

On the Computation of Nash Equilibria in Games on Graphs

Patricia Bouyer 

LSV, CNRS, ENS Paris-Saclay, Université Paris-Saclay, France

<http://www.lsv.fr/~bouyer/>

bouyer@lsv.fr

Abstract

In this talk, I will show how one can characterize and compute Nash equilibria in multiplayer games played on graphs. I will present in particular a construction, called the *suspect game construction*, which allows to reduce the computation of Nash equilibria to the computation of winning strategies in a two-player zero-sum game.

2012 ACM Subject Classification Theory of computation; Theory of computation → Solution concepts in game theory; Theory of computation → Verification by model checking

Keywords and phrases Multiplayer games, Nash equilibria

Digital Object Identifier 10.4230/LIPIcs.TIME.2019.3

Category Invited Talk

Funding Work supported by ERC project EQualIS between 2013 and 2019.

Acknowledgements I would like to thank all my co-authors since I started working on multiplayer games played on graphs, that is, Nicolas Markey, Romain Brenguier [3], Daniel Stan [9], Michael Ummels and Nathan Thomasset.

1 Introduction

Multiplayer concurrent games over graphs allow to model rich interactions between players. Those games are played as follows. In a state, each player chooses privately and independently an action, defining globally a move (one action per player); the next state of the game is then defined as the successor (on the graph) of the current state using that move; players continue playing from that new state, and form a(n infinite) play. Each player then gets a reward given by a payoff function (one function per player). In particular, objectives of the players may not be contradictory: those games are non-zero-sum games, contrary to two-player games used for controller or reactive synthesis [10, 7].

Using solution concepts borrowed from game theory, one can describe the interactions between the players, and in particular describe their rational behaviours. One of the most basic solution concepts is that of Nash equilibria [8]. A Nash equilibrium is a strategy profile where no player can improve her payoff by unilaterally changing her strategy. The outcome of a Nash equilibrium can therefore be seen as a rational behaviour of the system. While very much studied by game theoretists (e.g. over matrix games), such a concept (and variants thereof) has been only rather recently studied over games on graphs. Probably the first works in that direction are [5, 4, 11, 12].

Computing Nash equilibria requires to (i) find a good behaviour of the system; (ii) detect deviations from that behaviour, and identify deviating players (called deviators); (iii) punish them. Variants of Nash equilibria (like subgame-perfect equilibria, robust equilibria, *etc*) require slightly different ingredients, but they are mostly of a similar vein.



© Patricia Bouyer;

licensed under Creative Commons License CC-BY

26th International Symposium on Temporal Representation and Reasoning (TIME 2019).

Editors: Johann Gamper, Sophie Pinchinat, and Guido Sciavicco; Article No. 3; pp. 3:1–3:3

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In this talk, we will first recall some basics of game theory over matrix games. Those games are not sufficient in a verification context: indeed, explicit states are very useful when modelling systems or programs, but are missing in matrix games. However stability notions like Nash equilibria or other solution concepts borrowed from game theory, are very relevant. We will thus present the model of concurrent multiplayer games (played on graphs), which extends in a natural way standard models used in verification with multiplayer interactions. We will explain how Nash equilibria can be characterized and computed in such general games. We will also discuss some existence results for Nash equilibria.

A reference note for this talk is [1]. Related notes are [10], which discussed the use of two-player zero-sum games in verification, and [6], which discussed solution concepts in multiplayer turn-based games on graphs.

To give a taste of the approach, we informally discuss below a simple scenario. The general case with concurrent games and more general payoff functions will be handled by the suspect game abstraction, which somehow generalizes the simple scenario below. For readers not familiar at all with games, you can skip the discussion and wait until the talk.

2 Discussion on a Simple Scenario

We fix a turn-based and deterministic game \mathcal{G} with set of players \mathcal{P} , and we assume that the payoff function for each player $A \in \mathcal{P}$ is given by a Boolean prefix-independent objective ϕ_A (that is, the player gets +1 if the play satisfies ϕ_A , and 0 otherwise). Each player A plays using a (deterministic) *strategy* σ_A , and once a strategy is fixed for every player, we have a *strategy profile* $\sigma = (\sigma_A)_{A \in \mathcal{P}}$. In such a game, the player objective is to make the generated play satisfy her formula. Hence, if the unique outcome of σ satisfies ϕ_A , then player A is satisfied. Otherwise player A will try to be more satisfied by changing her strategy; the new strategy is then called a (single-player) *deviation*. The strategy profile σ will then be a (pure) *Nash equilibrium* if it is resistant to single-player deviations.

In our simple setting, it will be rather easy to characterize deviations that are *profitable* to player A : once a deviation by player A has occurred, we assume that all other players (which we note as a coalition $(\neg A)$) play optimally in σ for the objective $\neg\phi_A$.¹ That way, if the outcome of the strategy does only visit winning states for $(\neg A)$ (which we denote $W_{(\neg A)}$), then no deviation for player A can be profitable; conversely if the outcome visits a winning state for A (by determinacy, this is the negation of the previous case), then there will be a profitable deviation for player A . One can then characterize Nash equilibria as follows:

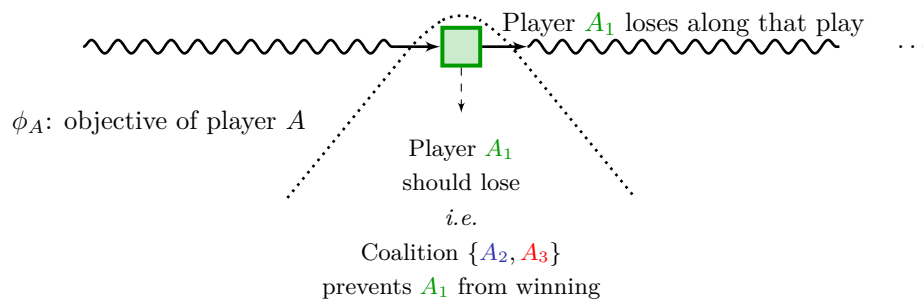
► **Proposition 1.** *Let ρ be an infinite path in \mathcal{G} from the initial vertex v_{init} . Then, $\rho \models \Phi_{NE}$ if and only if there is a Nash equilibrium σ from v_{init} such that the outcome of σ is ρ , where*

$$\Phi_{NE} = \bigwedge_{A \in \mathcal{P}} \left(\neg\phi_A \Rightarrow \mathbf{G}W_{(\neg A)} \right)$$

The situation is illustrated on Figure 1. Note that by determinacy, “Player A_1 should lose” can be replaced by “Coalition $\{A_2, A_3\}$ prevents A_1 from winning”.

A solution to compute Nash equilibria is then to compute for every $A \in \mathcal{P}$ the set $W_{(\neg A)}$ (or equivalently W_A), and to compute an infinite path in the game which satisfies formula Φ_{NE} (which can be done for instance by enumerating the possible set of losing players, and then finding an adequate ultimately periodic play). Obviously, for specific winning conditions, more efficient algorithms can be designed, see for instance [13, 2].

¹ Such a strategy is sometimes called a *threat* or a *trigger* strategy.



■ **Figure 1** General shape of a Nash equilibrium in the simple setting (example with three players A_1 , A_2 and A_3).

References

- 1 Patricia Bouyer. A Note on Game Theory and Verification. In *Proc. 17th International Symposium on Automated Technology for Verification and Analysis (ATVA'19)*, Lecture Notes in Computer Science. Springer, 2019. To appear.
- 2 Patricia Bouyer, Romain Brenguier, Nicolas Markey, and Michael Ummels. Pure Nash Equilibria in Concurrent Games. *Logical Methods in Computer Science*, 11(2:9), 2015.
- 3 Romain Brenguier. *Nash Equilibria in Concurrent Games – Application to Timed Games*. PhD thesis, ENS Cachan, France, 2012.
- 4 Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdziński. Games with Secure Equilibria. *Theoretical Computer Science*, 365(1-2):67–82, 2006.
- 5 Krishnendu Chatterjee, Rupak Majumdar, and Marcin Jurdziński. On Nash Equilibria in Stochastic Games. In *Proc. 18th International Workshop on Computer Science Logic (CSL'04)*, volume 3210 of *Lecture Notes in Computer Science*, pages 26–40. Springer, 2004.
- 6 Erich Grädel and Michael Ummels. Solution Concepts and Algorithms for Infinite Multiplayer Games. In *New Perspectives on Games and Interaction*, volume 4 of *Texts in Logic and Games*, pages 151–178. Amsterdam University Press, 2008.
- 7 Thomas A. Henzinger. Games in system design and verification. In *Proc. 10th Conference on Theoretical Aspects of Rationality and Knowledge (TARK'05)*, pages 1–4, 2005.
- 8 John F. Nash. Equilibrium Points in n -Person Games. *Proceedings of the National Academy of Sciences of the United States of America*, 36(1):48–49, 1950.
- 9 Daniel Stan. *Randomized Strategies in Concurrent Games*. PhD thesis, Université Paris-Saclay, France, 2017.
- 10 Wolfgang Thomas. Infinite Games and Verification. In *Proc. 14th International Conference on Computer Aided Verification (CAV'02)*, volume 2404 of *Lecture Notes in Computer Science*, pages 58–64. Springer, 2002. Invited Tutorial.
- 11 Michael Ummels. Rational Behaviour and Strategy Construction in Infinite Multiplayer Games. In *Proc. 26th Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'06)*, volume 4337 of *Lecture Notes in Computer Science*, pages 212–223. Springer, 2006.
- 12 Michael Ummels. The Complexity of Nash Equilibria in Infinite Multiplayer Games. In *Proc. 11th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'08)*, volume 4962 of *Lecture Notes in Computer Science*, pages 20–34. Springer, 2008.
- 13 Michael Ummels. *Stochastic Multiplayer Games – Theory and Algorithms*. PhD thesis, RWTH Aachen, Germany, 2010.