

Approximate Unitary $n^{2/3}$ -Designs Give Rise to Quantum Channels with Super Additive Classical Holevo Capacity

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Abstract

In a breakthrough, Hastings [10] showed that there exist quantum channels whose classical Holevo capacity is superadditive i.e. more classical information can be transmitted by quantum encoding strategies entangled across multiple channel uses as compared to unentangled quantum encoding strategies. Hastings' proof used Haar random unitaries to exhibit superadditivity. In this paper we show that a unitary chosen uniformly at random from an approximate $n^{2/3}$ -design gives rise to a quantum channel with superadditive classical Holevo capacity, where n is the dimension of the unitary exhibiting the Stinespring dilation of the channel superoperator. We do so by showing that the minimum output von Neumann entropy of a quantum channel arising from an approximate unitary $n^{2/3}$ -design is subadditive, which by Shor's work [26] implies superadditivity of classical Holevo capacity of quantum channels.

We follow the geometric functional analytic approach of Aubrun, Szarek and Werner [3] in order to prove our result. More precisely we prove a sharp Dvoretzky-like theorem stating that, with high probability under the choice of a unitary from an approximate t -design, random subspaces of large dimension make a Lipschitz function take almost constant value. Such theorems were known earlier only for Haar random unitaries. We obtain our result by appealing to Low's technique [16] for proving concentration of measure for an approximate t -design, combined with a stratified analysis of the variational behaviour of Lipschitz functions on the unit sphere in high dimension. The stratified analysis is the main technical advance of this work.

Haar random unitaries require at least $\Omega(n^2)$ random bits in order to describe them with good precision. In contrast, there exist exact $n^{2/3}$ -designs using only $O(n^{2/3} \log n)$ random bits [15]. Thus, our work can be viewed as a partial derandomisation of Hastings' result, and a step towards the quest of finding an explicit quantum channel with superadditive classical Holevo capacity.

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1 Introduction

For the past two decades, additivity conjectures have been extensively studied in quantum information theory e.g. [5, 22, 1, 20, 26, 11]. In this paper, we concentrate on the issue of additivity of classical Holevo capacity of a quantum channel Φ , denoted henceforth by $C(\Phi)$.



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The quantity $C(\Phi)$ is the number of classical bits of information per channel use that can reliably be transmitted in the limit of infinitely many independent uses of Φ . Capacities of classical memoryless channels are known to be additive, that is, the capacity of two channels Φ and Ψ , used independently, is the sum of the individual capacities. In other words, $C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi)$. This additivity property leads to a single letter characterization of the capacity of classical channels viz. the capacity is nothing but the mutual information between the input and channel output maximised over all possible input distributions for one channel use [27]. For a long time, in analogy with the classical setting, it was generally believed that the classical Holevo capacity of a quantum channel is additive. In fact, this belief was proven to be true for several classes of quantum channels e.g. [13, 8, 12, 25, 14]. Thus, it came as a major surprise to the community when Hastings, in a major breakthrough, showed that there are indeed quantum channels with superadditive classical Holevo capacity [10] i.e. there are quantum channels Φ, Ψ such that $C(\Phi \otimes \Psi) > C(\Phi) + C(\Psi)$.

Hastings' proof proceeds by showing that a Haar random unitary leads to such channels with high probability, in the sense that the unitary, when viewed suitably, is the Stinespring dilation of a quantum channel with superadditive classical Holevo capacity. The drawback of using Haar random unitaries is that they are inefficient to implement. In fact, it takes at least $\Omega(n^2 \log(1/\epsilon))$ random bits in order to pick an $n \times n$ Haar random unitary to within a precision of ϵ in the ℓ_2 -distance [28]. Hence, it is of considerable interest to find an explicit efficiently implementable unitary that gives rise to a quantum channel with superadditive classical Holevo capacity.

In this paper, we take the first step in this direction. We show that with high probability a uniformly random $n \times n$ unitary from an approximate $n^{2/3}$ -design leads to a quantum channel with superadditive classical Holevo capacity. Though no efficient algorithms for implementing approximate $n^{2/3}$ -designs are known, nevertheless, it is known that a uniformly random unitary from an exact $n^{2/3}$ -design can be sampled using only $O(n^{2/3} \log n)$ random bits [15, Theorem 3.3]. Also, efficient constructions of approximate $(\log n)^{O(1)}$ -designs are known [24, 6]. Thus, our work can be viewed as a partial derandomisation of Hastings' result, and a step towards the quest of finding an explicit quantum channel with superadditive classical Holevo capacity.

Hastings' proof was considerably simplified by Aubrun, Szarek and Werner [3] who showed that existence of channels with subadditive minimum output von Neumann entropy follows from a sharp Dvoretzky-like theorem which states that, under the Haar measure, random subspaces of large dimension make a Lipschitz function take almost constant value. Dvoretzky's original theorem [7] stated that any centrally symmetric convex body can be embedded with low distortion into a section of a high dimensional unit ℓ_2 -sphere. Milman [17] extended Dvoretzky's theorem by proving that, with high probability, Haar random subspaces of an appropriate dimension make a Lipschitz function take almost constant value. Dvoretzky's theorem becomes the special case of Milman's theorem where the Lipschitz function happens to be norm induced by the centrally symmetric convex body i.e. the norm under which the convex body becomes the unit ball. Milman's work started a whole body of research sharpening the various parameters of the extended Dvoretzky theorem e.g. [23, 9] etc. However, all these works use Haar random subspaces. A Haar random subspace of \mathbb{C}^n of dimension d can be obtained by applying a Haar random unitary to a fixed subspace of dimension d e.g. the subspace spanned by the first d standard basis vectors of \mathbb{C}^n . Our work is the first one to replace the Haar random unitary in any Dvoretzky-type theorem by a uniformly random unitary chosen from an approximate t -design for a suitable value of t . In other words, our main technical result is an Aubrun-Szarek-Werner style

result for approximate t -designs instead of Haar random unitaries. As a corollary, we obtain the subadditivity of minimum output von Neumann entropy for unitaries chosen from an approximate $n^{2/3}$ -design. As another corollary, we obtain the subadditivity of minimum output Rényi p -entropy for all $p > 1$ for quantum channels arising from unitaries chosen from approximate unitary $p^3 n^{\frac{4}{9p} + \frac{5}{9} + \frac{2p}{3}} \log n$ -design. Such a unitary can in fact be chosen from an exact $p^3 n^{\frac{4}{9p} + \frac{5}{9} + \frac{2p}{3}} \log n$ -design using only $p^3 n^{\frac{4}{9p} + \frac{5}{9} + \frac{2p}{3}} (\log n)^2$ random bits [15], which is much less than $\Omega(n^2)$ random bits required to choose a Haar random unitary. Subadditivity of minimum output Rényi p -entropy for all $p > 1$ was originally proved for Haar random unitaries by Hayden and Winter [11].

To prove our main technical result, we use a concentration of measure result by Low [16] for approximate unitary t -designs, combined with a stratified analysis of the variational behaviour of Lipschitz functions on the unit sphere in high dimension. We need such a fine grained stratified analysis for the following reason. Aubrun, Szarek and Werner [3] worked with the function $f(M) := \|MM^\dagger - (I/k)\|_2$, where the argument M is a k^3 -tuple rearranged to form a $k \times k^2$ matrix. They found subspaces of dimension k^2 where f took almost constant value. For this, they had to do a two step analysis. The global Lipschitz constant of f was 2 which, under naive Dvoretzky type arguments, would only guarantee the existence of subspaces of dimension $\frac{k^2}{\log k}$ where f is almost constant. This does not suffice to find a counter example to minimum output von Neumann entropy. In order to shave off the $\log k$ term in the denominator, they had to use several sophisticated arguments. One of them was the observation that there is a high probability subset T of $\mathbb{S}_{\mathbb{C}^{k^3}}$ on which the Lipschitz constant of f was $k^{-1/2}$. They exploited this by their two step analysis, where they separately analysed the behaviour of f on T and on T^c , and managed to shave off the $\log k$ term. For us, since we are working with designs, we need the function to be a polynomial. Hence, instead of f , we have to work with f^2 . This seemingly trivial change introduces severe technical difficulties. The main reason behind them is that the Lipschitz constant of f^2 is about twice the Lipschitz constant for f but the variation that we are looking to bound is around square of the earlier variation! This contradiction lies at the heart of the technical difficulty. In order to overcome this, we have to partition $\mathbb{S}_{\mathbb{C}^{k^3}}$ into a number of sets $\Omega_1, \Omega_2, \dots, \Omega_{\log k}$, called ‘layers’, with local Lipschitz constants for f^2 running as $k^{-3/2}, 2^3 k^{-3/2}, 3^3 k^{-3/2}, \dots, (\log k)^3 k^{-3/2}$. We have to bound the variation of f^2 individually on Ω_i as well as put them together to bound the variation on large subspheres of $\mathbb{S}_{\mathbb{C}^{k^3}}$. This leads to a challenging stratified analysis, which forms the main technical advance of this paper.

Another tool developed in this work which should find use in other situations also, is a systematic way to approximate a monotonic differentiable function and its derivative using moderate degree polynomials. This tool is crucially used to prove strict subadditivity of Rényi p -entropy for any $p > 1$ for channels whose unitary Stinespring dilation is chosen from an approximate design instead of a Haar random unitary.

The power of our stratified analysis shows up in the consequence that the dimension of the subspace on which the Lipschitz function is almost constant depends only on the smallest local Lipschitz constant, provided some mild niceness conditions are satisfied. This gives larger dimensional subspaces than a naive analysis which would depend on the global Lipschitz constant. In fact, the stratified analysis allows us to prove a sharper Dvoretzky-type theorem even for the Haar measure. As a result, we can recover Aubrun, Szarek and Werner’s result for the function f directly and elegantly instead of applying their Dvoretzky-type result twice which is rather messy. Another powerful consequence of our stratified analysis is that with probability exponentially close to one random, via Haar or t -design, subspaces

make the Lipschitz function almost constant. In contrast, Aubrun, Szarek and Werner could only guarantee constant probability close to one.

The rest of the paper is organised as follows. Section 2 contains notations, symbols definitions and preliminary tools required for the paper. Section 3 states and proves the main technical theorems viz. the stratified analyses for Haar measure and approximate t -designs. Section 4 describes the application to subadditivity of minimum output von Neumann entropy. Section B describes the application to subadditivity of minimum output Rényi p -entropy for $p > 1$. The detailed proofs of all the lemmas and propositions can be found in the full version [19].

2 Preliminaries

All Hilbert spaces used in this paper are finite dimensional. The n dimensional space over complex numbers, \mathbb{C}^n , is endowed with the standard inner product aka the dot product: $\langle x, y \rangle := \sum_{i=1}^n x_i^* y_i$. The unit radius sphere in \mathbb{C}^n is denoted by $\mathbb{S}_{\mathbb{C}^n}$. The symbol $\mathcal{M}_{k,d}$ denotes the Hilbert space of $k \times d$ linear operators over the complex field under the Hilbert-Schmidt inner product $\langle M, N \rangle := \text{Tr}[M^\dagger N]$, and $\mathcal{M}_d := \mathcal{M}_{d,d}$. Let $\mathcal{U}(n)$ denote the set of $n \times n$ unitary matrices with complex entries. For a composite Hilbert space $\mathbb{C}^k \otimes \mathbb{C}^d$, the notation $\text{Tr}_{\mathbb{C}^d}[\cdot]$ denotes the operation of taking partial trace i.e. tracing out the mentioned subsystem \mathbb{C}^d . We use $\text{Tr}[\cdot]$ to denote the trace of the underlying operator. Fix standard bases for Hilbert spaces $A \cong \mathbb{C}^k$, $B \cong \mathbb{C}^d$. Let $|e_i\rangle^A$, $|e_i\rangle^B$ denote standard basis vectors of A , B respectively. Any vector $x \in A \otimes B$ can be written as $x = \sum_{i,j} \alpha_{ij} |e_i\rangle^A \otimes |e_j\rangle^B$. We use $\text{op}_{d \rightarrow k}(x)$ to denote the operator $\sum_{i,j} \alpha_{ij} |e_i\rangle^A \langle e_j|^B$ in $\mathcal{M}_{k,d}$. Conversely, given an operator $M = \sum_{ij} m_{ij} |e_i\rangle^A \langle e_j|^B$ in $\mathcal{M}_{k,d}$, we let $\text{vec}(M) := \sum_{ij} m_{ij} |e_i\rangle^A \otimes |e_j\rangle^B$ denote the vector in $\mathbb{C}^k \otimes \mathbb{C}^d$.

For Hermitian positive semidefinite operators M , we define M^α for any $\alpha > 0$ to be the unique Hermitian operator obtained by keeping the eigenbasis same and taking the α th power of the eigenvalues. We can define $\log M$ similarly. For $p > 1$, the notation $\|M\|_p$ denotes the Schatten p -norm of the matrix M , which is nothing but the ℓ_p -norm of the vector of its singular values. Alternatively, $\|M\|_p = (\text{Tr}[(M^\dagger M)^{p/2}])^{1/p}$. Then $p = 2$ gives the Hilbert Schmidt norm aka the Frobenius norm which is nothing but $\|M\|_2 = \|\text{vec}(M)\|_2$. Also, $p = \infty$ gives the operator norm aka spectral norm which is nothing but $\|M\|_\infty = \max_{v: \|v\|_2=1} \|Mv\|_2$.

Unless stated otherwise, the symbol ρ denotes a quantum state aka density matrix which is nothing but a Hermitian, positive semidefinite matrix with unit trace. A rank one density matrix is called a pure state. By the spectral theorem, any density matrix is a convex combination of pure states. The notation $\mathcal{D}(\mathbb{C}^d)$ denotes the convex set of all $d \times d$ density matrices. We use $|\cdot\rangle$ to denote a unit vector. By a slight abuse of notation, we shall often use a unit vector $|\psi\rangle$ to denote a pure state $|\psi\rangle\langle\psi|$. A linear mapping $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_d$ is called a superoperator. A superoperator is trace preserving if $\text{Tr} \Phi(M) = \text{Tr} M$ for all $M \in \mathcal{M}_m$. It is said to be positive if $\Phi(M)$ is positive semidefinite for all positive semidefinite M . Furthermore, Φ is said to be completely positive if $\Phi \otimes \mathbb{I}$ is a positive superoperator for identity superoperators \mathbb{I} of all dimensions. Completely positive and trace preserving (CPTP) superoperators are referred to as quantum channels. Unless stated otherwise, Φ , Ψ are used to denote quantum channels.

A compact convex set \mathcal{S} in \mathbb{C}^n is called a convex body. The radius $r(\mathcal{S})$ of a convex body \mathcal{S} is defined as

$$r(\mathcal{S}) := \min_{x \in \mathcal{S}} \max_{y \in \mathcal{S}} \|x - y\|_2.$$

Any point $x \in \mathcal{S}$ achieving the minimum above is said to be a centre of \mathcal{S} . The convex body \mathcal{S} is said to be centrally symmetric iff for every $x \in \mathbb{C}^n$, $x \in \mathcal{S} \leftrightarrow -x \in \mathcal{S}$. The zero vector is a centre of a centrally symmetric convex body. A centrally symmetric convex body lying in \mathbb{C}^n can be thought of as the unit sphere of a suitable notion of norm in \mathbb{C}^n . Conversely for any norm in \mathbb{C}^n , the unit sphere under the norm forms a centrally symmetric convex body.

2.1 Entropies and norms

► **Definition 1.** *The von Neumann entropy of a quantum state ρ is defined as*

$$S(\rho) := -\text{Tr}[\rho \log \rho].$$

For all $p > 1$, the Rényi p -entropy of a quantum state ρ is defined as

$$S_p(\rho) := \frac{1}{1-p} \log \text{Tr} \rho^p = -\frac{p}{p-1} \log \|\rho\|_p.$$

It turns out that $S(\rho) = \lim_{p \downarrow 1} S_p(\rho) =: S_1(\rho)$.

Also, it can be shown that for $p \geq 1$, $S_p(\cdot)$ is concave in its argument.

► **Definition 2.** *For $p \geq 1$, the minimum output Rényi p -entropy of a quantum channel Φ is defined as :*

$$S_p^{\min}(\Phi) := \min_{\rho \in \mathcal{D}(\mathbb{C}^m)} S_p(\Phi(\rho))$$

By an easy concavity argument it can be seen that above minimum is achieved on a pure state. Equivalently, to obtain $S_p^{\min}(\Phi)$ for $p > 1$ we must maximise $\|\Phi(\rho)\|_p$ for all input states ρ . This quantity is also known as the $1 \rightarrow p$ superoperator norm of superoperator $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_d$:

$$\|\Phi\|_{1 \rightarrow p} := \max_{M \in \mathcal{M}_m: \|M\|_1=1} \|\Phi(M)\|_p.$$

By an easy convexity argument it can be seen that the above maximum is achieved on a pure state i.e.

$$\|\Phi\|_{1 \rightarrow p} = \max_{x \in \mathbb{C}^m: \|x\|_2=1} \|x\rangle\langle x\|_p.$$

Thus, the additivity conjecture for minimal output p -Rényi p -entropy, $p > 1$, for quantum channels Φ and Ψ is equivalent to multiplicativity of $1 \rightarrow p$ -norms of quantum channels viz. $\|\Phi \otimes \Psi\|_{1 \rightarrow p} \stackrel{?}{=} \|\Phi\|_{1 \rightarrow p} \cdot \|\Psi\|_{1 \rightarrow p}$. This equivalence will be used in Section B to give a counter example to additivity conjecture for all $p > 1$ where the Stinespring dilation of the quantum channel will be described from a unitary chosen uniformly at random from an approximate t -design. The equivalent result for Haar random unitaries was originally proved by Hayden and Winter [11].

We heavily use the one-one correspondence between quantum channels and subspaces of composite Hilbert spaces, originally proved by Aubrun, Szarek and Werner [4], in this paper. Let \mathcal{W} be a subspace of $\mathbb{C}^k \otimes \mathbb{C}^d$ of dimension m . Identify \mathcal{W} with \mathbb{C}^m through an isometry $V : \mathbb{C}^m \rightarrow \mathbb{C}^k \otimes \mathbb{C}^d$ whose range is \mathcal{W} . Then, the corresponding quantum channel $\Phi_{\mathcal{W}} : \mathcal{M}_m \rightarrow \mathcal{M}_k$ is defined by $\Phi_{\mathcal{W}}(\rho) := \text{Tr}_{\mathbb{C}^d}(V\rho V^\dagger)$. Using this equivalence and the fact that for $p > 1$ the $1 \rightarrow p$ -superoperator norm is achieved on pure input states, we can write [4]

$$\|\Phi_{\mathcal{W}}\|_{1 \rightarrow p} = \max_{x \in \mathcal{W}: \|x\|_2=1} \|\text{Tr}_{\mathbb{C}^d} |x\rangle\langle x|\|_p = \max_{x \in \mathcal{W}: \|x\|_2=1} \|\text{op}_{d \rightarrow k}(x)\|_{2p}^2. \quad (1)$$

In an important paper, Shor [26] proved that several additivity conjectures for quantum channels were in fact equivalent to the additivity of minimum output von Neumann entropy of a quantum channel. More specifically, Shor showed that if there is a quantum channel Φ whose minimum output von Neumann entropy is subadditive, then there are quantum channels Ψ_1, Ψ_2 exhibiting superadditive classical Holevo capacity viz. $C(\Psi_1 \otimes \Psi_2) > C(\Psi_1) + C(\Psi_2)$. This equivalence was used as a starting point by Hastings [10] in his proof that there are channels with superadditive classical Holevo capacity. Aubrun, Szarek and Werner [3], as well as this paper also have the same starting point. For this, we need the following fact.

► **Fact 3** ([3, Lemma 2]). *Let a quantum channel $\Phi_{\mathcal{W}} : \mathcal{M}_m \rightarrow \mathcal{M}_k$ be described by a subspace $\mathcal{W} \leq \mathbb{C}^k \otimes \mathbb{C}^d$ of dimension m . Then,*

$$\begin{aligned} S_{\min}(\Phi_{\mathcal{W}}) &= \log k - k \cdot \max_{\rho \in \mathcal{D}(\mathbb{C}^m)} \|\Phi(\rho) - \frac{\mathbb{1}}{k}\|_2^2 \\ &= \log k - k \cdot \max_{x \in \mathcal{W}: \|x\|_2=1} \|(\text{op}_{d \rightarrow k}(x))(\text{op}_{d \rightarrow k}(x))^\dagger - \frac{\mathbb{1}}{k}\|_2^2. \end{aligned}$$

We will need the following result proved by Hayden and Winter [11] that upper bounds $S_p^{\min}(\Phi \otimes \bar{\Phi})$ where $\bar{\Phi}$ denotes the CPTP superoperator obtained by taking complex conjugate of the CPTP superoperator Φ .

► **Fact 4.** *Let $V : \mathbb{C}^m \rightarrow \mathbb{C}^k \otimes \mathbb{C}^d$ be an isometry describing the quantum channel $\Phi : \rho \mapsto \text{Tr}_{\mathbb{C}^d}[V\rho V^\dagger]$. Let $|\phi\rangle$ denote the maximally entangled state in $\mathbb{C}^m \otimes \mathbb{C}^m$. Suppose $m \leq d$. Then $(\Phi \otimes \bar{\Phi})(|\phi\rangle\langle\phi|)$ has a singular value not less than $\frac{m}{kd}$. Hence for all $p > 1$,*

$$\|\Phi \otimes \bar{\Phi}\|_{1 \rightarrow p} \geq \|\Phi \otimes \bar{\Phi}\|_{1 \rightarrow \infty} \geq \frac{m}{kd}.$$

Moreover,

$$S_{\min}(\Phi \otimes \bar{\Phi}) \leq 2 \log k - \frac{m}{kd} \log k + O\left(\frac{m}{kd} \log \frac{d}{m} + \frac{1}{k}\right).$$

2.2 Concentration results for Lipschitz functions

We now state some basic definitions and facts from geometric functional analysis that will be used in the proof of our main result.

► **Definition 5.** *A function $f : X \rightarrow \mathbb{C}$ defined over a metric space X is said to be L -Lipschitz if $\forall x, y \in X$ it satisfies the following inequality:*

$$|f(x) - f(y)| \leq L \cdot d(x, y).$$

► **Definition 6.** *Let X be a compact metric space. An ϵ -net \mathcal{N} of X is a finite set of points such that for any point $x \in X$, there is a point $x' \in \mathcal{N}$ such that $d(x, x') \leq \epsilon$.*

Note that compactness guarantees that finite sized ϵ -nets exist for all $\epsilon > 0$.

We will need the following definition and fact from [3].

► **Definition 7.** *A function $f : X \rightarrow \mathbb{C}$ defined over a normed linear space X is said to be circled if $f(e^{i\theta}x) = f(x)$ for all $\theta \in \mathbb{R}$ and $x \in X$.*

► **Fact 8.** *Let $f : X \rightarrow \mathbb{R}$ be a function defined on a metric space X . Suppose there exists a subset $Y \subseteq X$ such that f restricted to Y is L -Lipschitz. Then there is a function $\hat{f} : X \rightarrow \mathbb{R}$ that is L -Lipschitz on all of X satisfying $\hat{f}(y) = f(y)$ for all $y \in Y$. If X is a normed linear space over real or complex numbers and f is circled then the extension \hat{f} is also circled.*

Proof Sketch. Define $\hat{f}(x) := \inf_{y \in Y} [f(y) + Ld(x, y)]$. ◀

In this paper, we endow \mathbb{C}^n with the ℓ_2 -metric and $\mathbb{U}(n)$ with the Schatten ℓ_2 -metric aka Frobenius metric. The following fact gives a reasonably tight upper bound on the size of an ϵ -net of $\mathbb{S}_{\mathbb{C}^n}$.

► **Fact 9** ([28, Corollary 4.2.13]). *Let $\epsilon > 0$. There exists an ϵ -net of $\mathbb{S}_{\mathbb{C}^n}$ of size less than $(\frac{3}{\epsilon})^{2n}$.*

A fundamental result about concentration of Lipschitz functions defined on the unit sphere or the unitary group, known as Levy's lemma, lies at the heart of all proofs of Dvoretzky-type theorems via the probabilistic method. We now state the version of Levy's lemma that will be used in this paper.

► **Fact 10** (Levy's lemma, [2, Corollary 4.4.28]). *Consider the Haar probability measure on $\mathbb{S}_{\mathbb{C}^n}$. Let $f : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{C}$ be an L -Lipshitz function. Let $\mu := \mathbb{E}_x[f(x)]$ and $\lambda > 0$. Then*

$$\Pr_x(|f(x) - \mu| \geq \lambda) \leq 2 \exp(-\frac{n\lambda^2}{4L^2}).$$

An elementary proof of the above fact, without explicitly calculated constants, can be found in [28, Theorem 5.1.4].

For our work, we need a measure concentration inequality like Levy's lemma for difference of function values on two distinct arbitrary points which is sensitive to the distance between those points. Such an inequality is stated in the following fact.

► **Fact 11** ([3, Lemma 9]). *Let $f : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{C}$ be a circled L -Lipshitz function. Consider the Haar probability measure on $\mathbb{U}(n)$. Then for any $x, y \in \mathbb{S}_{\mathbb{C}^n}$, $x \neq y$ and for any $\lambda > 0$,*

$$\Pr_U[|f(Ux) - f(Uy)| > \lambda] \leq 2 \exp(-\frac{\lambda^2 n}{8L^2 \|x - y\|_2^2}).$$

The derandomisation in our paper is carried out by replacing the Stinespring dilation unitary of a quantum channel, which is chosen from the Haar measure in [3], with a unitary chosen uniformly at random from a finite cardinality approximate unitary t -design for a suitable value of t . The next few statements lead us to the definition of an approximate unitary t -design.

► **Definition 12** ([16, Definition 2.2]). *A monomial in the entries of a matrix U is of degree (r, s) if it contains r conjugated elements and s unconjugated elements. The evaluation of monomial M at the entries of a matrix U is denoted by $M(U)$. We call a monomial balanced if $r = s$, and say that it has degree t if it is of degree (t, t) . A polynomial is said to be balanced of degree t if it is a sum of balanced monomials of degree at most t .*

► **Definition 13** ([16, Definition 2.3]). *A probability distribution ν supported on a finite set of $d \times d$ unitary matrices is said to be an exact unitary t -design if for all balanced monomials M of degree at most t , $\mathbb{E}_{U \sim \nu}[M(U)] = \mathbb{E}_{U \sim \text{Haar}}[M(U)]$.*

► **Definition 14** ([16, Definition 2.6]). *A probability distribution ν supported on a finite set of $d \times d$ unitary matrices is said to be an ϵ -approximate unitary t -design if for all balanced monomials M of degree at most t*

$$|\mathbb{E}_{U \sim \nu}(M(U)) - \mathbb{E}_{U \sim \text{Haar}}(M(U))| \leq \frac{\epsilon}{d^t}.$$

We will need the following fact.

► **Fact 15** ([16, Lemma 3.4]). *Let $Y : \mathbb{U}(n) \rightarrow \mathbb{C}$ be a balanced polynomial of degree a in the entries of the unitary matrix U that is provided as input. Let $\alpha(Y)$ denote the sum of absolute values of the coefficients of Y . Let r, t be positive integers satisfying $2ar < t$. Let ν be an ϵ -approximate unitary t -design. Then*

$$\mathbb{E}_{U \sim \nu}[|Y_U|^{2r}] \leq \mathbb{E}_{U \sim \text{Haar}}[|Y_U|^{2r}] + \frac{\epsilon \alpha(Y)^{2r}}{n^t}.$$

3 Sharp Dvoretzky-like theorems via stratified analysis

In this section, we prove our main technical results viz. sharp Dvoretzky-like theorems for Haar measure as well as approximate t -designs using stratified analysis. We start by stating the following three lemmas, with their proofs in Appendix A which are ‘baby stratified’ analogues of Fact 11 for Haar measure and approximate unitary t -designs.

► **Lemma 16.** *Let $Y : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ be a circled function with global Lipschitz constant L_1 . Suppose that there exists a subset $\Omega \subseteq \mathbb{S}_{\mathbb{C}^n}$ such that Y restricted to Ω has a smaller Lipschitz constant L_2 . Let $x, y \in \mathbb{S}_{\mathbb{C}^n}$. Let $Y_x := Y(Ux)$, $Y_y := Y(Uy)$ be two correlated random variables, under the choice of a Haar random unitary U . Let $\lambda > 0$. Then*

$$\Pr_{U \sim \text{Haar}}[|Y_x - Y_y| > \lambda] \leq 2 \exp\left(-\frac{n\lambda^2}{8L_2^2\|x - y\|_2^2}\right) + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c].$$

► **Lemma 17.** *Let $Y : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ be a circled function with global Lipschitz constant L_1 . Suppose that there exists a subset $\Omega \subseteq \mathbb{S}_{\mathbb{C}^n}$ such that Y restricted to Ω has a smaller Lipschitz constant L_2 . Let $Z : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ be a balanced polynomial of degree a in entries of the vector $x \in \mathbb{C}^n$ that is provided as input. Let $\delta > 0$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Suppose that $|Z(x) - f(Y(x))| < \delta$ for all $x \in \mathbb{S}_{\mathbb{C}^n}$. Let $x, y \in \mathbb{S}_{\mathbb{C}^n}$. Let $Z_x := Z(Ux)$, $Z_y := Z(Uy)$ be two correlated random variables, under the choice of a unitary U chosen uniformly at random from an ϵ -approximate unitary t -design ν . Let r be a positive integer satisfying $2ar \leq t$. Let $0 < \epsilon < \frac{n^{t-r}(4rL_2^2\|x-y\|_2^2)^r}{\alpha(Z)^{2r}}$. Then*

$$\mathbb{E}_{U \sim \nu}[|Z_x - Z_y|^{2r}] \leq 3 \left(\frac{4rL_2^2\|x - y\|_2^2}{n}\right)^r + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c] \cdot (L_1^2\|x - y\|_2^2)^r.$$

► **Lemma 18.** *Let $Y : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ be a balanced polynomial of degree a in entries of the vector $x \in \mathbb{C}^n$ that is provided as input. Let $\alpha(Y)$ denote the sum of absolute values of the coefficients of Y . Suppose Y has global Lipschitz constant L_1 . Suppose that there exists a subset $\Omega \subseteq \mathbb{S}_{\mathbb{C}^n}$ such that Y restricted to Ω has a smaller Lipschitz constant L_2 . Let $x, y \in \mathbb{S}_{\mathbb{C}^n}$. Let $Y_x := Y(Ux)$, $Y_y := Y(Uy)$ be two correlated random variables, under the choice of a unitary U chosen uniformly at random from an ϵ -approximate unitary t -design ν . Let r be a positive integer satisfying $2ar \leq t$. Let $0 < \epsilon < \frac{n^{t-r}(4rL_2^2\|x-y\|_2^2)^r}{\alpha(Y)^{2r}}$. Then*

$$\mathbb{E}_{U \sim \nu}[|Y_x - Y_y|^{2r}] \leq 3 \left(\frac{4rL_2^2\|x - y\|_2^2}{n}\right)^r + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c] \cdot (L_1^2\|x - y\|_2^2)^r.$$

We also need a so-called *chaining inequality* for probability similar to Dudley’s inequality in geometric functional analysis [3, 21]. The original Dudley’s inequality bounds the expectation of the supremum, over pairs of correlated random variables, of the difference between them in terms of an integral, over η , of a certain function of the size of an η -net of $\mathbb{S}_{\mathbb{C}^n}$. Our

chaining lemma differs from it in two important respects. First, instead of the expectation it bounds a tail probability of the supremum, over pairs of correlated random variables, of the difference between them. Second, it replaces the integral by a finite summation over η -nets of $\mathbb{S}_{\mathbb{C}^n}$ with geometrically decreasing η . Despite the fancy name, our chaining lemma is a simple consequence of the union bound of probabilities. Nevertheless, it is crucial to proving our main result as it allows us to efficiently invoke powerful measure concentration results in order to bound the variation of a Lipschitz function on subspaces of \mathbb{C}^n .

► **Lemma 19** (Chaining). *Let $\{X_s\}_{s \in \mathcal{S}}$ be a family of correlated complex valued random variables indexed by elements of a compact metric space \mathcal{S} . Let $\lambda, L_1 > 0$. The family is said to be L_1 -Lipschitz if for all $s, t \in \mathcal{S}$, $|X_s - X_t| \leq L_1 d(s, t)$ for all points of the sample space. Define i_0 to be the unique integer such that the radius of \mathcal{S} lies in the interval $(2^{-i_0-1}, 2^{-i_0}]$. Define $i_1 := \max\{i_0, \lceil \log \frac{2L_1}{\lambda} \rceil\}$. Let $p: \mathbb{Z} \rightarrow \mathbb{R}_+$ be a non-decreasing function. Suppose the infinite series $\sum_{i > i_0} \frac{\sqrt{|i|p(i)}}{2^i}$ is convergent with value C . Then,*

$$\Pr\left[\sup_{s, t \in \mathcal{S}} |X_s - X_t| > \lambda\right] \leq \sum_{i=i_0+1}^{i_1+1} \sum_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i: d(u, u') < 2^{-i+2}} \Pr\left[|X_u - X_{u'}| > \frac{\lambda \sqrt{|i|p(i)}}{4C \cdot 2^i}\right],$$

for a sequence of 2^{-i} -nets \mathcal{N}_i , $i_0 \leq i \leq i_1$, $|\mathcal{N}_{i_0}| = 1$, of \mathcal{S} .

Proof. For every $i \in \mathbb{Z}$, let \mathcal{N}_i be a 2^{-i} -net of \mathcal{S} . Let i_0 be such that radius of \mathcal{S} lies in $(2^{-(i_0+1)}, 2^{-i_0}]$. The net \mathcal{N}_{i_0} consists of a single element, say s_0 . For every $s \in \mathcal{S}$ and $i \in \mathbb{Z}$, let $\pi_i(s)$ be an element of \mathcal{N}_i satisfying $d(s, \pi_i(s)) \leq 2^{-i}$. We have the following chaining equation for every $s \in \mathcal{S}$:

$$X_s = X_{s_0} + \left(\sum_{i=i_0}^{i_1} (X_{\pi_{i+1}(s)} - X_{\pi_i(s)}) \right) + (X_s - X_{\pi_{i_1+1}(s)}).$$

Lipschitz property of the family implies that

$$\begin{aligned} \sup_{s, t \in \mathcal{S}} |X_s - X_t| &\leq 2 \sum_{i=i_0}^{i_1} \sup_{s \in \mathcal{S}} |X_{\pi_{i+1}(s)} - X_{\pi_i(s)}| + L_1 2^{-i_1} \\ &\leq 2 \sum_{i=i_0}^{i_1} \sup_{(u, u') \in \mathcal{N}_i \times \mathcal{N}_{i+1}: d(u, u') < 2^{-i+1}} |X_u - X_{u'}| + L_1 2^{-i_1} \\ &\leq 2 \sum_{i=i_0+1}^{i_1+1} \sup_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i: d(u, u') < 2^{-i+2}} |X_u - X_{u'}| + \frac{\lambda}{2}. \end{aligned}$$

Now if $\sup_{s, t \in \mathcal{S}} |X_s - X_t| > \lambda$, there must exist an i , $i_0 + 1 \leq i \leq i_1 + 1$ such that

$$\sup_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i: d(u, u') < 2^{-i+2}} |X_u - X_{u'}| > \frac{\lambda \sqrt{|i|p(i)}}{4C \cdot 2^i}.$$

Applying the union bound on probability leads us to the conclusion of the lemma. ◀

We now prove our sharp Dvoretzky-like theorem for subspaces chosen from the Haar measure using stratified analysis.

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► **Theorem 20.** *Let $p : \mathbb{N} \rightarrow \mathbb{R}_+$ be a non-decreasing function. Suppose the infinite series $\sum_{i>0} \frac{\sqrt{ip(i)}}{2^i}$ is convergent with value C . Let $f : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ have global Lipschitz constant L_1 . Let $L_2, c_1, c_2, c_3, \lambda > 0$. Define $m := \lceil \frac{c_1 n \lambda^2}{L_2^2} \rceil$. Suppose there is an increasing sequence of subsets $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ of $\mathbb{S}_{\mathbb{C}^n}$ such that with probability at least $1 - c_2 e^{-c_3 m i}$, a Haar random subspace of dimension m lies in Ω_i and f restricted to Ω_i has Lipschitz constant $L_2 \sqrt{p(i)}$. Then there exists a constant c depending on $c_3, C, 0 < c < 1$, such that for $m' := cm$ with probability at least $1 - (c_2 + 1)2^{-m'}$, a subspace W of dimension m' chosen with respect to Haar measure satisfies the property that $|f(w) - \mu| < \lambda$ for all points $w \in W \cap \mathbb{S}_{\mathbb{C}^n}$.*

Proof. In this proof $\mathbb{S}_{\mathbb{C}^n}$ denotes the unit ℓ_2 -length sphere in \mathbb{C}^n together with the origin point 0. The radius of $\mathbb{S}_{\mathbb{C}^n}$ is one which makes $i_0 = 0$ in Lemma 19. Consider a canonical embedding of $\mathbb{S}_{\mathbb{C}^{m'}}$ into $\mathbb{S}_{\mathbb{C}^m}$ and further into $\mathbb{S}_{\mathbb{C}^n}$. Define

$$B_i := \{U \in \mathbb{U}(n) : \forall z \in \mathbb{S}_{\mathbb{C}^m}, Uz \in \Omega_i\}.$$

For $s \in \mathbb{S}_{\mathbb{C}^{m'}}$, define the random variable $Y_s := f(Us) - \mu$, where the randomness arises solely from the choice of $U \in \mathbb{U}(n)$. Then $\Pr_{U \sim \text{Haar}}[B_i] \geq 1 - c_2 e^{-c_3 m i}$.

Let $i_1 := \lceil \log \frac{2L_1}{\lambda} \rceil$. Let $\mathcal{N}_i, i = 0, 1, \dots, i_1$ be a sequence of 2^{-i} -nets in $\mathbb{S}_{\mathbb{C}^{m'}}$ of minimum cardinality, where $\mathcal{N}_0 := \{0\}$ and $Y_0 := 0$. We can take $|\mathcal{N}_i| \stackrel{a}{\leq} 2^{2(i+2)m'}$ by Fact 9. By Lemma 19

$$\Pr_{U \sim \text{Haar}} \left[\sup_{s, t \in \mathbb{S}_{\mathbb{C}^{m'}}} |Y_s - Y_t| > \lambda \right] \leq 2 \sum_{i=1}^{i_1+1} \sum_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i : \|u - u'\|_2 < 2^{-i+2}} \Pr_{U \sim \text{Haar}} \left[|Y_u - Y_{u'}| > \frac{\lambda \sqrt{ip(i)}}{4C \cdot 2^i} \right].$$

Applying Lemma 16 to the set B_i gives, for u, u' satisfying $\|u - u'\|_2 < 2^{-i+2}$,

$$\begin{aligned} & \Pr_{U \sim \text{Haar}} \left[|Y_u - Y_{u'}| > \frac{\lambda \sqrt{ip(i)}}{4C \cdot 2^i} \right] \\ & \leq 2 \exp \left(-\frac{n \lambda^2 ip(i)}{2^7 C^2 2^{2i} L_2^2 p(i) \|u - u'\|_2^2} \right) + 2 \Pr_{z \sim \text{Haar}} [z \in \Omega_i^c] \\ & \leq 2 \exp \left(-\frac{ni \lambda^2}{2^9 C^2 L_2^2} \right) + 2 \Pr_{z \sim \text{Haar}} [z \in \Omega_i^c] \\ & \leq 2 \exp \left(-\frac{im}{2^9 C^2} \right) + 2c_2 \exp(-c_3 m i) \leq 2(c_2 + 1) \exp(-c_4 m i), \end{aligned}$$

for a constant c_4 depending only on C and c_3 .

This gives us

$$\begin{aligned} & \Pr_{U \sim \text{Haar}} \left[\sup_{s, t \in \mathbb{S}_{\mathbb{C}^{m'}}} |Y_s - Y_t| > \lambda \right] \\ & \leq 4(c_2 + 1) \sum_{i=1}^{i_1+1} \sum_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i : \|u - u'\|_2 < 2^{-i+2}} e^{-c_4 m i} \leq 4(c_2 + 1) \sum_{i=1}^{i_1+1} |\mathcal{N}_{i-1}| \cdot |\mathcal{N}_i| \cdot e^{-c_4 m i} \\ & \leq 4(c_2 + 1) \sum_{i=1}^{i_1+1} 2^{4m'(i+2)} e^{-c_4 m i} \leq (c_2 + 1)2^{-m'}, \end{aligned}$$

where the third inequality follows from (a) and the fourth inequality follows from the definition $m' := cm$ for an appropriate choice of c depending only on c_4 . In other words, c depends only on C and c_3 .

Taking $t = 0$, we see that with probability at least $1 - (c_2 + 1)2^{-m'}$ over the choice of a Haar random unitary, we have that for all $s \in \mathbb{S}_{\mathbb{C}^{m'}}$, $|Y_s| \leq \lambda$. This completes the proof of the theorem. ◀

Remark

The sets Ω_i and the Lipschitz constants $L_2\sqrt{p(i)}$ for $1 \leq i \leq \lceil \log \frac{2L_1}{\lambda} \rceil + 1$ formalise the idea of stratified analysis mentioned intuitively in the introduction. As i increases the relevant Lipschitz constant increases. So we need a finer net i.e. a 2^{-i} -net for the i th layer Ω_i in order to control the variation of f for subspaces lying inside Ω_i . With exponentially high probability, we thus get a Haar random subspace of dimension m' , slightly smaller than m , where f is almost constant. Note that the definition of m involves only the smallest local Lipschitz constant L_2 . Thus the dimension of the space m' that we obtain is larger than what would be obtained by a naive analysis which would be constrained by the global Lipschitz constant L_1 . Moreover, a naive analysis would not give exponentially high probability, just an arbitrary constant close one. These two properties underscore the power of our stratified analysis. However, applying the stratified analysis to a concrete function is not always straightforward. We need to define the layers $\Omega_1, \Omega_2, \dots$, properly and show separately that Haar random subspaces of dimension m lie in Ω_i with probability $1 - c_2e^{-c_3m^i}$. But for several interesting functions this can be done without much difficulty. This will become clearer in Section 4 where we will show how to recover Aubrun, Szarek and Werner’s result for the Haar measure directly from Theorem 20, without having to apply a Dvoretzky-style theorem twice in a messy fashion as in the original paper [3]. Moreover, we get success probability exponentially close to one unlike Aubrun, Szarek and Werner who could get only a constant close to one. Furthermore, our methods extend to approximate t -designs and allows us to prove exponentially close to one probability even for that setting.

We now prove our sharp Dvoretzky-like theorem for subspaces chosen from approximate t -designs using stratified analysis.

► **Theorem 21.** *Let $p : \mathbb{N} \rightarrow \mathbb{R}_+$ be a non-decreasing function. Suppose the infinite series $\sum_{i>0} \frac{\sqrt{ip(i)}}{2^i}$ is convergent with value C . Let $f : \mathbb{S}_{\mathbb{C}^n} \rightarrow \mathbb{R}$ be a balanced degree ‘ a ’ polynomial with global Lipschitz constant L_1 . Let $0 \leq L_2 \leq 1$, $c_1, c_2, c_3, \lambda > 0$. Define $m := \lceil \frac{c_1 n \lambda^2}{L_2^2} \rceil$. Suppose there is an increasing sequence of subsets $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ of $\mathbb{S}_{\mathbb{C}^n}$ such that with probability at least $1 - c_2e^{-c_3m^i}$, a Haar random subspace of dimension m lies in Ω_i and f restricted to Ω_i has Lipschitz constant $L_2\sqrt{p(i)}$. Suppose*

$$0 < \epsilon < \left(\frac{\lambda}{4L_1} \right)^{2m} \cdot \frac{n^{(2a-1)m} (L_2^2 p(1))^m}{\max\{\alpha(f)^{2m}, 1\}}.$$

Then there exists a constant c depending on $c_1, c_3, C, p(1), 0 < c < 1$ such that for

$$m' := cm \frac{\log \log \frac{C^2 L_1^2}{\lambda^2 p(1)}}{\lceil \log \frac{C^2 L_1^2}{\lambda^2 p(1)} \rceil},$$

with probability at least $1 - (c_2 + 1)2^{-m'}$, a subspace W of dimension m' chosen under an ϵ -approximate $(2am)$ -design ν satisfies the property that $|f(w) - \mu| < \lambda$ for all points $w \in W \cap \mathbb{S}_{\mathbb{C}^n}$.

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Proof. In this proof $\mathbb{S}_{\mathbb{C}^n}$ denotes the unit ℓ_2 -length sphere in \mathbb{C}^n together with the origin point 0. The radius of $\mathbb{S}_{\mathbb{C}^n}$ is one which makes $i_0 = 0$ in Lemma 18. Consider a canonical embedding of $\mathbb{S}_{\mathbb{C}^{m'}}$ into $\mathbb{S}_{\mathbb{C}^m}$ and further into $\mathbb{S}_{\mathbb{C}^n}$. Define

$$B_i := \{U \in \mathbb{U}(n) : \forall z \in \mathbb{S}_{\mathbb{C}^m}, Uz \in \Omega_i\}.$$

For $s \in \mathbb{S}_{\mathbb{C}^{m'}}$, define the random variable $Y_s := f(Us) - \mu$, where the randomness arises solely from the choice of $U \in \mathbb{U}(n)$. Then $\Pr_{U \sim \text{Haar}}[B_i] \geq 1 - c_2 e^{-c_3 m i}$.

Let $i_1 := \lceil \log \frac{2L_1}{\lambda} \rceil$. Let \mathcal{N}_i , $i = 0, 1, \dots, i_1$ be a sequence of 2^{-i} -nets in $\mathbb{S}_{\mathbb{C}^{m'}}$ of minimum cardinality, where $\mathcal{N}_0 := \{0\}$ and $Y_0 := 0$. We can take $|\mathcal{N}_i| \stackrel{a}{\leq} 2^{2(i+2)m'}$ by Fact 9. By Lemma 19

$$\begin{aligned} & \Pr_{U \sim \nu} \left[\sup_{s, t \in \mathbb{S}_{\mathbb{C}^{m'}}} |Y_s - Y_t| > \lambda \right] \leq \\ & 2 \sum_{i=1}^{i_1+1} \sum_{(u, u') \in \mathcal{N}_{i-1} \times \mathcal{N}_i : \|u - u'\|_2 < 2^{-i+2}} \Pr_{U \sim \nu} [|Y_u - Y_{u'}| > \frac{\lambda \sqrt{ip(i)}}{4C \cdot 2^i}]. \end{aligned} \quad (2)$$

Let r be a positive integer such that $r(i_1 + 1) < m$. Applying Lemma 18 to the set B_i gives, for u, u' satisfying $\|u - u'\|_2 < 2^{-i+2}$,

$$\begin{aligned} & \Pr_{U \sim \nu} [|Y_u - Y_{u'}| > \frac{\lambda \sqrt{ip(i)}}{4C \cdot 2^i}] \\ &= \Pr_{U \sim \nu} [|Y_u - Y_{u'}|^{2ri} > \left(\frac{\lambda^2 ip(i)}{2^4 C^2 2^{2i}} \right)^{ri}] \leq \left(\frac{2^{2i+4} C^2}{\lambda^2 ip(i)} \right)^{ri} \mathbb{E}_{U \sim \nu} [|Y_u - Y_{u'}|^{2ri}] \\ &\leq 3 \left(\frac{2^{2i+4} C^2}{\lambda^2 ip(i)} \right)^{ri} \left(\left(\frac{4ri L_2^2 p(i) \|u - u'\|_2^2}{n} \right)^{ri} + c_2 e^{-c_3 m i} \cdot (L_1^2 \|u - u'\|_2^2)^{ri} \right) \\ &\leq 3 \left(\frac{2^{2i+6} C^2 r L_2^2 \|u - u'\|_2^2}{n \lambda^2} \right)^{ri} + 3c_2 e^{-c_3 m i} \left(\frac{2^{2i+4} C^2 L_1^2 \|u - u'\|_2^2}{\lambda^2 ip(i)} \right)^{ri} \\ &\leq \underbrace{3 \left(\frac{2^{10} C^2 r L_2^2}{n \lambda^2} \right)^{ri}}_{=: \text{I}} + \underbrace{3c_2 e^{-c_3 m i} \left(\frac{2^8 C^2 L_1^2}{\lambda^2 p(1)} \right)^{ri}}_{=: \text{II}}. \end{aligned}$$

We now analyse the two terms in the above expression. Take

$$r := \frac{c_4 n \lambda^2}{2^{10} C^2 L_2^2} \cdot \frac{1}{\lceil \log \frac{2^8 C^2 L_1^2}{\lambda^2 p(1)} \rceil}$$

for a constant c_4 , $0 < c_4 < 1$, c_4 depending only on $C, c_1, c_3, p(1)$ chosen to be small enough so that $r(i_1 + 1) < m$ and $\frac{c_4 n \lambda^2}{2^{10} C^2 L_2^2} \leq \frac{c_3 m}{2}$. Substitute r back in I and II to get

$$\text{I} \leq 3 \cdot 2^{-ri \log \log \frac{2^8 C^2 L_1^2}{\lambda^2 p(1)}}, \quad \text{II} \leq 3c_2 e^{-c_3 m i} 2^{\frac{c_3 m i}{2}} < 3c_2 e^{-c_3 m i / 2}.$$

We choose

$$m'' := r \log \log \frac{2^8 C^2 L_1^2}{\lambda^2 p(1)} < \frac{c_3 m}{2}.$$

This gives us

$$\text{I} \leq 3 \cdot 2^{-m'' i}, \quad \text{II} \leq 3c_2 e^{-m'' i}.$$

Thus, we have shown that

$$\Pr_{U \sim \nu} [|Y_u - Y_{u'}| > \frac{\lambda \sqrt{p(i)}}{4C \cdot 2^i}] \leq 3(c_2 + 1)2^{-m''i}.$$

Substituting above in Equation 2, we get

$$\begin{aligned} & \Pr_{U \sim \nu} [\sup_{s, t \in \mathbb{S}_{Cm'}} |Y_s - Y_t| > \lambda] \\ & \leq 2 \sum_{i=1}^{i_1+1} \sum_{u, u' \in \mathcal{N}_{i-1} \times \mathcal{N}_i: \|u - u'\| < 2^{-i+2}} 3(c_2 + 1)2^{-m''i} \\ & \leq 6(c_2 + 1) \sum_{i=1}^{i_1+1} |\mathcal{N}_{i-1}| \cdot |\mathcal{N}_i| \cdot 2^{-m''i} \leq 6(c_2 + 1) \sum_{i=1}^{i_1+1} 2^{4m'(i+2)} 2^{-m''i} \leq (c_2 + 1)2^{-m'}, \end{aligned}$$

if m' is chosen as indicated above for a small enough constant c , $0 < c < 1$, c depending only on c_4, c_1, C i.e. c depending only on $C, c_1, c_3, p(1)$.

Taking $t = 0$, we see that with probability at least $1 - (c_2 + 1)2^{-m'}$ over the choice of a uniformly random unitary from the approximate $(2am)$ -design, we have that for all $s \in \mathbb{S}_{Cm'}$, $|Y_s| \leq \lambda$. This completes the proof of the theorem. \blacktriangleleft

4 Strict subadditivity of minimum output von Neumann entropy for approximate t -designs

We first apply Theorem 20 in order to directly recover Aubrun, Szarek and Werner's result [3] that channels with Haar random unitary Stinespring dilations exhibit strict subadditivity of minimum output von Neumann entropy. In fact, we go beyond their result in the sense that we obtain exponentially high probability close to one as opposed to constant probability. After this warmup, we apply Theorem 21 in order to show that channels with approximate $n^{2/3}$ -design unitary Stinespring dilations exhibit strict subadditivity of minimum output von Neumann entropy with exponentially high probability close to one.

Let k be a positive integer. Consider the sphere $\mathbb{S}_{\mathbb{C}^{k^3}}$. Define the $k \times k^2$ matrix M to be the rearrangement of a k^3 -tuple from $\mathbb{S}_{\mathbb{C}^{k^3}}$. Note that the ℓ_2 -norm on \mathbb{C}^{k^3} is the same as the Frobenius norm on $\mathbb{C}^{k \times k^2}$.

In Step I, we define the function $f : \mathbb{S}_{\mathbb{C}^{k^3}} \rightarrow \mathbb{R}$ as $f(M) := \|M\|_\infty$. The function f has global Lipschitz constant $L_1 = 1$ since

$$|f(M) - f(N)| \leq \|M - N\|_\infty \leq \|M - N\|_2.$$

For large enough k the mean μ of f , under the Haar measure, is less than $2k^{-1/2}$ [3, Corollary 7]. We use the notation of Theorem 20. Define $L_2 := 1$, $p(i) := 1$ for all $i \in \mathbb{N}$. Then $C < 2$. Define the layers $\Omega_1, \Omega_2, \dots$, to be all of $\mathbb{S}_{\mathbb{C}^{k^3}}$. Let j , $4 \leq j \leq k$ be a positive integer. Let $\lambda_j := \sqrt{\frac{j}{k}}$. Define $c_1 := 1$, $m = k^2$, $c_2 := 0$, $c_3 := 1$. Trivially, a Haar random subspace of dimension mj lies in Ω_i with probability at least $1 - c_2 e^{-c_3 m j^i}$. Theorem 20 tells us that there is a universal constant \hat{c}_1 such that for $m' := \hat{c}_1 k^2$, with probability at least $1 - 2^{-m'j}$, a Haar random subspace W of dimension $m'j$ satisfies

$$\|M\|_\infty < \frac{2}{\sqrt{k}} + \sqrt{\frac{j}{k}} < 2\sqrt{\frac{j}{k}}$$

for all $M \in W$.

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In Step II, we define the function $f : \mathbb{S}_{\mathbb{C}^{k^3}} \rightarrow \mathbb{R}$ as $f(M) := \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2$. The function f has global Lipschitz constant $L_1 = 2$ since

$$\begin{aligned} |f(M) - f(N)| &\leq \|MM^\dagger - NN^\dagger\|_2 \leq \|MM^\dagger - MN^\dagger\|_2 + \|MN^\dagger - NN^\dagger\|_2 \\ &\leq \|M\|_\infty \|M^\dagger - N^\dagger\|_2 + \|N^\dagger\|_\infty \|M - N\|_2 \\ &= (\|M\|_\infty + \|N\|_\infty) \|M - N\|_2 \leq 2\|M - N\|_2. \end{aligned}$$

The mean μ of f , under the Haar measure, is less than $c_0 k^{-1}$ for a universal constant c_0 [3, Corollary 7]. We use the notation of Theorem 20. Let j , $c_0 < j \leq k$ be a positive integer. Define $L_2 := 4\sqrt{\frac{j}{k}}$, $p(i) := i + 3$ for all $i \in \mathbb{N}$. Then $C \leq 4$. Define the layers $\Omega_1, \Omega_2, \dots$, to be the subsets

$$\Omega_i := \left\{ M \in \mathbb{S}_{\mathbb{C}^{k^3}} : \|M\|_\infty \leq 2\sqrt{\frac{j(i+3)}{k}} \right\}.$$

It is easy to see that f restricted to Ω_i has local Lipschitz constant at most $L_2 \sqrt{p(i)}$. Let $\lambda := \frac{j}{k}$. Define $c_1 := 16\hat{c}_1$, $m = \hat{c}_1 j k^2$, $c_2 := 1$, $c_3 := \ln 2$. By the previous paragraph, a Haar random subspace of dimension $m(i+3)$ lies in Ω_i with probability at least $1 - c_2 e^{-c_3 m(i+3)} \geq 1 - c_2 e^{-c_3 m i}$. Theorem 20 tells us that there is a universal constant \hat{c}_2 such that for $m' := \hat{c}_2 k^2$, with probability at least $1 - 2^{-m' j}$, a Haar random subspace W of dimension $m' j$ satisfies

$$f(M) = \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 < \frac{c_0}{k} + \frac{j}{k} < \frac{2j}{k}$$

for all $M \in W$. Setting $j = 1$ allows us to recover Aubrun, Szarek and Werner's technical result [3] with probability exponentially close to one viz. with probability at least $1 - 2^{-m'}$, a Haar random subspace W of dimension m' satisfies $\|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 < \frac{2}{k}$ for all $M \in W$. We will see below how this implies the existence of a channel with strictly subadditive minimum output von Neumann entropy.

In Step III, we define the function $f : \mathbb{S}_{\mathbb{C}^{k^3}} \rightarrow \mathbb{R}$ as $f(M) := \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2^2$ i.e. this f is the square of the f defined in the last paragraph. Now, f is a balanced polynomial of degree $a = 2$ and $1 < \alpha(f) < k^6$ as can be seen by considering $f(J)$ where J is the $k \times k^2$ all ones matrix. The function f has global Lipschitz constant $L_1 = 4$ since

$$\begin{aligned} |f(M) - f(N)| &\leq \left| \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 - \|NN^\dagger - \frac{\mathbf{1}}{k}\|_2 \right| \cdot \left(\|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 + \|NN^\dagger - \frac{\mathbf{1}}{k}\|_2 \right) \\ &\leq (\|M\|_\infty + \|N\|_\infty) \left(\|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 + \|NN^\dagger - \frac{\mathbf{1}}{k}\|_2 \right) \|M - N\|_2 \\ &\leq 4\|M - N\|_2. \end{aligned}$$

The mean μ of f under the Haar measure is less than $c_0^2 k^{-2}$ for the same universal constant c_0 [3, Corollary 7]. We use the notation of Theorem 21. Define $L_2 := 16k^{-3/2}$, $p(i) := i^3$ for all $i \in \mathbb{N}$. Then $C \leq 5$. Define the layers $\Omega_1, \Omega_2, \dots$, to be the subsets

$$\Omega_i := \left\{ M \in \mathbb{S}_{\mathbb{C}^{k^3}} : \|M\|_\infty \leq 2\sqrt{\frac{i}{k}}, \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2 < \frac{2i}{k} \right\}.$$

It is easy to see that f restricted to Ω_i has local Lipschitz constant at most $L_2 \sqrt{p(i)}$. Let $\lambda := k^{-2}$. Define $c_1 := 2^8 \hat{c}_2$, $m = \hat{c}_2 k^2 < \hat{c}_1 k^2$, $c_2 := 2$, $c_3 := \ln 2$. By the previous two paragraphs, a Haar random subspace of dimension $m i$ lies in Ω_i with probability at least $1 - c_2 e^{-c_3 m i}$. In particular, a Haar random subspace of dimension m lies in Ω_i with probability at least $1 - c_2 e^{-c_3 m i}$. Let

$$0 \leq \epsilon < \left(\frac{1}{16k^2} \right)^{2m} \frac{k^{9m} k^{-3m}}{k^{12m}} = (4k)^{-10\hat{c}_2 k^2}.$$

Theorem 21 tells us that there is a universal constant \hat{c}_3 such that for

$$m' := \hat{c}_3 k^2 \frac{\log \log k}{\log k},$$

with probability at least $1 - 3 \cdot 2^{-m'}$, a subspace W of dimension m' chosen from an ϵ -approximate $(4\hat{c}_2 k^2)$ -design ν satisfies

$$f(M) = \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2^2 < \frac{c_0^2}{k^2} + \frac{1}{k^2} = \frac{c_0^2 + 1}{k^2}$$

for all $M \in W$.

We shall now see how this result gives us a channel with strict subadditivity of minimum output von Neumann entropy. Consider the channel Φ corresponding to the subspace W . The output dimension is k . The input dimension is $\dim W = m'$. The Stinespring dilation of the channel Φ is the $k^3 \times k^3$ unitary matrix that defines the subspace W' . The subspace W' is obtained by taking the first m' columns of the unitary matrix. This unitary matrix is chosen uniformly at random from a $k^{-8\hat{c}_2 k^2}$ -approximate unitary $(4\hat{c}_2 k^2)$ -design. From Fact 3, we get

$$S_{\min}(\Phi) \geq \log k - k \max_{M \in W} f(M) \geq \log k - \frac{c_0^2 + 1}{k}.$$

And from Fact 4, with $d = k^2$, we get

$$\begin{aligned} S_{\min}(\Phi \otimes \bar{\Phi}) &\leq 2 \log k - \frac{m'}{kd} \log k + O\left(\frac{m'}{kd} \log \frac{d}{m'} + \frac{1}{k}\right) \\ &= 2 \log k - \frac{\hat{c}_3 \log \log k}{k} + O\left(\frac{(\log \log k)^2}{k \log k} + \frac{1}{k}\right) \\ &< S_{\min}(\Phi) + S_{\min}(\bar{\Phi}), \end{aligned}$$

for large enough k . Thus, we have shown that for large enough n approximate unitary $n^{2/3}$ -designs give rise to channels exhibiting strict subadditivity of minimum output von Neumann entropy, implying that classical Holevo capacity of quantum channels can be superadditive.

If instead we were to use the existence of a Haar random subspace W of dimension $m := \hat{c}_2 k^2$ proved at the end of Step II above, we will recover Aubrun, Szarek and Werner's result on strict subadditivity of minimum output von Neumann entropy with probability exponentially close to one. Consider the channel Φ corresponding to the subspace W . The output dimension is k . The input dimension is $\dim W = m$. The Stinespring dilation of the channel Φ is the $k^3 \times k^3$ unitary matrix that defines the subspace W . The subspace W is obtained by taking the first m columns of a Haar random unitary matrix. From Fact 3, we get

$$S_{\min}(\Phi) \geq \log k - k \max_{M \in W} \|MM^\dagger - \frac{\mathbf{1}}{k}\|_2^2 \geq \log k - \frac{4}{k}.$$

And from Fact 4, with $d = k^2$, we get

$$\begin{aligned} S_{\min}(\Phi \otimes \bar{\Phi}) &\leq 2 \log k - \frac{m}{kd} \log k + O\left(\frac{m}{kd} \log \frac{d}{m} + \frac{1}{k}\right) \\ &= 2 \log k - \frac{\hat{c}_2 \log k}{k} + O\left(\frac{1}{k}\right) \\ &< S_{\min}(\Phi) + S_{\min}(\bar{\Phi}), \end{aligned}$$

for large enough k . Thus, we have shown that for large enough n Haar random unitaries give rise to channels exhibiting strict subadditivity of minimum output von Neumann entropy, implying that classical Holevo capacity of quantum channels can be superadditive. Observe that the counter example we get for additivity conjecture for classical Holevo capacity of quantum channels, when the channel is chosen from an approximate unitary t -design has weaker parameters than a channel chosen from Haar random unitaries. Nevertheless, as explained in the introduction our work is the first partial derandomisation of a construction of quantum channels violating additivity of classical Holevo capacity.

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A

 Proofs of Lemmas used in stratified analysis

Proof of Lemma 16. By Fact 8, there is a circled function Y' that agrees with Y on Ω and is L_2 -Lipschitz on all of $\mathbb{S}_{\mathbb{C}^n}$. Define correlated random variables Y'_x, Y'_y in the natural manner. Then using Fact 11, we get

$$\begin{aligned}
 & \Pr_{U \sim \text{Haar}} [|Y_x - Y_y| > \lambda] \\
 &= \Pr_{U \sim \text{Haar}} [(Ux, Uy) \in \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}} [|Y_x - Y_y| > \lambda | (Ux, Uy) \in \Omega \times \Omega] \\
 &\quad + \Pr_{U \sim \text{Haar}} [(Ux, Uy) \notin \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}} [|Y_x - Y_y| > \lambda | (Ux, Uy) \notin \Omega \times \Omega] \\
 &= \Pr_{U \sim \text{Haar}} [(Ux, Uy) \in \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}} [|Y'_x - Y'_y| > \lambda | (Ux, Uy) \in \Omega \times \Omega] \\
 &\quad + \Pr_{U \sim \text{Haar}} [(Ux, Uy) \notin \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}} [|Y_x - Y_y| > \lambda | (Ux, Uy) \notin \Omega \times \Omega] \\
 &\leq \Pr_{U \sim \text{Haar}} [|Y'_x - Y'_y| > \lambda] + 2 \Pr_{z \sim \text{Haar}} [z \in \Omega^c] \\
 &\leq 2 \exp\left(-\frac{n\lambda^2}{8L_2^2 \|x - y\|_2^2}\right) + 2 \Pr_{z \sim \text{Haar}} [z \in \Omega^c].
 \end{aligned}$$

This finishes the proof of the lemma. ◀

Proof of Lemma 17. Since $Z_x - Z_y$ is a balanced polynomial in the entries of the unitary matrix U , from Fact 15 we have

$$\mathbb{E}_{U \sim \nu} [|Z_x - Z_y|^{2r}] \stackrel{a}{\leq} \mathbb{E}_{U \sim \text{Haar}} [|Z_x - Z_y|^{2r}] + \frac{\epsilon \alpha(Z)^{2r}}{n^t}.$$

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By choosing ϵ small enough to satisfy the constraint above, we get $\frac{\epsilon\alpha(Z)^{2r}}{n^t} \stackrel{\text{b}}{\leq} \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r$. Combining (a) and (b) gives

$$\mathbb{E}_{U \sim \nu}[|Z_x - Z_y|^{2r}] \stackrel{\text{c}}{\leq} \mathbb{E}_{U \sim \text{Haar}}(|Z_x - Z_y|^{2r}) + \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r.$$

Now we find $\mathbb{E}_{U \sim \text{Haar}}[|Z_x - Z_y|^{2r}]$.

By Fact 8, there is a circled function Y' that agrees with Y on Ω and is L_2 -Lipschitz on all of $\mathbb{S}_{\mathbb{C}^n}$. Define correlated random variables Y'_x, Y'_y in the natural manner. Then using Fact 11, we get

$$\begin{aligned} & \Pr_{U \sim \text{Haar}}[|Y_x - Y_y| > \lambda] \\ &= \Pr_{U \sim \text{Haar}}[(Ux, Uy) \in \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}}[|Y_x - Y_y| > \lambda | (Ux, Uy) \in \Omega \times \Omega] \\ & \quad + \Pr_{U \sim \text{Haar}}[(Ux, Uy) \notin \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}}[|Y_x - Y_y| > \lambda | (Ux, Uy) \notin \Omega \times \Omega] \\ &= \Pr_{U \sim \text{Haar}}[(Ux, Uy) \in \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}}[|Y'_x - Y'_y| > \lambda | (Ux, Uy) \in \Omega \times \Omega] \\ & \quad + \Pr_{U \sim \text{Haar}}[(Ux, Uy) \notin \Omega \times \Omega] \cdot \Pr_{U \sim \text{Haar}}[|Y_x - Y_y| > \lambda | (Ux, Uy) \notin \Omega \times \Omega] \\ &\leq \Pr_{U \sim \text{Haar}}[|Y'_x - Y'_y| > \lambda] + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c] \\ &\leq 2 \exp\left(-\frac{n\lambda^2}{8L_2^2\|x-y\|_2^2}\right) + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c]. \end{aligned}$$

This finishes the proof of the lemma. \blacktriangleleft

Proof of Lemma 18. Since $Y_x - Y_y$ is a balanced polynomial in the entries of the unitary matrix U , from Fact 15 we have

$$\mathbb{E}_{U \sim \nu}[|Y_x - Y_y|^{2r}] \stackrel{\text{a}}{\leq} \mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r}] + \frac{\epsilon\alpha(Y)^{2r}}{n^t}.$$

By choosing ϵ small enough to satisfy the constraint above, we get $\frac{\epsilon\alpha(Y)^{2r}}{n^t} \stackrel{\text{b}}{\leq} \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r$. Combining (a) and (b) gives

$$\mathbb{E}_{U \sim \nu}[|Y_x - Y_y|^{2r}] \stackrel{\text{c}}{\leq} \mathbb{E}_{U \sim \text{Haar}}(|Y_x - Y_y|^{2r}) + \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r.$$

Now we find $\mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r}]$. Since Y is a balanced polynomial, it is circled. By Fact 8, there is a circled function Y' such that Y' agrees with Y on Ω and Y' is L_2 -Lipschitz on all of $\mathbb{S}_{\mathbb{C}^n}$. Define correlated random variables Y'_x, Y'_y in the natural manner. Then

$$\begin{aligned} & \mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r}] \\ &= \Pr_{U \sim \text{Haar}}[(Ux, Uy) \in \Omega \times \Omega] \cdot \mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r} | (Ux, Uy) \in \Omega \times \Omega] \\ & \quad + \Pr_{U \sim \text{Haar}}[(Ux, Uy) \notin \Omega \times \Omega] \cdot \mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r} | (Ux, Uy) \notin \Omega \times \Omega] \\ &= \Pr_{U \sim \text{Haar}}[(Ux, Uy) \in \Omega \times \Omega] \cdot \mathbb{E}_{U \sim \text{Haar}}[|Y'_x - Y'_y|^{2r} | (Ux, Uy) \in \Omega \times \Omega] \\ & \quad + \Pr_{U \sim \text{Haar}}[(Ux, Uy) \notin \Omega \times \Omega] \cdot \mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r} | (Ux, Uy) \notin \Omega \times \Omega] \\ &\stackrel{\text{d}}{\leq} \mathbb{E}_{U \sim \text{Haar}}[|Y'_x - Y'_y|^{2r}] + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c] \cdot (L_1^2\|x-y\|_2^2)^r. \end{aligned}$$

Now we find $\mathbb{E}_{U \sim \text{Haar}}[|Y'_x - Y'_y|^{2r}]$ using Fact 11 and Low's method [16, Lemma 3.3].

$$\begin{aligned} & \mathbb{E}_{U \sim \text{Haar}}[|Y'_x - Y'_y|^{2r}] \\ &= \int_0^\infty \Pr_{U \sim \text{Haar}}[|Y'_x - Y'_y|^{2r} > \lambda] d\lambda = \int_0^\infty \Pr_{U \sim \text{Haar}}[|Y'_x - Y'_y| > \lambda^{1/(2r)}] d\lambda \\ &\leq 2 \int_0^\infty \exp\left(-\frac{n\lambda^{1/r}}{8L_2^2\|x-y\|_2^2}\right) d\lambda \stackrel{e}{\leq} 2 \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r. \end{aligned}$$

Combining inequalities (d) and (e), we have

$$\mathbb{E}_{U \sim \text{Haar}}[|Y_x - Y_y|^{2r}] \leq 2 \left(\frac{4rL_2^2\|x-y\|_2^2}{n}\right)^r + 2 \Pr_{z \sim \text{Haar}}[z \in \Omega^c] \cdot (L_1^2\|x-y\|_2^2)^r.$$

Further combining with (c) gives us the desired conclusion of the lemma. \blacktriangleleft

B Strict subadditivity of minimum output Rényi p -entropy for approximate t -designs

We now give a general proposition showing how to approximate a continuous non-decreasing function by a polynomial of moderate degree. The proof can be found in the full version in [19].

► **Proposition 22.** *Let $f : [0, A] \rightarrow [0, 1]$ be a continuous non-decreasing onto function that has left and right derivatives everywhere. Define the global Lipschitz constant $L := \max_{y \in [0, A]} f'(y)$, where henceforth we use $f'(y)$ to denote the maximum of the left and right derivatives of f at y . Fix $0 < \epsilon < 1$. Define the ϵ -smoothed local Lipschitz constant at x ,*

$$L_x^\epsilon := \max_{y \in f^{-1}((f(x)-\epsilon, f(x)+\epsilon))} f'(y).$$

Let n be the minimum positive odd integer satisfying $nA \leq \frac{\epsilon}{2} \sqrt{n}$, where $m := \frac{2L}{\epsilon} \sqrt{\ln \epsilon^{-2}}$. Define $m_x := \frac{2L_x^\epsilon}{\epsilon} \sqrt{\ln \epsilon^{-2}}$. Then there is a polynomial $p(x)$ of degree at most $2n + 1$ such that

$$p(x) - 2\epsilon \leq f(x) \leq p(x) + 3\epsilon, \quad -m\epsilon^2 < p'(x) < \epsilon m_x + m\epsilon^2, \quad \forall x \in [0, A].$$

Moreover the sum of absolute values of the coefficients of $p(x)$, denoted by $\alpha(p(x))$, is at most $e^{2((A+1)m)^2}$.

In this section, we apply Proposition 22 and Theorem 21 in order to show that channels with approximate $p^3 n^{\frac{4}{9p} + \frac{5}{9} + \frac{2p}{3}} \log n$ -design unitary Stinespring dilations exhibit strict subadditivity of minimum output Rényi p -entropy for $p > 1$ with exponentially high probability close to one.

Let k be a positive integer. Consider the sphere $\mathbb{S}_{\mathbb{C}^{k^3}}$. Define the $k \times k^2$ matrix M to be the rearrangement of a k^3 -tuple from $\mathbb{S}_{\mathbb{C}^{k^3}}$. Note that the ℓ_2 -norm on \mathbb{C}^{k^3} is the same as the Frobenius norm on $\mathbb{C}^{k \times k^2}$. Let $p > 1$.

In Step I, we define the function $f : \mathbb{S}_{\mathbb{C}^{k^3}} \rightarrow \mathbb{R}$ as $f(M) := \|M\|_{2p}$. The function f has global Lipschitz constant $L_1 = 1$ since

$$|f(M) - f(N)| \leq \|M - N\|_{2p} \leq \|M - N\|_2.$$

For large enough k the mean μ of f , under the Haar measure, is less than $2k^{\frac{1}{2p} - \frac{1}{2}}$ [4, Section VIII], [3, Corollary 7]. We use the notation of Theorem 20. Define $L_2 := 1$, $p(i) := 1$ for all $i \in \mathbb{N}$. Then $C < 2$. Define the layers $\Omega_1, \Omega_2, \dots$, to be all of $\mathbb{S}_{\mathbb{C}^{k^3}}$. Let j , $4 \leq j \leq k$ be

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a positive integer. Let $\lambda_j := j^{\frac{1}{2}} k^{\frac{1}{2p} - \frac{1}{2}}$. Define $c_1 := 1$, $m = k^{2 + \frac{1}{p}}$, $c_2 := 0$, $c_3 := 1$. Trivially, a Haar random subspace of dimension mj lies in Ω_i with probability at least $1 - c_2 e^{-c_3 m j^i}$. Theorem 20 tells us that there is a universal constant \hat{c}_1 such that for $m' := \hat{c}_1 k^{2 + \frac{1}{p}}$, with probability at least $1 - 2^{-m' j}$, a Haar random subspace W of dimension $m' j$ satisfies

$$\|M\|_\infty \leq \|M\|_{2p} < 2k^{\frac{1}{2p} - \frac{1}{2}} + j^{\frac{1}{2}} k^{\frac{1}{2p} - \frac{1}{2}} < 2j^{\frac{1}{2}} k^{\frac{1}{2p} - \frac{1}{2}}$$

for all $M \in W$. In particular, with probability at least $1 - 2^{-\hat{c}_1 j k^{\frac{4}{3p} + \frac{5}{3}} (\log k)^{-1}}$, a Haar random subspace W of dimension $\hat{c}_1 j k^{\frac{4}{3p} + \frac{5}{3}} (\log k)^{-1}$ satisfies

$$\|M\|_\infty \leq \|M\|_{2p} < 2j^{\frac{1}{2}} k^{\frac{1}{2p} - \frac{1}{2}}$$

for all $M \in W$.

Let j , $4 \leq j \leq k$ be a positive integer. Define the function $f : [0, 1] \rightarrow [0, 1]$ as $f(x) := x^p$. Set $\epsilon := k^{-p}$ in Proposition 22. Let n be the minimum positive odd integer satisfying $2pk^p \sqrt{\ln k^{2p}} \leq \frac{k^{-\frac{p}{n}} \sqrt{n}}{2}$; $n < 2^7 p^3 k^{2p} \log k$. Proposition 22 implies that there is a polynomial $p(x)$ of degree at most $2n + 1 < 2^9 p^3 k^{2p} \log k$ such that

$$\begin{aligned} p(x) - 2k^{-p} &\leq x^p \leq p(x) + 3k^{-p}, & \forall x \in [0, 1], \\ |p'(x)| &< 4p(j+1)^{p-1} \sqrt{\ln k^{2p}} k^{\frac{5}{3} - \frac{2}{3p} - p}, & \forall x \in [0, jk^{\frac{2}{3p} - 1}], \\ |p'(x)| &< 4p(5j)^{p-1} \sqrt{\ln k^{2p}} k^{2-p-\frac{1}{p}}, & \forall x \in (jk^{\frac{2}{3p} - 1}, 5jk^{\frac{1}{p} - 1}), \\ |p'(x)| &< 4p\sqrt{\ln k^{2p}}, & \forall x \in (5jk^{\frac{1}{p} - 1}, 1]. \end{aligned} \tag{3}$$

Also, Proposition 22 guarantees that $\alpha(p(x)) < e^{2^7 p^3 k^{2p} \log k}$.

In Step II, we define the function $f : \mathbb{S}_{\mathbb{C}^{k^3}} \rightarrow \mathbb{R}$ as $f(M) := \text{Tr}[p(MM^\dagger)]$, where p is the polynomial defined in Equation 3. Now, f is a balanced polynomial of degree $a = 2n + 1 < 2^9 p^3 k^{2p} \log k$ and

$$\alpha(f) = \text{Tr}[p(JJ^\dagger)] = k^3 \alpha(p(x)) < e^{2^8 p^3 k^{2p} \log k},$$

where J is the $k \times k^2$ all ones matrix. For a $k \times k$ matrix X , define $\text{Sing}(X)$ to be the $k \times k$ diagonal matrix consisting of the singular values of X arranged in decreasing order. The function f has global Lipschitz constant $L_1 = 2^4 p^{3/2} \sqrt{\log k}$ since

$$\begin{aligned} |f(M) - f(N)| &= |\text{Tr}[p(\text{Sing}(M)^2)] - \text{Tr}[p(\text{Sing}(N)^2)]| = |\text{Tr}[p(\text{Sing}(M)^2) - p(\text{Sing}(N)^2)]| \\ &\leq 8p^{3/2} \sqrt{\log k} \cdot \|\text{Sing}(M)^2 - \text{Sing}(N)^2\|_1 \\ &\leq 8p^{3/2} \sqrt{\log k} \cdot \|\text{Sing}(M) - \text{Sing}(N)\|_2 \cdot \|\text{Sing}(M) + \text{Sing}(N)\|_2 \\ &\leq 2^{7/2} p^{3/2} \sqrt{\log k} \cdot \|\text{Sing}(M) - \text{Sing}(N)\|_2 \cdot \sqrt{\|M\|_2^2 + \|N\|_2^2} \\ &\leq 2^4 p^{3/2} \sqrt{\log k} \cdot \|M - N\|_2. \end{aligned}$$

Above, the first inequality follows from Equation 3, the second inequality is Cauchy-Schwarz and the last inequality follows from [18, Section 4]. By setting $j = 4$ in Step I, we conclude that the mean μ of f under the Haar measure is less than $2^{4p} k^{1-p}$. We use the notation of Theorem 21. Let $\lambda := k^{1-p}$. Define

$$L_2 := 2^{4p+3} p^{3/2} \sqrt{\log k} \cdot k^{\frac{5}{3} - p - \frac{2}{3p}},$$

$p(i) := (i+4)^{2p-1}$ for all $i \in \mathbb{N}$. Then $C \leq p^{2p}$. Define the layers $\Omega_1, \Omega_2, \dots$, to be the subsets

$$\Omega_i := \left\{ M \in \mathbb{S}_{\mathbb{C}^{k^3}} : \|M\|_{2p} \leq 2(i+3)^{\frac{1}{2}} k^{\frac{1}{2p} - \frac{1}{2}} \right\}.$$

We will now show that f restricted to Ω_i has local Lipschitz constant at most $L_2\sqrt{p(i)}$. Note that for any $M \in \Omega_i$, $\|M\|_\infty \leq 2(i+3)^{\frac{1}{2}}k^{\frac{1}{2p}-\frac{1}{2}}$. Let B denote the number of singular values of M larger than $(i+3)^{\frac{1}{2}}k^{\frac{1}{3p}-\frac{1}{2}}$. Let b_1, \dots, b_k be the singular values of M in descending order. Then

$$2^{2p}(i+3)^p k^{1-p} \geq \|M\|_{2p}^{2p} \geq \sum_{i=1}^B b_i^{2p} \geq \left(\sum_{i=1}^B b_i^2 \right) (i+3)^{p-1} k^{\frac{5}{3}-\frac{2}{3p}-p},$$

which gives $\sum_{i=1}^B b_i^2 \leq 2^{2p}(i+3)k^{\frac{2}{3p}-\frac{2}{3}}$. Let C denote the number of singular values of N larger than $(i+3)^{\frac{1}{2}}k^{\frac{1}{3p}-\frac{1}{2}}$. Without loss of generality, $B \geq C$. Restricting M, N to belong to Ω_i , we get from Equation 3 that

$$\begin{aligned} & |f(M) - f(N)| \\ &= |\text{Tr}[p(\text{Sing}(M)^2) - p(\text{Sing}(N)^2)]| \\ &\leq \sum_{i=1}^C |p(b_i^2) - p(c_i^2)| + \sum_{i=C+1}^B |p(b_i^2) - p(c_i^2)| + \sum_{i=B+1}^k |p(b_i^2) - p(c_i^2)| \\ &\leq 8p^{3/2}(5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sum_{i=1}^C |b_i^2 - c_i^2| \\ &\quad + 8p^{3/2}(5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sum_{i=C+1}^B |b_i^2 - c_i^2| \\ &\quad + 8p^{3/2}((i+4))^{p-1} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \sum_{i=B+1}^k |p(b_i^2) - p(c_i^2)| \\ &\leq 8p^{3/2}(5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sqrt{\sum_{i=1}^C (b_i - c_i)^2} \cdot \sqrt{\sum_{i=1}^C (b_i + c_i)^2} \\ &\quad + 8p^{3/2}(5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sqrt{\sum_{i=C+1}^B (b_i - c_i)^2} \cdot \sqrt{\sum_{i=C+1}^B (b_i + c_i)^2} \\ &\quad + 8p^{3/2}((i+4))^{p-1} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \sqrt{\sum_{i=B+1}^k (b_i - c_i)^2} \cdot \sqrt{\sum_{i=B+1}^k (b_i + c_i)^2} \\ &\leq 2^{7/2} p^{3/2} (5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sqrt{\sum_{i=1}^k (b_i - c_i)^2} \cdot \sqrt{\sum_{i=1}^C (b_i^2 + c_i^2)} \\ &\quad + 2^{7/2} p^{3/2} (5(i+3))^{p-1} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \sqrt{\sum_{i=1}^k (b_i - c_i)^2} \cdot \sqrt{\sum_{i=C+1}^B (b_i^2 + c_i^2)} \\ &\quad + 2^{7/2} p^{3/2} ((i+4))^{p-1} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \sqrt{\sum_{i=1}^k (b_i - c_i)^2} \cdot \sqrt{\sum_{i=1}^k (b_i^2 + c_i^2)} \\ &\leq 2^4 p^{3/2} 2^p 5^{p-1} (i+3)^{p-\frac{1}{2}} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \cdot k^{\frac{1}{3p}-\frac{1}{3}} \cdot \|\text{Sing}(M) - \text{Sing}(N)\|_2 \\ &\quad + 2^4 p^{3/2} 2^p 5^{p-1} (i+3)^{p-\frac{1}{2}} \sqrt{\log k} \cdot k^{2-p-\frac{1}{p}} \cdot k^{\frac{1}{3p}-\frac{1}{3}} \cdot \|\text{Sing}(M) - \text{Sing}(N)\|_2 \\ &\quad + 2^4 p^{3/2} ((i+4))^{p-1} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \|\text{Sing}(M) - \text{Sing}(N)\|_2 \\ &\leq 2^6 p^{3/2} 2^p 5^{p-1} (i+4)^{p-\frac{1}{2}} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \cdot \|\text{Sing}(M) - \text{Sing}(N)\|_2 \\ &\leq 2^{4p+3} p^{3/2} (i+4)^{p-\frac{1}{2}} \sqrt{\log k} \cdot k^{\frac{5}{3}-p-\frac{2}{3p}} \cdot \|M - N\|_2. \end{aligned}$$

9:22 Super Additivity with t -Designs

This completes the proof of the claim above that f restricted to Ω_i has local Lipschitz constant at most $L_2\sqrt{p(i)}$. Define $c_1 := 2^{8p+6}p^3\hat{c}_1$, $m = \hat{c}_1k^{\frac{4}{3p}+\frac{5}{3}}(\log k)^{-1}$, $c_2 := 1$, $c_3 := \ln 2$. By Step I, a Haar random subspace of dimension mi lies in Ω_i with probability at least $1 - c_2e^{-c_3mi}$. Let

$$0 \leq \epsilon < \left(\frac{k^{1-p}}{4L_1}\right)^{2m} \frac{k^{3(2a-1)m}5^{(2p-1)m}L_2^{2m}}{\alpha(f)^{2m}}.$$

Theorem 21 tells us that there is a universal constant \hat{c}_3 such that for

$$m' := \hat{c}_3k^{\frac{4}{3p}+\frac{5}{3}} \frac{\log \log k}{(\log k)^2},$$

with probability at least $1 - 2 \cdot 2^{-m'}$, a subspace W of dimension m' chosen from an ϵ -approximate $(2am)$ -design ν satisfies

$$f(M) = \text{Tr}[p(MM^\dagger)] < 2^{4p}k^{1-p} + k^{1-p} < 2^{4p+1}k^{1-p}$$

for all $M \in W$. By Equation 3, this implies that

$$\text{Tr}[(MM^\dagger)^p] < \text{Tr}[p(MM^\dagger)] + 3k^{1-p} < 2^{4p+3}k^{1-p}$$

for all $M \in W$. In other words, $\|M\|_{2p}^2 < 2^7k^{\frac{1}{p}-1}$ for all $M \in W$.

We shall now see how this result gives us a channel with strict supermultiplicativity of the $\|\cdot\|_{1 \rightarrow p}$ -norm or equivalently, strict subadditivity of minimum output Rényi p -entropy for $p > 1$. Consider the channel Φ corresponding to the subspace W . The output dimension is k . The input dimension is $\dim W = m'$. The Stinespring dilation of the channel Φ is the $k^3 \times k^3$ unitary matrix that defines the subspace W' . The subspace W' is obtained by taking the first m' columns of the unitary matrix. This unitary matrix is chosen uniformly at random from an ϵ -approximate unitary $\hat{c}_12^{10}p^3k^{\frac{4}{3p}+\frac{5}{3}+2p} \log k$ -design. From Equation 1, we get

$$\|\Phi\|_{1 \rightarrow p} = \max_{M \in W: \|M\|_2=1} \|M\|_{2p}^2 \leq 2^7k^{\frac{1}{p}-1}.$$

From Fact 4,

$$\|\Phi \otimes \bar{\Phi}\|_{1 \rightarrow p} \geq \frac{m'}{k^3} = \hat{c}_3k^{\frac{4}{3p}-\frac{4}{3}} \frac{\log \log k}{(\log k)^2} > (\|\Phi\|_{1 \rightarrow p})^2$$

for large enough k . Thus, we have shown that for large enough n approximate unitary $p^3n^{\frac{4}{3p}+\frac{5}{3}+\frac{2p}{3}} \log n$ -designs give rise to channels exhibiting strict subadditivity of minimum output Rényi p -entropy for any $p > 1$.