \sum_{\substack{x_I \in \mathcal{H}^- \\ I \neq J}} s_I(x_J)g_I(x_J)$$

thus

$$\sum_{x_J \in \mathcal{H}^-(f)} f(x_J)(1 - c_J) = \sum_{x_J \in \mathcal{H}^-(f)} s_0(x_J) + \sum_{x_J \in \mathcal{H}^-(f)} \left(\sum_{\substack{x_I \in \mathcal{H}^- \\ I \neq J}} s_I(x_J)g_I(x_J) \right).$$

Finally, we have

$$\sum_{x_J \in \mathcal{H}^+(f)} f(x_J) + \sum_{x_J \in \mathcal{H}^-(f)} f(x_J)(1 - c_J) = \sum_{x_J \in \mathcal{H}} s_0(x_J) + \sum_{x_J \in \mathcal{H}} \left(\sum_{\substack{x_I \in \mathcal{H}^- \\ I \neq J}} s_I(x_J)g_I(x_J) \right),$$

where

$$\sum_{x_J \in \mathcal{H}} \left(\sum_{\substack{x_I \in \mathcal{H}^- \\ I \neq J}} s_I(x_J)g_I(x_J) \right) \geq \sum_{x_I \in \mathcal{H}^-} \left(\left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) s_I(x_I) \left(\min_{\substack{x_J \in \mathcal{H}^- \\ J \neq I}} g_I(x_J) \right) \right)$$

and the last inequality follows by Corollary 3.1. Since, $\sum_{x_J \in \mathcal{H}} s_0(x_J) \geq 0$ and $s_I(x_I) = c_I \cdot \frac{f(x_I)}{g_I(x_I)}$, the claim follows. ◀

► **Theorem 10.** *If every SoS certificate of f over system \mathcal{G} of the form $f = s_0 + \sum_{i=1}^m s_i g_i$, is such that the polynomial $s(x) := \sum_{i=1}^m s_i(x)$ necessarily has to take at least value $1 - c$ for $x = (0, 0, \dots, 0)$ and at most value c , for every other $x \in \{0, 1\}^n$, for some constant $2^{-n} \leq c < 1/2$, then the degree of the certificate is at least $\Omega(\sqrt{n} + \sqrt{n \log 1/c})$.*

Proof. Assume by contradiction that there exists a SoS certificate of degree smaller than $\Omega(\sqrt{n} + \sqrt{n \log 1/c})$. Note that s is a real polynomial that approximates the NOR function in ℓ_∞ -norm within the constant error c . Since s is of degree smaller than $\Omega(\sqrt{n} + \sqrt{n \log 1/c})$ it contradicts Theorem 7. ◀

Finally, we show an argument for an upper bounds on the SoS rank.

► **Lemma 11.** *Let f be of degree at most $d + 1$ and let g_i for $i \in [m]$ be linear. If there exist SoS polynomials s_1, \dots, s_m of degree at most d such that*

$$f(x) \geq \sum_{i=1}^m s_i(x)g_i(x), \quad \text{for every } x \in \{0, 1\}^n,$$

then there exists a SoS certificate for f of degree $\lceil \frac{n+d}{2} \rceil$.

Proof. Note that $f(x) - \sum_{i=1}^m s_i(x)g_i(x) \geq 0$, for every $x \in \{0, 1\}^n$. By Theorem 4 and the fact that we consider g_i linear, for $i \in [m]$, we get that there exists a SoS polynomial s_0 of degree at most $\lceil \frac{n+(d+1)-1}{2} \rceil$ such that: $f(x) - \sum_{i=1}^m s_i(x)g_i(x) = s_0$. ◀

4 Application to the EIH problem

In this section we show how to use the results presented in Section 3.1 to derive lower and upper bounds on the SoS rank for the Empty Integral Hull problem.

4.1 SoS rank lower bounds

In this section we prove a lower bounds on the SoS rank for the Empty Integral Hull problem.

► **Theorem 12.** *The SoS rank for the Empty Integral Hull problem parametrized by constant B , is greater or equal the minimum d , which satisfies:*

$$\frac{B}{B-1} \geq \frac{2^n}{\sum_{k=0}^d \binom{n}{k}}.$$

Proof. We directly apply Theorem 9. We want to prove that for $f(x) = -1$ there is no SoS certificate of degree d , for d such that $\frac{B}{B-1} < \frac{2^n}{\sum_{k=0}^d \binom{n}{k}}$. Following the notation from Theorem 9 note that for every $g_I \in \mathcal{G}$, the smallest nonnegative value over the hypercube $\{0, 1\}^n$ of g_I is $1 - 1/B$. By Theorem 9 there is no degree d SoS certificate for $f = -1$, if for every $c_I \geq 1$, $I \subseteq [n]$, the following is satisfied:

$$(-1) \sum_{I \subseteq [n]} (1 - c_I) < \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) \sum_{I \subseteq [n]} c_I \frac{-1}{-\frac{1}{B}} \cdot \left(1 - \frac{1}{B} \right).$$

Let $c = 1/2^n \sum_{I \subseteq [n]} c_I$. The above is satisfied if

$$\frac{c-1}{c} < \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) (B-1).$$

Since $c \geq 1$, we have $\frac{c-1}{c} < 1$ and the above inequality holds if $\frac{1}{B-1} < \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right)$. ◀

We get the following corollary.

► **Corollary 4.1.** *The SoS rank for the EIH problem with $B = 2$, is at least $\lceil n/2 \rceil$.*

Proof. By Theorem 12 the SoS rank of the EIH for $B = 2$ is greater or equal to the minimum d that satisfies: $2 \geq \frac{2^n}{\sum_{k=0}^d \binom{n}{k}}$. which holds for $d \geq \lceil n/2 \rceil$. ◀

One can apply Theorem 12 to easily reprove the result from [31], saying that for $B = 2^{n+1}$ the SoS rank is at least n .

► **Corollary 4.2.** *For $B = 2^{n+1}$ the EIH problem has the SoS rank n .*

4.2 SoS rank upper bounds

In this section we prove an upper bound on the SoS rank for the Empty Integral Hull problem.

► **Lemma 13.** *The SoS rank for the Empty Integral Hull problem parametrized by the constant B , is less or equal the minimum d , that satisfies*

$$\frac{Bn}{Bn-1} > \frac{2^n}{\sum_{k=0}^d \binom{n}{k}}.$$

Proof. We follow the notation and reasoning from Theorem 9. Let $c_I \geq 1$, for $I \subseteq [n]$ be large enough constants, whose value is specified later.

Our goal is to construct a SoS certificate for $f = -1$ using the SoS polynomials of degree at most $2d$, for d given in the statement of this lemma. In our construction we take $s_0 = 0$ and consider s_J , for $J \subseteq [n]$, to be 2^n polynomials squared, each of which constructed as in Lemma 6, such that for every $I, J \subseteq [n]$ we have $\sum_{K \subseteq [n]} s_I(x_K)g_I(x_K) = \sum_{K \subseteq [n]} s_J(x_K)g_J(x_K)$.

We want to prove that there exists a certificate of the form

$$-1 = f(x_J) = \underbrace{s_J(x_J)g_J(x_J)}_{=c_J f(x_J)=-c_J} + \sum_{\substack{x_I \in \mathcal{H} \\ I \neq J}} \underbrace{s_I(x_I)g_I(x_I)}_{\geq 0}, \quad \text{for every } J \subseteq [n], \quad (4.1)$$

where, for every $J \subseteq [n]$, by the construction of the polynomial s_J , we know that:

$$\left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) s_J(x_J) \underbrace{\left(\min_{\substack{x_I \in \mathcal{H} \\ I \neq J}} g_I(x_I) \right)}_{=1-1/B} \leq \sum_{\substack{x_I \in \mathcal{H} \\ I \neq J}} \underbrace{s_I(x_I)g_I(x_I)}_{\geq 0} \leq \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) s_J(x_J) \underbrace{\left(\max_{\substack{x_I \in \mathcal{H} \\ I \neq J}} g_I(x_I) \right)}_{=n-1/B}.$$

Let

$$\sum_{\substack{x_I \in \mathcal{H} \\ I \neq J}} \underbrace{s_I(x_I)g_I(x_I)}_{\geq 0} = \alpha \left(1 - \frac{1}{B} \right) \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) s_J(x_J), \quad \text{for every } J \subseteq [n],$$

for some $1 \leq \alpha \leq \frac{n-1/B}{1-1/B}$. Since $s_J(x_J) = c_J \cdot f(x_J)/g_J(x_J) = c_J \cdot \frac{-1}{-1/B} = c_J \cdot B$, thus satisfying Equation 4.1 requires:

$$\frac{c_J - 1}{c_J} = \alpha \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) (B - 1), \quad \text{for every } J \subseteq [n].$$

79:10 Sum-Of-Squares Bounds via Boolean Function Analysis

Now, put all c_J equal and note that c_J can be chosen arbitrarily, so there exists SoS certificate of degree d if we can satisfy,

$$\alpha \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) (B-1) < 1.$$

which holds if

$$\frac{n-1/B}{1-1/B} (B-1) \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) = (Bn-1) \left(\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} - 1 \right) < 1.$$

Finally, the above is satisfied if $\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} < \frac{Bn}{Bn-1}$. ◀

For $B = 2$ we get the following corollary.

► **Corollary 4.3.** *The SoS rank for the EIH problem for $B = 2$ is at most $\lceil \frac{n}{2} + \sqrt{n \log 2n} \rceil$.*

Proof. By Lemma 13 we know that the SoS rank is smaller or equal to the smallest d that satisfies

$$\frac{2^n}{\sum_{k=0}^d \binom{n}{k}} < \frac{2n}{2n-1}.$$

Since $2^n = \sum_{k=0}^d \binom{n}{k} + \sum_{k=d+1}^n \binom{n}{k}$ we get the requirement

$$\frac{\sum_{k=d+1}^n \binom{n}{k}}{2^n - \sum_{k=d+1}^n \binom{n}{k}} < \frac{2n}{2n-1} - 1$$

and finally

$$\sum_{k=d+1}^n \binom{n}{k} < \frac{2^n}{2n}.$$

By the standard Chernoff bound, see e.g. [10], for $d > n/2$ we know that $\sum_{k=d+1}^n \binom{n}{k} = \sum_{k=0}^{n-d-1} \binom{n}{k} \leq 2^n \exp \frac{-(2d+2-n)^2}{4n}$. Thus for $d > \frac{1}{2} (n-2 + 2\sqrt{n \log 2n})$ there exists a SoS certificate for EIH problem for $B = 2$. ◀

5 Application to the MK problem

In this section we prove SoS rank lower and upper bounds for the MK problem.

5.1 SoS rank lower bound

We start with presenting the lower bound for the MK Problem.

► **Lemma 14.** *The SoS rank lower bound for the MK problem for is at least $\Omega(\sqrt{n} + \sqrt{n \log P})$.*

Proof. By contradiction, assume there exists a SoS certificate of degree smaller than $\Omega(\sqrt{n} + \sqrt{n \log P})$ for the function $\sum_{i=1}^n x_i - 1$, namely

$$\sum_{i=1}^n x_i - 1 = s_0(x) + s_1(x) \left(\sum_{i=1}^x x_i - \frac{1}{P} \right), \quad \text{for every } x \in \{0, 1\}^n.$$

for s_0 and s_1 SoS of degree smaller than $\Omega(\sqrt{n} + \sqrt{n \log P})$. Since $s_0(x) \geq 0$, for every $x \in \{0, 1\}^n$, we know that

$$s_1(0, \dots, 0) \geq \frac{-1}{-\frac{1}{P}} = P \quad \text{and} \quad s_1(x) \leq \frac{\sum_{i=1}^n x_i - 1}{\sum_{i=1}^n x_i - \frac{1}{P}} \leq 1 \text{ for every other } x \in \{0, 1\}^n.$$

This implies the existence of the function $\bar{s}(x) = s(x)/P$ of degree smaller than $\Omega(\sqrt{n} + \sqrt{n \log P})$ that approximates the NOR function within ℓ_∞ -norm within the error $1/P$ which contradicts Theorem 10. \blacktriangleleft

A direct application of Lemma 14 gives the following corollary:

► **Corollary 5.1.** *The SoS rank lower bound for MK problem for $P = 2$ is at least $\Omega(\sqrt{n})$.*

5.2 SoS rank upper bounds

In this section we prove an upper bound on the SoS rank for the MK problem for $P = 2$. We start with proving the following Lemma.

► **Lemma 15.** *For every $n \in \mathbb{N}$ there exists a function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ of degree $2\lceil \sqrt{n} \rceil$ such that $f(0, \dots, 0) \geq 2$, $f(x) = 0$ for every $x \in \{0, 1\}^n$ such that $|x| = 1$ and for every other $x \in \{0, 1\}^n$ the function takes the value $|f(x)| \leq 1$.*

Proof. Our construction provides a symmetric polynomial thus in the following we construct a univariate polynomial $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x_1, \dots, x_n) := g(\sum_{i=1}^n x_i)$ satisfies the claimed properties. We start with presenting steps for constructing polynomial g .

Step 1: Consider a Chebyshev polynomial $T_d(x) : [-1, 1] \rightarrow [-1, 1]$ of degree $d = 2\lceil \sqrt{n} \rceil$.

Prove that the smallest root (left-most one) appears before point $x = -1 + 1/n$.

Step 2: Shift and spread the polynomial $T_d(x)$ such that the domain is $[0, n]$, namely consider $T'_d(x) = T_d(2x/n - 1)$. Note that the smallest zero of T'_d appears before point $x = 1/2$.

Step 3: Extend the domain of $T'_d(x)$ to be $[-1/2, n]$ and then prove that $T'_d(-1/2) = T_d(-1 - 1/n)$ takes the absolute value at least 2.

Step 4: Consider a polynomial T''_d obtained from shifting the polynomial T'_d to the right such that the smallest zero is exactly at point $x = 1$. Note that $g := T''_d$ satisfies all the required properties.

It remains to prove the claims from Step 1 and 3. Let us start with the claim from Step 1.

It is known that Chebyshev polynomial of degree d , $T_d(x)$, has zeros at points $\cos\left(\frac{2j-1}{d}\frac{\pi}{2}\right)$ for $j = 1, \dots, d$ see e.g. [47][Equation 1.17], thus smallest zero is at the point $\cos\left(\pi - \frac{\pi}{2d}\right)$.

We consider $d = 2\lceil \sqrt{n} \rceil$, thus it is sufficient to prove that $\cos\left(\pi - \frac{\pi}{4\sqrt{n}}\right) + 1 - \frac{1}{n} \leq 0$ for every $n \in \mathbb{R}_+$. To do so, consider the function

$$h(z) = \cos\left(\pi - \frac{\pi}{4z}\right) + 1 - \frac{1}{z^2}$$

and note that $h(z)$ is increasing in z for $z \in \mathbb{R}_+$. Indeed the derivative of $h(z)$ is equal to

$$h'(z) = \frac{2}{z^3} - \frac{\pi}{4z^2} \sin\left(\pi - \frac{\pi}{4z}\right) = \frac{8 - \pi z \sin\left(\frac{\pi}{4z}\right)}{4z^3}.$$

Since, for all $z \in \mathbb{R}_+$, $\sin z \leq z$, it is enough to show that $8 - \pi z \frac{\pi}{4z} \geq 0$, which is clearly the case. Next we show that $\lim_{z \rightarrow \infty} h(z) = 0$. Indeed, by the algebraic limit theorem

$$\lim_{z \rightarrow \infty} \left(\cos\left(\pi - \frac{\pi}{4z}\right) + 1 - \frac{1}{z^2} \right) = \lim_{z \rightarrow \infty} \cos\left(\pi - \frac{\pi}{4z}\right) - \lim_{z \rightarrow \infty} \frac{1}{z^2} + 1 = 0.$$

79:12 Sum-Of-Squares Bounds via Boolean Function Analysis

It remains to prove the claim from Step 3. To do so, we use the characterization of Chebyshev polynomial given in [47, Equation 1.12]:

$$T_d(x) = \frac{1}{2} \left(\left(x - \sqrt{x^2 - 1} \right)^d + \left(\sqrt{x^2 - 1} + x \right)^d \right).$$

For $d = 2\lceil\sqrt{n}\rceil$ and $x = -1 - 1/n$ we get:

$$\begin{aligned} T_{2\lceil\sqrt{n}\rceil} \left(-1 - \frac{1}{n} \right) &\geq \frac{1}{2} \left(\left(-1 - \frac{1}{n} - \sqrt{\left(-1 - \frac{1}{n} \right)^2 - 1} \right)^{2\lceil\sqrt{n}\rceil} \right) \\ &\geq \frac{1}{2} \left(\left(-1 - \frac{\sqrt{2}}{\sqrt{n}} \right)^{2\sqrt{n}} \right) \end{aligned}$$

where $\frac{1}{2} \left(\left(-1 - \frac{\sqrt{2}}{\sqrt{n}} \right)^{2\sqrt{n}} \right)$ is increasing in n and for $n = 1$ takes value bigger than 2. ◀

Finally, we prove the SoS rank upper bound for the MK problem for $P = 2$.

► **Lemma 16.** *The SoS rank for the MK for $P = 2$ is at most $d = \lceil \frac{n+4\lceil\sqrt{n}\rceil}{2} \rceil$.*

Proof. We show that there exist a SoS certificate of degree $d = \lceil \frac{n+4\lceil\sqrt{n}\rceil}{2} \rceil$ for the function $\sum_{i=1}^m x_i - 1$. Consider the polynomial f of degree $2\lceil\sqrt{n}\rceil$ constructed in Lemma 15. Take $s_1 = f^2$ and note that $\sum_{i=1}^n x_i - 1 - 0.8f^2(x) (\sum_{i=1}^m x_i - 1/2) \geq 0$ for every $x \in \{0, 1\}^n$, since $(0.8 \cdot 2)^2 > 2$ and $0.8^2 < 2/3$. Applying Theorem 4 we get that there exists a SoS certificate for $\sum_{i=1}^n x_i - 1$ of degree $d = \lceil \frac{n+4\lceil\sqrt{n}\rceil}{2} \rceil$. ◀

6 Application to the SC problem

In the last section we prove the SoS rank lower bound for the SC problem.

► **Lemma 17.** *The SoS rank for the SC problem is at least $\Omega(\sqrt{n})$.*

Proof. By contradiction, assume there exists a SoS certificate of degree smaller than $\Omega(\sqrt{n})$ for the function $\sum_{i=1}^n x_i - 2$, namely: $\sum_{i=1}^n x_i - 2 = s_0(x) + \sum_{j=1}^n s_j(x) \left(\sum_{i \neq j}^n x_i - 1 \right)$.

We follow the idea in the proof of Lemma 14. Let $s(x) = \sum_{i=1}^n s_i(x)$ and take $k = \lfloor n/3 \rfloor$. For every $j \in [n]$ and $x \in \{0, 1\}^n$ such that $|x| \geq 3$ we know that $\sum_{i \neq j}^n x_i - 1 \geq \sum_{i=1}^n x_i - 2$.

Thus, note that for every $\ell \in [k]$ and for every $x \in \{0, 1\}^n$ such that $|x| = 3\ell$ we have:

$$s(0, \dots, 0) \geq 2 \quad \text{and} \quad s(x) \leq 1, \quad \text{for every } x \in \{0, 1\}^n, \text{ s.t. } |x| = 3\ell, \text{ for } \ell \in [k].$$

Now consider a symmetric function \bar{s} over k -dimensional boolean hypercube $\{0, 1\}^k$, $\bar{s} := \{0, 1\}^k \rightarrow \mathbb{R}$ that takes the values:

$$\bar{s}(z) := \binom{n}{3\ell}^{-1} \sum_{\substack{x \in \{0, 1\}^n \\ |x|=3\ell}} s(x), \quad \text{for every } z \in \{0, 1\}^k, |z| = \ell.$$

Clearly, $\deg(\bar{s})$ is at most $\deg(s)$ and note that it holds:

$$\bar{s}(0, \dots, 0) \geq 2 \quad \text{and} \quad \bar{s}(z) \leq 1 \quad \text{for every other } z \in \{0, 1\}^k$$

and the function $\bar{s}(z)/3$ is of degrees smaller than $\Omega(\sqrt{n})$ and approximates the NOR function in ℓ_∞ -norm within the error $1/3$, which contradicts Theorem 10. ◀

References

- 1 S. Arora, S. Rao, and U. V. Vazirani. Expander flows, geometric embeddings and graph partitioning. *J. ACM*, 56(2):5:1–5:37, 2009.
- 2 B. Barak, S. B. Hopkins, J. A. Kelner, P. Kothari, A. Moitra, and A. Potechin. A Nearly Tight Sum-of-Squares Lower Bound for the Planted Clique Problem. In *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA*, pages 428–437, 2016.
- 3 B. Barak, J. A. Kelner, and D. Steurer. Dictionary Learning and Tensor Decomposition via the Sum-of-Squares Method. In *STOC 2015, Portland, OR, USA, June 14-17, 2015*, pages 143–151, 2015.
- 4 B. Barak and A. Moitra. Noisy Tensor Completion via the Sum-of-Squares Hierarchy. In *COLT 2016, New York, USA, June 23-26, 2016*, pages 417–445, 2016.
- 5 B. Barak, P. Raghavendra, and D. Steurer. Rounding Semidefinite Programming Hierarchies via Global Correlation. In *FOCS*, pages 472–481, 2011.
- 6 B. Barak and D. Steurer. Proofs, beliefs, and algorithms through the lens of sum-of-squares, 2016. URL: <https://www.sumofsquares.org>.
- 7 M. Hossein Bateni, M. Charikar, and V. Guruswami. MaxMin allocation via degree lower-bounded arborescences. In *STOC*, pages 543–552, 2009.
- 8 D. Bienstock and M. Zuckerberg. Subset Algebra Lift Operators for 0-1 Integer Programming. *SIAM Journal on Optimization*, 15(1):63–95, 2004.
- 9 S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.
- 10 H. Chernof. A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the sum of Observations. *Ann. Math. Statist.*, 23(4):493–507, December 1952.
- 11 K. K. H. Cheung. Computation of the Lasserre Ranks of Some Polytopes. *Math. Oper. Res.*, 32(1):88–94, 2007.
- 12 E. Chlamtac. Approximation Algorithms Using Hierarchies of Semidefinite Programming Relaxations. In *FOCS*, pages 691–701, 2007.
- 13 E. Chlamtac and G. Singh. Improved Approximation Guarantees through Higher Levels of SDP Hierarchies. In *APPROX-RANDOM*, pages 49–62, 2008.
- 14 E. Chlamtac and M. Tulsiani. *Convex Relaxations and Integrality Gaps*, pages 139–169. Springer US, Boston, MA, 2012.
- 15 W. Cook and S. Dash. On the matrix-cut rank of polyhedra. *Math. Oper. Res.*, 26(1):19–30, 2001.
- 16 G. Cornuéjols and Y. Li. On the Rank of Mixed 0, 1 Polyhedra. In *Proceedings of the 8th International Conference on Integer Programming and Combinatorial Optimization, IPCO 2001, Utrecht, The Netherlands*, pages 71–77, 2001.
- 17 M. Cygan, F. Grandoni, and M. Mastrolilli. How to Sell Hyperedges: The Hypermatching Assignment Problem. In *SODA*, pages 342–351, 2013.
- 18 W. F. de la Vega and C. Kenyon-Mathieu. Linear programming relaxations of maxcut. In *SODA*, pages 53–61, 2007.
- 19 M. X. Goemans and L. Tunçel. When Does the Positive Semidefiniteness Constraint Help in Lifting Procedures? *Math. Oper. Res.*, 26(4):796–815, 2001.
- 20 M. X. Goemans and D. P. Williamson. Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, 1995.
- 21 D. Grigoriev. Complexity of Positivstellensatz proofs for the knapsack. *Comput. Complexity*, 10(2):139–154, 2001.
- 22 D. Grigoriev, E. A. Hirsch, and D. V. Pasechnik. Complexity of Semi-algebraic Proofs. In *STACS*, pages 419–430, 2002.
- 23 D. Grigoriev and N. Vorobjov. Complexity of Null-and Positivstellensatz proofs. *Ann. Pure App. Logic*, 113(1-3):153–160, 2001.

- 24 V. Guruswami and A. K. Sinop. Lasserre hierarchy, higher eigenvalues, and approximation schemes for graph partitioning and quadratic integer programming with psd objectives. In *FOCS*, pages 482–491, 2011.
- 25 S. B. Hopkins, T. Schramm, J. Shi, and D. Steurer. Fast spectral algorithms from sum-of-squares proofs: tensor decomposition and planted sparse vectors. In *STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 178–191, 2016.
- 26 P. Kothari, J. Steinhardt, and D. Steurer. Robust Moment Estimation and Improved Clustering via Sum of Squares. In *STOC 2018*, 2018.
- 27 P. K. Kothari, R. Mori, R. O’Donnell, and D. Witmer. Sum of squares lower bounds for refuting any CSP. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 132–145, 2017.
- 28 A. Kurpisz, S. Leppänen, and M. Mastrolilli. Sum-of-Squares Hierarchy Lower Bounds for Symmetric Formulations. In *Integer Programming and Combinatorial Optimization - 18th International Conference, IPCO 2016, Liège, Belgium, June 1-3, 2016, Proceedings*, pages 362–374, 2016.
- 29 A. Kurpisz, S. Leppänen, and M. Mastrolilli. Tight Sum-Of-Squares Lower Bounds for Binary Polynomial Optimization Problems. In *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, pages 78:1–78:14, 2016.
- 30 A. Kurpisz, S. Leppänen, and M. Mastrolilli. An unbounded Sum-of-Squares hierarchy integrality gap for a polynomially solvable problem. *Math. Program.*, 166(1-2):1–17, 2017.
- 31 A. Kurpisz, S. Leppänen, and M. Mastrolilli. On the Hardest Problem Formulations for the 0/1 Lasserre Hierarchy. *Math. Oper. Res.*, 42(1):135–143, 2017.
- 32 J. B. Lasserre. An Explicit Exact SDP Relaxation for Nonlinear 0-1 Programs. In *Integer Programming and Combinatorial Optimization, 8th International IPCO Conference, Utrecht, The Netherlands, June 13-15, 2001, Proceedings*, pages 293–303, 2001.
- 33 M. Laurent. A Comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre Relaxations for 0-1 Programming. *Math. Oper. Res.*, 28(3):470–496, 2003.
- 34 M. Laurent. Lower Bound for the Number of Iterations in Semidefinite Hierarchies for the Cut Polytope. *Math. Oper. Res.*, 28(4):871–883, 2003.
- 35 M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In *Emerging applications of algebraic geometry*, volume 149 of *IMA Vol. Math. Appl.*, pages 157–270. Springer, New York, 2009.
- 36 J. R. Lee, P. Raghavendra, and D. Steurer. Lower Bounds on the Size of Semidefinite Programming Relaxations. In *STOC*, pages 567–576, 2015.
- 37 T. Lee, A. Prakash, R. Wolf, and H. Yuen. On the Sum-of-Squares Degree of Symmetric Quadratic Functions. In *31st Conference on Computational Complexity, CCC 2016, May 29 to June 1, 2016, Tokyo, Japan*, pages 17:1–17:31, 2016.
- 38 A. Magen and M. Moharrami. Robust Algorithms for on Minor-Free Graphs Based on the Sherali-Adams Hierarchy. In *APPROX-RANDOM*, pages 258–271, 2009.
- 39 M. Mastrolilli. High Degree Sum of Squares Proofs, Bienstock-Zuckerberg Hierarchy and CG Cuts. In *Integer Programming and Combinatorial Optimization - 19th International Conference, IPCO 2017, Waterloo, ON, Canada, June 26-28, 2017, Proceedings*, pages 405–416, 2017.
- 40 Raghav Meka, Aaron Potechin, and Avi Wigderson. Sum-of-squares Lower Bounds for Planted Clique. In *Proceedings of the 47th Annual ACM Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, pages 87–96*, 2015.
- 41 Y. Nesterov. *Global quadratic optimization via conic relaxation*, pages 363–384. Kluwer Academic Publishers, 2000.
- 42 R. O’Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.
- 43 P. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.

- 44 R. Paturi. On the Degree of Polynomials that Approximate Symmetric Boolean Functions (Preliminary Version). In *Proceedings of the 24th Annual ACM Symposium on Theory of Computing, May 4-6, 1992, Victoria, British Columbia, Canada*, pages 468–474, 1992.
- 45 A. Potechin and D. Steurer. Exact tensor completion with sum-of-squares. In *COLT 2017, Amsterdam, The Netherlands, 7-10 July 2017*, pages 1619–1673, 2017.
- 46 P. Raghavendra and N. Tan. Approximating CSPs with global cardinality constraints using SDP hierarchies. In *SODA*, pages 373–387, 2012.
- 47 T. Rivlin. The chebyshev polynomials. *SERBIULA (sistema Librum 2.0)*, February 1974.
- 48 S. Sakaue, A. Takeda, S. Kim, and N. Ito. Exact Semidefinite Programming Relaxations with Truncated Moment Matrix for Binary Polynomial Optimization Problems. *SIAM Journal on Optimization*, 27(1):565–582, 2017.
- 49 T. Schramm and D. Steurer. Fast and robust tensor decomposition with applications to dictionary learning. In *COLT 2017, Amsterdam, The Netherlands, 7-10 July 2017*, pages 1760–1793, 2017.
- 50 Alexander A. Sherstov. Approximate Inclusion-Exclusion for Arbitrary Symmetric Functions. *Computational Complexity*, 18(2):219–247, 2009. doi:10.1007/s00037-009-0274-4.
- 51 N. Shor. Class of global minimum bounds of polynomial functions. *Cybernetics*, 23(6):731–734, 1987.
- 52 J. Thapper and S. Zivny. The Power of Sherali-Adams Relaxations for General-Valued CSPs. *SIAM J. Comput.*, 46(4):1241–1279, 2017.
- 53 R. Wolf. A note on quantum algorithms and the minimal degree of ϵ -error polynomials for symmetric functions. *Quantum Information & Computation*, 8(10):943–950, 2010.