

Approximating Activation Edge-Cover and Facility Location Problems

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Abstract

What approximation ratio can we achieve for the FACILITY LOCATION problem if whenever a client u connects to a facility v , the opening cost of v is at most θ times the service cost of u ? We show that this and many other problems are a particular case of the ACTIVATION EDGE-COVER problem. Here we are given a multigraph $G = (V, E)$, a set $R \subseteq V$ of terminals, and thresholds $\{t_u^e, t_v^e\}$ for each uv -edge $e \in E$. The goal is to find an assignment $\mathbf{a} = \{a_v : v \in V\}$ to the nodes minimizing $\sum_{v \in V} a_v$, such that the edge set $E_{\mathbf{a}} = \{e = uv : a_u \geq t_u^e, a_v \geq t_v^e\}$ activated by \mathbf{a} covers R . We obtain ratio $1 + \max_{x \geq 1} \frac{\ln x}{1 + x/\theta} \approx \ln \theta - \ln \ln \theta$ for the problem, where θ is a problem parameter. This result is based on a simple generic algorithm for the problem of minimizing a sum of a decreasing and a sub-additive set functions, which is of independent interest. As an application, we get the same ratio for the above variant of FACILITY LOCATION. If for each facility all service costs are identical then we show a better ratio $1 + \max_{k \in \mathbb{N}} \frac{H_k - 1}{1 + k/\theta}$, where $H_k = \sum_{i=1}^k 1/i$. For the MIN-POWER EDGE-COVER problem we improve the ratio 1.406 of [4] (achieved by iterative randomized rounding) to 1.2785. For unit thresholds we improve the ratio $73/60 \approx 1.217$ of [4] to $\frac{1555}{1347} \approx 1.155$.

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1 Introduction

Let $G = (V, E)$ be an undirected multigraph where each edge $e \in E$ has an **activating function** f^e from some range $L^e \subseteq \mathbb{R}_+^2$ to $\{0, 1\}$. Given a non-negative **assignment** $\mathbf{a} = \{a_v : v \in V\}$ to the nodes, we say that a uv -edge $e \in E$ is **activated** by \mathbf{a} if $f^e(a_u, a_v) = 1$. Let $E_{\mathbf{a}} = \{e \in E : f^e(a_u, a_v) = 1\}$ denote the set of edges activated by \mathbf{a} . The **value** of an assignment \mathbf{a} is $\mathbf{a}(V) = \sum_{v \in V} a_v$. In ACTIVATION NETWORK DESIGN problems the goal is to find an assignment \mathbf{a} of minimum value, such that the edge set $E_{\mathbf{a}}$ activated by \mathbf{a} satisfies a prescribed property. We make the following two assumptions, which are standard in the literature; see the paper of Panigrahi [20] that introduced the problem, and a recent survey [19] on various activation problems.

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Monotonicity Assumption. For every $e \in E$, f^e is monotone non-decreasing, namely, $f^e(x_u, x_v) = 1$ implies $f^e(y_u, y_v) = 1$ if $y_u \geq x_u$ and $y_v \geq x_v$.

Polynomial Domain Assumption. Every $v \in V$ has a polynomial size in $n = |V|$ set L_v of “levels” and $L^e = L_u \times L_v$ for every uv -edge $e \in E$.

Given a set $R \subseteq V$ of **terminals** we say that an edge set J is an R -**cover** or that J **covers** R if every $v \in R$ has some edge in J incident to it. In EDGE-COVER problems we seek an R -cover J that minimizes a given value function, e.g., the edge cost of J . The min-cost EDGE-COVER problem can be solved in polynomial time [9], and it is one of the most fundamental problems in Combinatorial Optimization, cf. [23].

We consider the ACTIVATION EDGE-COVER problem. Since we consider multigraphs, $e = uv$ means that e is a uv -edge, namely, that u, v are the endnodes of e ; $e = uv \in E$ means that e is a uv -edge that belongs to E . Under the two assumptions above, the problem can be formulated without activating functions. For this, replace each edge $e = uv$ by a set of at most $|L_u| \cdot |L_v|$ uv -edges $\{e(t_u, t_v) : (t_u, t_v) \in L_u \times L_v, f^e(t_u, t_v) = 1\}$. Then for any $J \subseteq E$ the optimal assignment \mathbf{a} activating J is given by $a_u = \max\{t_u^e : e \in J \text{ is incident to } u\}$; here and everywhere a maximum or a minimum taken over an empty set is assumed to be zero. Consequently, the problem can be restated as follows.

ACTIVATION EDGE-COVER

Input: A graph $G = (V, E)$, a set of terminals $R \subseteq V$, and thresholds $\{t_u^e, t_v^e\}$ for each uv -edge $e \in E$.

Output: An assignment \mathbf{a} of minimum value $\mathbf{a}(V) = \sum_{v \in V} a_v$, such that the edge set $E_{\mathbf{a}} = \{e = uv \in E : a_u \geq t_u^e, a_v \geq t_v^e\}$ activated by \mathbf{a} covers R .

As we will explain later, ACTIVATION EDGE-COVER problems are among the most fundamental problems in network design, that include NP-hard problems such as SET-COVER, FACILITY LOCATION, covering problems that arise in wireless networks (node weighted/min-power/installation problems), and many other problems. The ACTIVATION EDGE-COVER problem admit ratio $O(\ln |R|)$ by a factor 2 reduction to the FACILITY LOCATION problem.

To state our main result we define assignments \mathbf{q} and \mathbf{c} , where $c_v = q_v = 0$ if $v \in V \setminus R$ and for $u \in R$:

- $q_u = \min_{e=uv \in E} t_u^e$ is the minimum threshold at u of an edge in E incident to u .
- $c_u = \min_{e=uv \in E} (t_u^e + t_v^e) - q_u$, so $c_u + q_u$ is the minimum value of an edge in E incident to u .

Following [19], the quantity $\max_{u \in R} c_u/q_u$ is called the **slope** of the instance. We say that an ACTIVATION EDGE-COVER instance is θ -**bounded** if the instance slope is at most θ , namely if $c_u \leq \theta q_u$ for all $u \in R$; moreover, we assume by default that $\theta = \max_{u \in R} c_u/q_u$ is the instance slope. Let **opt** denote the optimal solution value of a problem instance at hand. For each $u \in R$ let e_u be some minimum value edge covering u . Then $\{e_u : u \in R\}$ is an R -cover of value at most $\sum_{u \in R} (c_u + q_u) = (\mathbf{c} + \mathbf{q})(R)$. From this and the definition of θ we get

$$0 \leq \text{opt} - \mathbf{q}(R) \leq \mathbf{c}(R) \leq \theta \mathbf{q}(R) \leq \theta \text{opt}$$

In particular, $(\mathbf{c} + \mathbf{q})(R) \leq (\theta + 1)\text{opt}$. Using this, it is possible to design a greedy algorithm with ratio $1 + \ln(\theta + 1)$. We will show how to obtain a better ratio (the difference is quite significant when $\theta \leq 10^4$ – see Table 1). In what follows, let H_k denote the k -th harmonic

number. Our approximation ratios are expressed in the terms of the following two functions $\omega(\theta)$ and $\bar{\omega}(\theta)$ defined for $\theta > 0$, where $\bar{\omega}$ can be viewed as a “discrete version” of ω :

$$\omega(\theta) = \max_{x \geq 1} \frac{\ln x}{1 + x/\theta} \quad \bar{\omega}(\theta) = \max_{k \in \mathbb{N}} \frac{H_k - 1}{1 + k/\theta}$$

■ **Table 1** Some numerical bounds on $1 + \omega(\theta)$, $1 + \bar{\omega}(\theta)$, $\ln \theta - \ln \ln \theta$, and $1 + \ln(\theta + 1)$.

| θ | 1 | 2 | 3 | 5 | 10 | 100 | 1000 | 10000 | 1000000 |
|-------------------------------|--------|--------|--------|--------|--------|--------|--------|---------|---------|
| $1 + \omega(\theta)$ | 1.2785 | 1.4631 | 1.6036 | 1.8146 | 2.1569 | 3.6360 | 5.4214 | 7.3603 | 11.4673 |
| $1 + \bar{\omega}(\theta)$ | 1.2167 | 1.3667 | 1.4834 | 1.6637 | 1.9645 | 3.3428 | 5.0808 | 6.9967 | 11.0820 |
| $\ln \theta - \ln \ln \theta$ | - | 1.0597 | 1.0046 | 1.1336 | 1.4686 | 3.0780 | 4.9752 | 6.9901 | 11.1898 |
| $1 + \ln(\theta + 1)$ | 1.6932 | 2.0987 | 2.3863 | 2.7918 | 3.3979 | 5.6152 | 7.9088 | 10.2105 | 14.8156 |

In Lemma 8 we show that $\omega(\theta) = W(\theta/e)$, where $W(z)$ is the **Lambert W Function** (a.k.a. ProductLog Function), which is the inverse function of $z(W) = We^W$, c.f. [7, 15]. We are not aware of any known formula for $\bar{\omega}(\theta)$, but since $H_k - 1 \leq \ln k$, $\bar{\omega}(\theta) < \omega(\theta)$ for $\theta > 0$. We also observe in Section 4 that in the definition of $\bar{\omega}(\theta)$ the maximum is attained for the smallest integer k such that $H_k \geq 2 + \frac{\theta-1}{k+1}$. It follows from [15] that $\lim_{\theta \rightarrow \infty} [1 + W(\theta/e) - (\ln \theta - \ln \ln \theta)] = 0$, so $1 + \omega(\theta)$ (and it seems that also $1 + \bar{\omega}(\theta)$) is close to $\ln \theta - \ln \ln \theta$ for large values of θ , although the convergence is very slow; see Table 1.

These functions were implicitly used before to bound approximation ratios. E.g., Robins & Zelikovsky [22] proved that their algorithm for the STEINER TREE problem in quasi-bipartite graph achieves ratio $1 + \omega(1) + \epsilon < 1.2785$; this was improved to $1 + \bar{\omega}(1) + \epsilon = \frac{73}{60} + \epsilon$ in [3]. See also [12] and a survey on the STEINER TREE problem in [13]. In [4] is established ratio $\frac{73}{60}$ for the MIN-POWER EDGE-COVER problem in bipartite graphs and for unit thresholds.

Our main result is:

► **Theorem 1.** *ACTIVATION EDGE-COVER admits ratio $1 + \omega(\theta)$ for θ -bounded instances. The problem also admits ratio $1 + \ln(\Delta + 1)$, and ratio $1 + \ln \Delta$ if R is an independent set in G , where Δ is the maximum number of terminal neighbors of a node in G .*

This result is based on a generic simple approximation algorithm for the problem of minimizing a sum of a decreasing and a sub-additive set functions, which is of independent interest; it is described in the next section. This result is inspired by the algorithm of Robins & Zelikovsky [22] for the STEINER TREE problem, and the analysis in [13] of this algorithm.

We note that Slavik [25] proved that the greedy algorithm for SET-COVER achieves ratio $\ln n - \ln \ln n + \Theta(1)$, while our ratio for ACTIVATION EDGE-COVER is asymptotically $\ln \theta - \ln \ln \theta + \Theta(1)$; but we do not see an immediate relation between the two results.

Let us say that $v \in V$ is a **steady node** if the thresholds t_v^e of the edges e incident to v are all equal to the same number w_v , which we call the **weight of v** . Note that we may assume that all non-terminals are steady, by replacing each $v \in V \setminus R$ by L_v new nodes; see the so called “Levels Reduction” in [19]. This implies that no two parallel edges are incident to the same non-terminal. Clearly, we may assume that $V \setminus R$ is an independent set in G . Let BIPARTITE ACTIVATION EDGE-COVER be the restriction of ACTIVATION EDGE-COVER to instances when also R is an independent set, namely, when G is bipartite with sides $R, V \setminus R$. Note that in this case G is a simple graph and all non-terminals are steady.

We now mention some threshold types in ACTIVATION EDGE-COVER problems, known problems arising from these types, and some implications of Theorem 1 for these problems.

Weighted Set-Cover.

This is a particular case of BIPARTITE ACTIVATION EDGE-COVER when all nodes are steady and nodes in R have weight 0. Note that in this case θ is infinite, and we can only deduce from Theorem 1 the known ratio $1 + \ln \Delta$. Consider a modification of the problem, which we call θ -BOUNDED WEIGHTED SET-COVER: when we pick a set $v \in V \setminus R$, we pay w_v for v , and also pay w_v/θ for each element in R covered by v that was not yet covered. Then the corresponding ACTIVATION EDGE-COVER instance is θ -bounded.

Facility Location.

Here we have a bipartite graph with sides R (clients) and $V \setminus R$ (facilities), weights (opening costs) $\mathbf{w} = \{w_v : v \in V \setminus R\}$, and distances (service costs) $\mathbf{d} = \{d_{uv} : u \in R, v \in V \setminus R\}$. We need to choose $S \subseteq V \setminus R$ with $\mathbf{w}(S) + \sum_{u \in R} d(u, S)$ minimal, where $d(u, S) = \min_{v \in S} d_{uv}$ is the minimal distance from u to S . This is equivalent to BIPARTITE ACTIVATION EDGE-COVER. Note however that if for some constant θ we have $w_v \leq \theta d_{uv}$ for all $uv \in E$ with $u \in R$ and $v \in V \setminus R$, then the corresponding BIPARTITE ACTIVATION EDGE-COVER instance is θ -bounded, and achieves a low constant ratio even for large values of θ , by Theorem 1.

Installation Edge-Cover.

Suppose that the installation cost of a wireless network is proportional to the total height of the towers for mounting antennas. An edge uv is activated if the towers at u and v are tall enough to overcome obstructions and establish line of sight between the antennas. This is modeled as each pair $u, v \in V$ has a height demand h^{uv} and constants γ_{uv}, γ_{vu} , such that a uv -edge is activated by \mathbf{a} if the scaled heights $\gamma_{uv}a_u, \gamma_{vu}a_v$ sum to at least h^{uv} . In the INSTALLATION EDGE-COVER problem, we need to assign heights to the antennas such that each terminal can communicate with some other node, while minimizing the total sum of the heights. The problem is SET-COVER hard even for 0,1 thresholds and bipartite G [20]. But in a practical scenario, the quotient of the maximum tower height over the minimum tower height is a small constant; say, if possible tower heights are 5, 15, 20, then $\theta = 4$.

Min-Power Edge-Cover.

This problem is a particular case of ACTIVATION EDGE-COVER when $t_u^e = t_v^e$ for every edge $e = uv \in E$; note that $\theta = 1$ in this case (in fact, the case $\theta = 1$ is much more general). The motivation is to assign energy levels to the nodes of a wireless network while minimizing the total energy consumption, and enabling communication for every terminal. The MIN-POWER EDGE-COVER problem is NP-hard even if $R = V$, or if R is an independent set in the input graph G and unit thresholds [14]. The problem admits ratio 2 by a trivial reduction to the min-cost case. This was improved to 1.5 in [17], and then to 1.406 in [4], which also establishes the ratio 73/60 for the bipartite case and for unit thresholds.

From Theorem 1 and the discussion above we get:

► **Corollary 2.** MIN-POWER EDGE-COVER admits ratio $1 + \omega(1) < 1.2785$, and the θ -bounded versions of each of the problems WEIGHTED SET-COVER, FACILITY LOCATION, and INSTALLATION EDGE-COVER, admits ratio $1 + \omega(\theta)$.

Let us illustrate this result on the FACILITY LOCATION problem. One might expect a constant ratio for any $\theta > 0$, but our ratio $1 + \omega(\theta)$ is surprisingly low. Even if $\theta = 100$ (service costs are at least 1% of opening costs) then we get a small ratio $1 + \omega(100) < 3.636$. Even for $\theta = 10^4$ we still get a reasonable ratio $1 + \omega(10^4) < 7.3603$. All previous results for the problem are usually summarized by just two observations: the problem is SET-COVER hard

(so has a logarithmic approximation threshold by [21, 10]), and that it admits a matching logarithmic ratio $1 + \ln |R|$ [6]; see surveys on FACILITY LOCATION problems by Vygen [26] and Shmoys [24]. Due to this, almost all work focused on the more tractable METRIC FACILITY LOCATION problem. Our Theorem 1 implies that many practical non-metric FACILITY LOCATION instances admit a reasonable small constant ratio.

Note that our Theorem 1 ratio 1.2785 for $\theta = 1$ significantly improves the previous best ratio 1.406 of [4] for MIN-POWER EDGE-COVER on general graphs achieved by iterative randomized rounding; we do not match the ratio 73/60 of [4] for the bipartite case, but note that the case $\theta = 1$ is much more general than the min-power case considered in [4].

For the case of “locally uniform” thresholds – when for each non-terminal (facility) all thresholds (service costs) are identical, we show a better ratio, see also Table 1.

► **Theorem 3.** BIPARTITE ACTIVATION EDGE-COVER *with locally uniform thresholds admits ratio $1 + \bar{\omega}(\theta)$.*

In addition, we consider unit thresholds, and using some new ideas in addition to [4], improve the “natural” previous best ratio $73/60 \approx 1.217$ of [4] as follows.

► **Theorem 4.** ACTIVATION EDGE-COVER *with unit thresholds admits ratio $\frac{1555}{1347} < 1.155$.*

We note that the proofs of some of our results are non-trivial, although we invested an effort to simplify matters. In any case, the focus in this paper is not technical, but rather conceptual. Our main contribution is giving a unified algorithm for a large class of problems that we identify – θ -BOUNDED ACTIVATION EDGE-COVER problems, either substantially improving known ratios, or showing that many seemingly SET-COVER hard problems may be tractable in practice. Let us also point out that our main result is more general than the applications listed in Corollary 2. The generalization to θ -bounded ACTIVATION EDGE-COVER problems is different from earlier results; besides finding a unifying algorithmic idea that generalizes and improves previous results, we are also able to find tractable special cases in a new direction.

The rest of this paper is organized as follows. In Section 2 we define the GENERALIZED MIN-COVERING problem and analyze a greedy algorithm for it, see Theorem 5. In Section 3 we use Theorem 5 to prove Theorem 1. Theorems 3 and 4 are proved using a modified method in Sections 4 and 5, respectively.

2 The Generalized Min-Covering problem

A set function f is **increasing** if $f(A) \leq f(B)$ whenever $A \subseteq B$; f is **decreasing** if $-f$ is increasing, and f is **sub-additive** if $f(A \cup B) \leq f(A) + f(B)$ for any subsets A, B of the ground-set. Let us consider the following algorithmic problem:

GENERALIZED MIN-COVERING

Input: Non-negative set functions ν, τ on subsets of a ground-set U such that ν is decreasing, τ is sub-additive, and $\tau(\emptyset) = 0$.

Output: $A \subseteq U$ such that $\nu(A) + \tau(A)$ is minimal.

The “ordinary” MIN-COVERING problem is $\min\{\tau(A) : \nu(A) = 0\}$; it is a particular case of the GENERALIZED MIN-COVERING problem when we seek to minimize $M\nu(A) + \tau(A)$ for a large enough constant M . Under certain assumptions, the MIN-COVERING problem admits ratio $1 + \ln \nu(\emptyset)$ [16]. Various generic covering problems are considered in the literature, among them the SUBMODULAR COVERING problem [27], and several other types, cf. [5].

The variant we consider is inspired by the algorithms of Robins & Zelikovsky [22] for the STEINER TREE problem, and the analysis in [13] of this algorithm; but, to the best of our knowledge, the explicit formulation of the GENERALIZED MIN-COVERING problem given here is new. Interestingly, our ratio for ACTIVATION EDGE-COVER with $\theta = 1$ is the same as that of [22] for STEINER TREE in quasi-bipartite graphs.

We call ν the **potential** and τ the **payment**. The idea behind this interpretation and the subsequent greedy algorithm is as follows. Given an optimization problem, the potential $\nu(A)$ is the value of some “simple” augmenting feasible solution for A . We start with an empty set solution, and iteratively try to decrease the potential by adding a set $B \subseteq U \setminus A$ of minimum “density” – the price paid for a unit of the potential. The algorithm terminates when the price ≥ 1 , since then we gain nothing from adding B to A . The ratio of such an algorithm is bounded by $1 + \ln \frac{\nu(\emptyset)}{\text{opt}}$ (assuming that during each iteration a minimum density set can be found in polynomial time). So essentially the greedy algorithm converts ratio $\alpha = \frac{\nu(\emptyset)}{\text{opt}}$ into ratio $1 + \ln \alpha$. However, sometimes a tricky definition of the potential and the payment functions may lead to a smaller ratio.

Let **opt** be the optimal solution value of a problem instance at hand. Fix an optimal solution A^* . Let $\nu^* = \nu(A^*)$, $\tau^* = \tau(A^*)$, so **opt** = $\tau^* + \nu^*$. The quantity $\frac{\tau(B)}{\nu(A) - \nu(A \cup B)}$ is called the **density** of B (w.r.t. A); this is the price paid by B for a unit of potential. The GREEDY ALGORITHM (a.k.a. Relative Greedy Heuristic) for the problem starts with $A = \emptyset$ and while $\nu(A) > \nu^*$ repeatedly adds to A a non-empty augmenting set $B \subseteq U$ that satisfies the following condition, while such B exists:

$$\text{Density Condition: } \frac{\tau(B)}{\nu(A) - \nu(A \cup B)} \leq \min \left\{ 1, \frac{\tau^*}{\nu(A) - \nu^*} \right\}.$$

Note that since ν is decreasing $\nu(A) - \nu(A \cup A^*) \geq \nu(A) - \nu(A^*) = \nu(A) - \nu^*$; hence if $\nu(A) > \nu^*$, then $\frac{\tau(A^*)}{\nu(A) - \nu(A \cup A^*)} \leq \frac{\tau^*}{\nu(A) - \nu^*}$ and there exists an augmenting set B that satisfies the condition $\frac{\tau(B)}{\nu(A) - \nu(A \cup B)} \leq \frac{\tau^*}{\nu(A) - \nu^*}$, e.g., $B = A^*$. Thus if B^* is a minimum density set and $\frac{\tau(B^*)}{\nu(A) - \nu(A \cup B^*)} \leq 1$, then B^* satisfies the Density Condition; otherwise, the density of B^* is larger than 1 so no set can satisfy the Density Condition.

► **Theorem 5.** *The GREEDY ALGORITHM achieves approximation ratio*

$$1 + \frac{\tau^*}{\text{opt}} \ln \frac{\nu_0 - \nu^*}{\tau^*} = 1 + \frac{\tau^*}{\text{opt}} \cdot \ln \left(1 + \frac{\nu_0 - \text{opt}}{\tau^*} \right).$$

Proof. Let ℓ be the number of iterations. Let $A_0 = \emptyset$ and for $i = 1, \dots, \ell$ let A_i be the intermediate solution at the end of iteration i and $B_i = A_i \setminus A_{i-1}$. Let $\nu_i = \nu(A_i)$, $i = 0, \dots, \ell$. Then:

$$\frac{\tau(B_i)}{\nu_{i-1} - \nu_i} \leq \min \left\{ 1, \frac{\tau^*}{\nu_{i-1} - \nu^*} \right\} \quad i = 1, \dots, \ell$$

Since ν is decreasing

$$\sum_{i=1}^{\ell} \tau(B_i) \leq \sum_{i=1}^{\ell} \min \left\{ 1, \frac{\tau^*}{\nu_{i-1} - \nu^*} \right\} (\nu_{i-1} - \nu_i)$$

This is the lower Darboux sum of the function $f(\nu) = \begin{cases} 1 & \text{if } \nu \leq \tau^* + \nu^* \\ \frac{\tau^*}{\nu - \nu^*} & \text{if } \nu > \tau^* + \nu^* \end{cases}$ in the interval $[\nu_\ell, \nu_0]$ w.r.t. the partition $\nu_\ell < \nu_{\ell-1} < \dots < \nu_0$. We claim that $\tau^* + \nu^* \geq \nu_\ell$. Since

the algorithm stopped with A_ℓ , at least one of the following holds: (i) $A_\ell = U$; (ii) $\nu_\ell \leq \nu^*$; (iii) A^* has density > 1 . If (i) holds then (ii) holds (by the monotonicity of ν), but case (ii) is trivial. So assume that only (iii) holds. Then $\frac{\tau(A^*)}{\nu(A_\ell) - \nu(A_\ell \cup A^*)} > 1$, thus since ν is decreasing $\tau(A^*) \geq \nu_\ell - \nu(A \cup A^*) \geq \nu_\ell - \nu(A^*)$. Consequently, $\sum_{i=1}^{\ell} \tau(B_i)$ is bounded by

$$\int_{\nu_\ell}^{\nu_0} f(\nu) d\nu = \int_{\nu_\ell}^{\tau^* + \nu^*} 1 d\nu + \int_{\tau^* + \nu^*}^{\nu_0} \frac{\tau^*}{\nu - \nu^*} d\nu = \tau^* + \nu^* - \nu_\ell + \tau^* \ln \frac{\nu_0 - \nu^*}{\tau^*}$$

Let $A = A_\ell = \bigcup_{i=1}^{\ell} B_i$ be the set computed by the algorithm. Since τ is sub-additive

$$\tau(A) \leq \sum_{i=1}^{\ell} \tau(B_i) \leq \tau^* + \nu^* - \nu(A) + \tau^* \ln \frac{\nu_0 - \nu^*}{\tau^*}$$

Thus the approximation ratio is bounded by $\frac{\tau(A) + \nu(A)}{\text{opt}} \leq 1 + \frac{\tau^*}{\text{opt}} \ln \frac{\nu_0 - \nu^*}{\tau^*}$. ◀

3 Algorithm for general thresholds (Theorem 1)

Given an instance $G = (V, E)$, \mathbf{t}, R of ACTIVATION EDGE-COVER define the corresponding GENERALIZED MIN-COVERING instance U, τ, ν as follows. We put at each node $u \in V$ a large set of “assignment units”, and let U be the union of these sets of “assignment units”. Note that to every $A \subseteq U$ naturally corresponds the assignment \mathbf{a} where a_u is the number of units in A put at u . It would be more convenient to define ν and τ in terms of assignments, by considering instead of a set $A \subseteq U$ the corresponding assignment \mathbf{a} .

To define ν and τ , let us recall the assignments \mathbf{q} and \mathbf{c} from the Introduction. We have $c_v = q_v = 0$ if $v \in V \setminus R$ and for $u \in R$:

- $q_u = \min_{e=uv \in E} t_u^e$ is the minimum threshold at u of an edge in E incident to u .
- $c_u = \min_{e=uv \in E} (t_u^e + t_v^e) - q_u$, so $c_u + q_u$ is the minimum value of an edge in E incident to u .

We let $Q = \mathbf{q}(V) = \mathbf{q}(R)$ and $C = \mathbf{c}(R)$. Note that $\mathbf{c}(R') \leq \theta \mathbf{q}(R')$ for any $R' \subseteq R$; in particular, $C \leq \theta Q$. For an assignment \mathbf{a} that “augments” \mathbf{q} let $R_{\mathbf{q}+\mathbf{a}}$ denote the set of terminals covered by the edge set $E_{\mathbf{q}+\mathbf{a}}$ activated by the assignment $\mathbf{q}+\mathbf{a}$. A natural definition of the potential and the payment functions would be $\tau(\mathbf{a}) = \mathbf{a}(V)$ and $\nu(\mathbf{a}) = (\mathbf{c}+\mathbf{q})(R \setminus R_{\mathbf{q}+\mathbf{a}})$ but this enable us to prove only ratio $1 + \ln(\theta + 1)$. We show a better ratio by adding to the potential in advance the “fixed” part Q . We define

$$\tau(\mathbf{a}) = \mathbf{a}(V) \quad \nu(\mathbf{a}) = Q + \mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}})$$

It is easy to see that ν is decreasing, τ is sub-additive, and $\tau(\mathbf{0}) = 0$.

The next lemma shows that the obtained GENERALIZED MIN-COVERING instance is equivalent to the original ACTIVATION EDGE-COVER instance.

► **Lemma 6.** *If $\mathbf{q} + \mathbf{a}$ is a feasible solution for ACTIVATION EDGE-COVER then $\tau(\mathbf{a}) + \nu(\mathbf{a}) = Q + \mathbf{a}(V)$. If \mathbf{a} is a feasible solution for GENERALIZED MIN-COVERING then one can construct in polynomial time a feasible solution for ACTIVATION EDGE-COVER of value at most $\tau(\mathbf{a}) + \nu(\mathbf{a})$. In particular, both problems have the same optimal value, and GENERALIZED MIN-COVERING has an optimal solution \mathbf{a}^* such that $\nu(\mathbf{a}^*) = Q$ and thus $\text{opt} = \tau(\mathbf{a}^*) + Q$.*

Proof. If $\mathbf{q} + \mathbf{a}$ is a feasible ACTIVATION EDGE-COVER solution then $R_{\mathbf{q}+\mathbf{a}} = R$ and thus $\nu(\mathbf{a}) = Q$. Consequently, $\tau(\mathbf{a}) + \nu(\mathbf{a}) = \mathbf{a}(V) + Q$.

Let now \mathbf{a} be a GENERALIZED MIN-COVERING solution. The assignment $\mathbf{q} + \mathbf{a}$ has value $Q + \mathbf{a}(V)$ and activates the edge set $E_{\mathbf{q}+\mathbf{a}}$ that covers $R_{\mathbf{q}+\mathbf{a}}$. To cover $R \setminus R_{\mathbf{q}+\mathbf{a}}$, pick for

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every $u \in R \setminus R_{\mathbf{q}+\mathbf{a}}$ an edge uv with $t_u^{uv} + t_v^{uv}$ minimum. Let \mathbf{b} be an assignment defined by $b_u = c_u$ if $u \in R \setminus R_{\mathbf{q}+\mathbf{a}}$ and $b_u = 0$ otherwise. The set of picked edges can be activated by an assignment $\mathbf{q} + \mathbf{b}$ that has value $Q + \mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}})$. The assignment $\mathbf{q} + \mathbf{a} + \mathbf{b}$ activates both edge sets and has value $Q + \mathbf{a}(V) + \mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}}) = \tau(\mathbf{a}) + \nu(\mathbf{a})$, as required. \blacktriangleleft

For the obtained GENERALIZED MIN-COVERING instance, let us fix an optimal solution \mathbf{a}^* as in Lemma 6, so $\nu^* = Q$ and $\text{opt} = \tau^* + Q$. Denote $\nu_0 = \nu(\mathbf{0}) = Q + \mathbf{c}(R)$, and note that $\mathbf{c}(R) \leq \theta Q$. To apply Theorem 5 we need several bounds given in the next lemma.

► **Lemma 7.** $\frac{\text{opt}}{\tau^*} \geq 1 + \frac{1}{\theta}$, $\frac{\nu_0}{\tau^*} \leq (\theta + 1) \left(\frac{\text{opt}}{\tau^*} - 1 \right)$, and $\frac{\nu_0 - \nu^*}{\tau^*} \leq \Delta + 1$.

Proof. Note that

$$\tau^* + Q = \text{opt} \leq \nu_0 \leq (\theta + 1)Q.$$

In particular, $\tau^* \leq \theta Q$, and this implies the first bound of the lemma

$$\frac{\text{opt}}{\tau^*} = 1 + \frac{Q}{\tau^*} \geq 1 + \frac{1}{\theta}.$$

The second bound of the lemma holds since $\nu_0 \leq (\theta + 1)Q = (\theta + 1)(\text{opt} - \tau^*)$.

The last bound of the lemma is equivalent to the bound $\mathbf{c}(R) \leq \tau^*(\Delta + 1)$. Let J be an inclusion minimal edge cover of R activated by $\mathbf{q} + \mathbf{a}^*$. Then J is a collection \mathcal{S} of node disjoint rooted stars with leaves in R . Let $S \in \mathcal{S}$. By the definition of \mathbf{c} , $\mathbf{a}^*(S) \geq \max_{u \in R \cap S} c_u$, thus $\mathbf{c}(R \cap S) \leq |R \cap S| \mathbf{a}^*(S) \leq (\Delta + 1) \mathbf{a}^*(S)$. Consequently, $\mathbf{c}(R) = \sum_{S \in \mathcal{S}} \mathbf{c}(R \cap S) \leq (\Delta + 1) \sum_{S \in \mathcal{S}} \mathbf{a}^*(S) \leq (\Delta + 1) \mathbf{a}^*(V)$. \blacktriangleleft

We will show later that the GREEDY ALGORITHM can be implemented to run in polynomial time; now we focus on showing that it achieves the approximation ratios stated in Theorem 1. Denote $x = \theta \left(\frac{\text{opt}}{\tau^*} - 1 \right)$ and $f(x) = \frac{\ln x}{1+x/\theta}$. By Lemma 7 first bound, $x \geq 1$. Substituting Lemma 7 second bound in Theorem 5 second bound we get that the ratio is bounded by

$$1 + \frac{\tau^*}{\text{opt}} \cdot \ln \left(1 + \frac{\nu_0}{\tau^*} - \frac{\text{opt}}{\tau^*} \right) \leq 1 + \frac{\ln x}{1+x/\theta} = 1 + f(x)$$

Consequently, the ratio is bounded by $1 + \max\{f(x) : x \geq 1\} = 1 + \omega(\theta)$.

Substituting Lemma 7 third bound in Theorem 5 first bound and observing that $\tau^* \leq \text{opt}$ we get that the ratio is bounded by $1 + \ln(\Delta + 1)$. In the case when R is an independent set in G , it is easy to see that the third bound in Lemma 7 improves to $\frac{\nu_0 - \nu^*}{\tau^*} \leq \Delta$, and we get ratio $1 + \ln \Delta$ in this case.

In the next lemma we show that $\omega(\theta) = W(\theta/e)$, where $W(z)$ is the Lambert W Function, which is the inverse function of $z(W) = We^W$.

► **Lemma 8.** For any $\theta > 0$, the equation $y + 1 = \ln(\theta/y)$ has a unique (real) root $y(\theta)$, and $0 < y(\theta) \leq \theta$. Furthermore, $\omega(\theta) = y(\theta) = W(\theta/e)$ for any $\theta > 0$.

Proof. Since the function $y + 1$ is strictly increasing and the function $\ln(\theta/y)$ is strictly decreasing, the equation has at most one root; we claim that this root exists and is in the interval $(0, \theta]$. To see this consider the function $h(y) = y + 1 - \ln(\theta/y)$, and note that h is continuous and that $h(\theta) = \theta + 1 > 0$ while $h(\epsilon) = \epsilon + 1 - \ln(\theta/\epsilon) < 0$ for $\epsilon > 0$ small enough.

Let $f(x) = \frac{\ln x}{1+x/\theta}$. Since $f(1) = 0$ and $\lim_{x \rightarrow \infty} f(x) = 0$ (by L'Hospital's Rule), $f(x)$ attains a maximum when

$$f'(x) = \frac{(1+x/\theta)/x - (\ln x)/\theta}{(1+x/\theta)^2} = \frac{1+x/\theta - (x/\theta) \ln x}{x(1+x/\theta)^2} = 0.$$

So $f'(x) = 0$ if and only if $1+x/\theta = (x/\theta) \ln x$, and then $f(x) = \theta/x$. Substituting $y \leftarrow \theta/x$ we get the equation $y+1 = \ln(\theta/y)$, where $0 < y \leq \theta$. Thus $\omega(\theta) = y(\theta)$.

To prove that $W(\theta/e) = y(\theta)$ we show that $W(\theta/e) + 1 = \ln(\theta/W(\theta/e))$, namely, that $W(\theta/e)$ is the root of the equation $y+1 = \ln(\theta/y)$ that defines $y(\theta)$. We have $W(z)e^{W(z)} = z$ for any $z > 0$ (c.f. Eq. 1.5 in [7]), which is equivalent to $W(z) + \ln W(z) = \ln z$. Thus $W(\theta/e) + \ln W(\theta/e) = \ln(\theta/e)$, which gives $W(\theta/e) + 1 = \ln(\theta/W(\theta/e))$, as claimed. ◀

Finally, we show that the GREEDY ALGORITHM algorithm can be implemented in polynomial time. As was mentioned in Section 2 before Theorem 5, we just need to perform in polynomial time the following two operations for any assignment \mathbf{a} : to check the condition $\nu(\mathbf{a}) > \nu^*$, and to find an augmenting assignment \mathbf{b} of minimum density.

The assignments \mathbf{q} and \mathbf{c} can be computed in polynomial time, and thus the potential $\nu(\mathbf{a}) = Q + \mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}})$ can be computed in polynomial time, for any \mathbf{a} . Let \mathbf{a}^* be an optimal solution as in Lemma 6, and denote $\tau^* = \tau(\mathbf{a}^*)$ and $\nu^* = \nu(\mathbf{a}^*) = Q$. Then the condition $\nu(\mathbf{a}) > \nu^*$ is equivalent to $\nu(\mathbf{a}) \geq Q$ and thus can be checked in polynomial time.

Now we show how to find an augmenting assignment \mathbf{b} of minimum density. Note that the density of an assignment \mathbf{b} w.r.t. \mathbf{a} is

$$\frac{\tau(\mathbf{b})}{\nu(\mathbf{a}) - \nu(\mathbf{a} + \mathbf{b})} = \frac{\mathbf{b}(V)}{\mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}}) - \mathbf{c}(R \setminus R_{\mathbf{q}+\mathbf{a}+\mathbf{b}})} = \frac{\mathbf{b}(V)}{\mathbf{c}(R_{\mathbf{q}+\mathbf{a}+\mathbf{b}} \setminus R_{\mathbf{q}+\mathbf{a}})}.$$

▶ **Lemma 9.** *There exists a polynomial time algorithm that given an instance of ACTIVATION EDGE-COVER and an assignment \mathbf{a} finds an assignment \mathbf{b} of minimum density.*

Proof. A **star** is a rooted tree $S = (V_S, E_S)$ with at least one edge such that only its root s may have degree ≥ 2 . We say that a star S is a **proper star** if all the leaves of S are terminals. We denote the terminals in S by $R_S = R \cap V_S$.

Since \mathbf{q}, \mathbf{a} are given assignments, we simplify the notation by assuming that $R \leftarrow R \setminus R_{\mathbf{q}+\mathbf{a}}$ is our set of terminals, and that $\mathbf{a} \leftarrow \mathbf{q} + \mathbf{a}$ is our given assignment. Then the density of \mathbf{b} is just $\frac{\mathbf{b}(V)}{\mathbf{c}(R_{\mathbf{a}+\mathbf{b}})}$. Let \mathbf{b}^* be an assignment of minimum density, and let $J^* \subseteq E_{\mathbf{a}+\mathbf{b}^*}$ be an inclusion minimal $R_{\mathbf{a}+\mathbf{b}^*}$ -cover. Then J^* decomposes into a collection \mathcal{S} of node disjoint proper stars that collectively cover $R_{\mathbf{a}+\mathbf{b}^*}$. For $S \in \mathcal{S}$ let \mathbf{b}^S be the optimal assignment such that $\mathbf{a} + \mathbf{b}^S$ activates S . Since the stars in \mathcal{S} are node disjoint

$$\sum_{S \in \mathcal{S}} \mathbf{b}^S(V) \leq \mathbf{b}^*(V) \quad \text{and} \quad \sum_{S \in \mathcal{S}} \mathbf{c}(R_S) = \mathbf{c}(R_{\mathbf{a}+\mathbf{b}^*}).$$

By an averaging argument, $\frac{\mathbf{b}^S(V)}{\mathbf{c}(R_S)} \leq \frac{\mathbf{b}^*(V)}{\mathbf{c}(R_{\mathbf{a}+\mathbf{b}^*})}$ holds for some $S \in \mathcal{S}$, and since \mathbf{b}^* is a minimum density assignment, so is \mathbf{b}^S , and $\frac{\mathbf{b}^S(V)}{\mathbf{c}(R_S)} = \frac{\mathbf{b}^*(V)}{\mathbf{c}(R_{\mathbf{a}+\mathbf{b}^*})}$ holds. Consequently, it is sufficient to show how to find in polynomial time an assignment \mathbf{b} such that $\mathbf{a} + \mathbf{b}$ activates a proper star S and $\frac{\mathbf{b}(V)}{\mathbf{c}(R_S)}$ is minimal.

We may assume that we know the root v and the value $w = b_v$ of an optimal density pair S, \mathbf{b} ; there are at most $|V||E|$ choices and we can try all and return the best outcome. Let $R_w = \{u \in R : \text{there is a } uv\text{-edge } e \text{ with } t_v^e \leq a_v + w\}$. For $u \in R_w$ let b_u be the minimal non-negative number for which there is a uv -edge e with $a_v + w \geq t_v^e$ and $a_u + b_u \geq t_u^e$.

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Then our problem is equivalent to finding $R_S \subseteq R_w$ with $\sigma(R_S) = \frac{w + \mathbf{b}(R_S)}{\mathbf{c}(R_S)}$ minimum. This problem can be solved in polynomial time, by starting with $R_S = \emptyset$ and while there is $u \in R_w \setminus R_S$ with $\sigma(R_S + u) < \sigma(R_S)$, adding $u \in R_w \setminus R_S$ to R_S with b_u/c_u minimum. ◀

The proof of Theorem 1 is complete.

4 Locally uniform thresholds (Theorem 3)

Here we consider the BIPARTITE ACTIVATION EDGE-COVER problem with locally uniform thresholds. This means that each non-terminal $v \in V \setminus R$ has weight w_v and all edges incident to v have the same threshold t^v ; in the θ -bounded version $w_v \leq \theta t^v$. We consider a natural greedy algorithm that repeatedly picks a star S that minimizes the average price paid for each terminal (the quotient of the optimal activation value of S over $|R_S|$), and then removes R_S . Each time we choose a star S we distribute its activation value uniformly among its terminals, paying in the computed solution the average price for each terminal of S .

We now apply a standard “set-cover” analysis, cf. [28]. In some optimal solution fix an inclusion maximal star S^* with center v and terminals R_{S^*} covered by the algorithm in the order r_k, r_{k-1}, \dots, r_1 , where r_k is covered first and r_1 last; we bound the algorithm payment for covering R_{S^*} . Note that $1 \leq k \leq \Delta$. Denote $w = w_v$ and let t be the threshold of the terminals in S^* . Let S_i^* be the substar of S^* with leaves r_i, \dots, r_1 . At the start of the iteration in which the algorithm covers r_i , the terminals of S_i^* are uncovered. Thus the algorithm pays for covering r_i at most the average price paid by S_i^* , namely $(w + it)/i = w/i + t$. Over all iterations, the algorithm pays for covering R_{S^*} at most $wH_k + kt$, while the optimum pays $w + kt$. Thus the quotient between them is bounded by

$$\frac{wH_k + kt}{w + kt} = \frac{w/tH_k + k}{w/t + k} \leq \frac{\theta H_k + k}{\theta + k} = 1 + \frac{\theta(H_k - 1)}{\theta + k} \leq 1 + \max_{1 \leq k \leq \Delta} \frac{H_k - 1}{1 + k/\theta} \leq 1 + \bar{\omega}(\theta).$$

Since any optimal solution decomposes into node disjoint stars, the last term bounds the approximation ratio, concluding the proof of Theorem 3.

Let $g(k) = \frac{H_k - 1}{1 + k/\theta}$ where $k \in \mathbb{N}$, so $\bar{\omega}(\theta) = \max_{k \in \mathbb{N}} g(k)$. We have

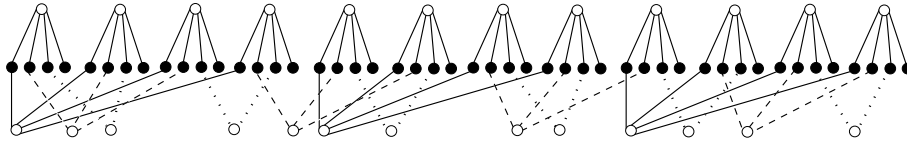
$$g(k+1) - g(k) = \frac{\theta}{(k+\theta)(k+1+\theta)} \left(2 - H_k + \frac{\theta-1}{k+1} \right).$$

Thus $g(k+1) \geq g(k)$ if and only if $2 - H_k + \frac{\theta-1}{k+1} > 0$. Consequently, $\bar{\omega}(\theta) = g(k(\theta))$ where $k(\theta)$ be the smallest integer k such that $H_k \geq 2 + \frac{\theta-1}{k+1}$.

For $\theta = 1$ we have $k(\theta) = 4$, so $1 + \bar{\omega}(1) = 73/60$. The greedy algorithm cannot do better even for unit thresholds, as shows the example in Fig. 1. The instance has 48 terminals (in black), and two sets of covering nodes: the upper 12 nodes that form an optimal cover, and the bottom 13 nodes. The bottom nodes have 3 nodes of degree 4, 4 of degree 3, and 6 of degree 2. The algorithm may start taking all bottom nodes, and only then add the upper ones, thus creating a solution of value 73, instead of the optimum 60.

5 Unit thresholds (Theorem 4)

Here we consider the case of unit thresholds when $t_u^e = t_v^e = 1$ for every uv -edge e . By a reduction from [4], we may assume that the instance is bipartite. Specifically, for any optimal assignment \mathbf{a} we have $a_u = 1$ for all $u \in R$, hence we can consider the residual instance obtained by removing the terminals covered by edges with both ends in R ; in the new obtained instance R is an independent set, and recall that we may assume that $V \setminus R$ is an independent set.



■ **Figure 1** Tight example of ratio $\frac{73}{60}$ for unit thresholds.

One can observe that in the obtained bipartite instance, \mathbf{a} is an optimal solution if and only if $a_v \in \{0, 1\}$ for all $v \in V$, $a_v = 1$ for all $v \in R$, and the set $C = \{v \in V \setminus R : a_v = 1\}$ covers R , meaning that R is the set of neighbors of C . Namely, our problem is equivalent to $\min\{|C| + |R| : C \subseteq V \setminus R, C \text{ covers } R\}$. On the other hand the problem $\min\{|C| : C \subseteq V \setminus R, C \text{ covers } R\}$ is essentially the (unweighted) SET-COVER problem, and C is a feasible solution to this SET-COVER instance if and only if $C \cup R$ is the characteristic set of a feasible assignment for the ACTIVATION EDGE-COVER instance. Note that both problems are equivalent w.r.t. their optimal solutions but may differ w.r.t. approximation ratios, since if C^* is an optimal SET-COVER solution then $\frac{|C| + |R|}{|C^*| + |R|}$ may be much smaller than $\frac{|C|}{|C^*|}$.

Recall that a standard greedy algorithm for SET-COVER repeatedly picks the center of a largest star and removes the star from the graph. This algorithm has ratio H_k for k -SET-COVER, where $k = \Delta$ is the maximum degree of a non-terminal (the maximum size of a set). However, the same algorithm achieves a much smaller ratio $\frac{73}{60}$ for ACTIVATION EDGE-COVER with unit thresholds; the ratio $\frac{73}{60}$ was established in [4], and it also follows from the case $\theta = 1$ in Theorem 3. In what follows we denote by α_k the best known ratio for k -SET-COVER. We have $\alpha_1 = \alpha_2 = 1$ ($k = 2$ is the EDGE-COVER problem) and $\alpha_3 = 4/3$ [8]. The current best ratios for $k \geq 4$ are due to [11] (see also [18, 1]). We summarize the current values of α_k for $k \leq 7$ in the following table.

■ **Table 2** Current values of α_k for $k \leq 7$.

| α_1 | α_2 | α_3 | α_4 | α_5 | α_6 | α_7 |
|------------|------------|---------------|-----------------|-----------------|-----------------|-------------------|
| 1 | 1 | $\frac{4}{3}$ | $\frac{73}{48}$ | $\frac{26}{15}$ | $\frac{28}{15}$ | $\frac{212}{105}$ |

We now show how these ratios for k -SET-COVER can be used to approximate the ACTIVATION EDGE-COVER problem with unit costs. We start by describing a simple algorithm with ratio $1\frac{67}{360} < \frac{73}{60}$, that uses only the $k = 2$ case.

■ **Algorithm 1** ratio $1\frac{67}{360}$

-
- 1 $A \leftarrow \emptyset$
 - 1 **while** there exists a star with at least 3 terminals **do**
 - 2 add to A and remove from G the node-set of a maximum size star
 - 3 add to A an optimal solution of the residual instance
-

We claim that the above algorithm achieves approximation ratio $1\frac{67}{360}$ for ACTIVATION EDGE-COVER (a similar analysis implies ratio $H_k - \frac{1}{6}$ for SET-COVER). In some optimal solution fix a star S^* with terminals covered in the order r_k, r_{k-1}, \dots, r_1 , where r_k is covered first and r_1 last; we bound the algorithm payment to cover these terminals. Let S_i^* be the substar of S^* with leaves r_i, \dots, r_1 . At the start of the iteration when r_i is covered, the terminals of S_i^* are uncovered. Thus the algorithm pays for covering r_i at most the density of S_i^* , namely, $(i + 1)i = 1 + 1/i$. Over all iterations, the algorithm pays for covering R_S at most $k + H_k$, while the optimum pays $k + 1$. If $k = 1$ then the algorithm pays at most

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the amount of the optimum. We claim that if $k \geq 2$ then in fact the payment is at most $k + H_k - 1/6$. If $k = 2$ then the payment is at most $3 < 3 + H_3 - 1/6$ (we pay 3 if the star “survives” all the iterations before the last). For $k \geq 3$, the pay for the last 3 terminals is either: $4/3$ for each of for r_3, r_2 and 2 for r_1 (a total of $14/3$), or $4/3$ for r_3 and 3 for r_2, r_1 (a total of $13/3$). The maximum is $14/3 = 3 + H_3 - 1/6$. Consequently, the ratio is bounded by

$$\max_{k \geq 2} \frac{k + H_k - 1/6}{k + 1} = 1 + \max_{k \geq 2} \frac{H_k - 7/6}{k + 1} = 1 + \max_{k \geq 2} g(k)$$

By fundamental computations we have $g(k + 1) - g(k) = \frac{13/6 - H_k}{(k+1)(k+2)}$. Thus $g(k)$ is increasing if and only if $H_k < \frac{13}{6}$. Since $H_4 = \frac{23}{12} < \frac{13}{6}$ and $H_5 = \frac{137}{60} > \frac{13}{6}$, we get that $\max_{k \geq 2} g(k) = g(5) = \frac{67}{360}$, so we have ratio $1 \frac{67}{360}$.

We now show ratio $\frac{8}{7} < \frac{1555}{1347} < \frac{7}{6}$. As in the greedy algorithm for SET-COVER, we repeatedly remove an inclusion maximal set of disjoint stars with maximum number of leaves and pick the set of roots of these stars. The difference is that each time stars with more than k leaves are exhausted, we compute an α_k -approximate solution A_k for the remaining k -SET-COVER instance; we let $A_0 = \emptyset$. This gives many SET-COVER solutions, each is a union of the centers of stars picked and A_k ; we choose the smallest one, and together with R this gives a feasible ACTIVATION EDGE-COVER solution. Formally, the algorithm is:

■ **Algorithm 2** ratio $\rho = \frac{1555}{1347} < 1.1545$

-
- 1 **for** $k \leftarrow \Delta$ **downto** 0 **do**
 - 2 remove from G a maximal collection of node disjoint $(k + 1)$ -stars
 let C_{k+1} be the set of the roots of the stars removed so far
 - 3 compute an α_k -approximate k -SET-COVER solution A_k in G
 - 4 **return** the smallest set $C_{k+1} \cup A_k$, $k \in \{\Delta, \dots, 0\}$
-

Since we claim ratio $\frac{1555}{1347} > 1.15 > \frac{8}{7}$, at iterations when $k \geq 7$ step 3 can be skipped, since then we can apply a standard “local ratio” analysis [2]. Indeed, when a star with $k \geq 7$ terminals is removed, the partial solution value increases by $k + 1$ while the optimum decreases by at least k . Hence for $k \geq 7$ it is a $\frac{k+1}{k} \leq \frac{8}{7}$ local ratio step. Consequently, we may assume that $\Delta \leq 6$, provided that we do not claim ratio better than $8/7$.

Let $r = |R|$. Let τ be the optimal value to the initial SET-COVER instance. At iteration k the algorithm computes a solution of value at most $\alpha_k \tau + r + |C_{k+1}|$. Thus we get ratio ρ if $\rho(r + \tau) \geq \alpha_k \tau + r + |C_{k+1}|$ holds for some $k \leq 6$. Otherwise,

$$\begin{aligned} \rho(r + \tau) &< \alpha_6 \tau + r \\ \rho(r + \tau) &< \alpha_5 \tau + r + |C_6| \\ \rho(r + \tau) &< \alpha_4 \tau + r + |C_5| \\ \rho(r + \tau) &< \alpha_3 \tau + r + |C_4| \\ \rho(r + \tau) &< \alpha_2 \tau + r + |C_3| \\ \rho(r + \tau) &< \alpha_1 \tau + r + |C_2| \\ \rho(r + \tau) &< r + |C_1| \end{aligned}$$

Denote $\sigma = \alpha_1 + \dots + \alpha_5 = \frac{1581}{240}$. Note that $|C_1| + \dots + |C_6| = r$, since in this sum the number of stars with k leaves is summed exactly k times, $k = 1, \dots, 6$. The first inequality, and the inequality obtained as the sum of the other six inequalities gives the following two

inequalities:

$$\begin{aligned}\rho(r + \tau) &< \alpha_6\tau + r \\ 6\rho(r + \tau) &< \sigma\tau + 7r\end{aligned}$$

Dividing both inequalities by τ and denoting $x = r/\tau$ gives:

$$\begin{aligned}\rho(x + 1) &< \alpha_6 + x \\ 6\rho(x + 1) &< \sigma + 7x\end{aligned}$$

Since $\rho > 1$ and $7 > 6\rho$ this is equivalent to:

$$\frac{6\rho - \sigma}{7 - 6\rho} < x < \frac{\alpha_6 - \rho}{\rho - 1}$$

We obtain a contradiction if ρ is the solution of the equation $\frac{6\rho - \sigma}{7 - 6\rho} = \frac{\alpha_6 - \rho}{\rho - 1}$, namely

$$\rho = \frac{7\alpha_6 - \sigma}{6\alpha_6 - \sigma + 1} = 1 + \frac{\alpha_6 - 1}{6\alpha_6 - \sigma + 1} = 1 + \frac{208}{1347} = \frac{1555}{1347}.$$

This concludes the proof of Theorem 4.

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