Strategy-Proof Approximation Algorithms for the Stable Marriage Problem with Ties and Incomplete Lists

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Abstract

In the stable marriage problem (SM), a mechanism that always outputs a stable matching is called a *stable mechanism*. One of the well-known stable mechanisms is the man-oriented Gale-Shapley algorithm (MGS). MGS has a good property that it is strategy-proof to the men's side, i.e., no man can obtain a better outcome by falsifying a preference list. We call such a mechanism a *man-strategy-proof mechanism*. Unfortunately, MGS is not a woman-strategy-proof mechanism. (Of course, if we flip the roles of men and women, we can see that the woman-oriented Gale-Shapley algorithm (WGS) is a woman-strategy-proof but not a man-strategy-proof mechanism.) Roth has shown that there is no stable mechanism that is simultaneously man-strategy-proof and woman-strategy-proof, which is known as Roth's impossibility theorem.

In this paper, we extend these results to the stable marriage problem with ties and incomplete lists (SMTI). Since SMTI is an extension of SM, Roth's impossibility theorem takes over to SMTI. Therefore, we focus on the one-sided-strategy-proofness. In SMTI, one instance can have stable matchings of different sizes, and it is natural to consider the problem of finding a largest stable matching, known as MAX SMTI. Thus we incorporate the notion of approximation ratios used in the theory of approximation algorithms. We say that a stable-mechanism is a *c-approximate-stable mechanism* if it always returns a stable matching of size at least $1/c$ of a largest one. We also consider a restricted variant of MAX SMTI, which we call MAX SMTI-1TM, where only men's lists can contain ties (and women's lists must be strictly ordered).

Our results are summarized as follows: (i) MAX SMTI admits both a man-strategy-proof 2-approximate-stable mechanism and a woman-strategy-proof 2-approximate-stable mechanism. (ii) MAX SMTI-1TM admits a woman-strategy-proof 2-approximate-stable mechanism. (iii) MAX SMTI-1TM admits a man-strategy-proof 1.5-approximate-stable mechanism. All these results are tight in terms of approximation ratios. Also, all these results apply for strategy-proofness against coalitions.

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1 Introduction

An instance of the *stable marriage problem* (*SM*) [5] consists of n men $m_1, m_2, \ldots, m_n$, n women $w_1, w_2, \ldots, w_n$, and each person's preference list, which is a total order of all the members of the opposite gender. If a person q_i precedes a person q_j in a person p's preference list, then we write $q_i \succ_p q_j$ and interpret it as "p prefers q_i to q_j". In this paper, we denote a preference list in the following form:

$$m_2 : w_3 \ w_1 \ w_4 \ w_2,$$

which means that m_2 prefers w_3 best, w_1 second, w_4 third, and w_2 last (this example is for $n = 4$).

A *matching* is a set of n (man, woman)-pairs in which no person appears more than once. For a matching M, $M(p)$ denotes the partner of a person p in M. If, for a man m and a woman w, both $w \succ_m M(m)$ and $m \succ_w M(w)$ hold, then we say that (m, w) is a *blocking pair* for M or (m, w) *blocks* M. Note that both m and w have incentive to be matched with each other ignoring the given partner, so it can be thought of as a threat for the current matching M. A matching with no blocking pair is a *stable matching*. It is known that any instance admits at least one stable matching, and one can be found by the *Gale-Shapley algorithm* (or *GS algorithm* for short) in $O(n^2)$ time [5]. There have been a plenty of research results on this problem from viewpoints of Economics, Computer Science, Mathematics, etc (see [7, 21, 14] e.g.).

1.1 Strategy-Proofness

The stable marriage problem can be seen as a game among participants, who have true preferences in mind, but may submit a falsified preference list hoping to obtain a better partner than the one assigned when true preference lists are used. Formally, let S be a *mechanism*, that is, a mapping from instances to matchings, and we denote $S(I)$ the matching output by S for an instance I. We say that S is a *stable mechanism* if, for any instance I, $S(I)$ is a stable matching for I. For a mechanism S, let I be an instance, M be a matching such that $M = S(I)$, and p be a person. We say that *p has a successful strategy in I* if there is an instance I' in which people except for p have the same preference lists in I and I', and p prefers M' to M (i.e., $M'(p) \succ_p M(p)$ with respect to p's preference list in I), where M' is a matching such that $M' = S(I')$. This situation is interpreted as follows: I is the set of true preference lists, and by submitting a falsified preference list (which changes the set of lists to I'), p can obtain a better partner $M'(p)$. We say that S is a *strategy-proof mechanism* if, when S is used, no person has a successful strategy in any instance. Also we say that S is a *man-strategy-proof mechanism* if, when S is used, no man has a successful strategy in any instance. A *woman-strategy-proof mechanism* is defined analogously. A mechanism is a *one-sided-strategy-proof mechanism* if it is either a man-strategy-proof mechanism or a woman-strategy-proof mechanism.

It is known that there is no strategy-proof stable mechanism for SM [18], which is known as *Roth's impossibility theorem*. By contrast, the man-oriented GS algorithm, *MGS* for short, (in which men send and women receive proposals; see Appendix A) is a man-strategy-proof stable mechanism for SM [18, 2]. Of course, by the symmetry of men and women, the woman-oriented GS algorithm (*WGS*) is a woman-strategy-proof stable mechanism.

1.2 Ties and Incomplete Lists

One of the most natural extensions of SM is the *Stable Marriage with Ties and Incomplete lists*, denoted *SMTI*. An instance of SMTI consists of n men, n women, and each person's preference list. A preference list may include *ties*, which represent indifference between two or more persons, and may be *incomplete*, meaning that a preference list may contain only a *subset* of people in the opposite gender. Such a preference list may be of the following form:

$m_2 : w_3\ (w_1\ w_4)$,

which represents that m_2 prefers w_3 best, w_1 and w_4 second with equal preference, but does not want to be matched with w_2. If a person q is included in p's preference list, we say that q is *acceptable* to p. A *matching* is a set of mutually acceptable (man, woman)-pairs in which no person appears more than once. The *size* of a matching M, denoted $|M|$, is the number of pairs in M. For a matching M, (m, w) is a *blocking pair* if (i) m and w are acceptable to each other, (ii) m is single in M or $w \succ_m M(m)$, and (iii) w is single in M or $m \succ_w M(w)$. A matching without blocking pairs is a *stable matching*. (When ties come into consideration, there are three definitions for stability, *super*, *strong*, and *weak* stabilities. Here we are considering weak stability which is the most natural notion among the three. In the case of super and strong stabilities, there exist instances that do not admit a stable matching. See [7, 14] for more details.)

Note that in the case of SM, the size of a matching is always n by definition, but it may be less than n in the case of SMTI. In fact, there is an SMTI-instance that admits stable matchings of different sizes, and the problem of finding a stable matching of the maximum size, called *MAX SMTI*, is NP-hard [10, 15]. There are a plenty of approximability and inapproximability results for MAX SMTI. The current best upper bound on the approximation ratio is 1.5 [16, 17, 11] and lower bounds are $33/29 \simeq 1.1379$ assuming P≠NP and $4/3 \simeq 1.3333$ assuming the Unique Games Conjecture (UGC) [22]. There are several attempts to obtain better algorithms (e.g., polynomial-time exact algorithms or polynomial-time approximation algorithms with better approximation ratio) for restricted instances; one of the most natural restrictions is to admit ties in preference lists of only one gender, which we call *SMTI-1T*. MAX SMTI-1T (i.e., the problem of finding a maximum cardinality stable matching in SMTI-1T) remains NP-hard, and as for the approximation ratio, the current best upper bound is $1 + 1/e \simeq 1.368$ [13] and lower bounds are $21/19 \simeq 1.1052$ assuming P≠NP and $5/4 = 1.25$ assuming UGC [8, 22].

1.3 Our Contributions

In this paper, we consider the strategy-proofness in MAX SMTI, and investigate the trade-off between strategy-proofness and approximability. In the case of incomplete preference lists, there may be unmatched (i.e., single) persons. Thus, we have to extend the definition of a person preferring one matching to another. We say that a person p prefers M' to M if either $M'(p) \succ_p M(p)$ holds or p is single in M but is matched in M' with some acceptable woman. Then the definition of strategy-proofness for SM naturally takes over to SMTI.

Let I be a MAX SMTI instance and M_{opt} be a maximum size stable matching for I. A stable matching M for I is called an *r-approximate solution for I* if $\frac{|M_{opt}|}{|M|} \le r$. A stable mechanism S is called an *r-approximate-stable mechanism* if $S(I)$ is an r-approximate solution for any MAX SMTI instance I.

Firstly, since SMTI is a generalization of SM, Roth's impossibility theorem for SM [18] holds also for MAX SMTI (regardless of approximability):

▶ **Proposition 1.** *There is no strategy-proof stable mechanism for MAX SMTI.*

Therefore, we focus on *one-sided*-strategy-proofness. We show that there is a 2-approximate-stable mechanism, which is achieved by a simple extension of the GS algorithm. We also show that this result is tight:

▶ **Theorem 2.** *MAX SMTI admits both a man-strategy-proof* 2*-approximate-stable mechanism and a woman-strategy-proof* 2*-approximate-stable mechanism. On the other hand, for any positive ϵ, MAX SMTI admits neither a man-strategy-proof* $(2-\epsilon)$*-approximate-stable mechanism nor a woman-strategy-proof* $(2-\epsilon)$*-approximate-stable mechanism.*

We next consider a restricted version, MAX SMTI-1T. Throughout the paper, we assume that ties appear in men's lists only (and women's lists must be strict). In the following, we use the name *MAX SMTI-1TM* to stress that only men's preference lists may contain ties. As for woman-strategy-proofness, we obtain the same result as for MAX SMTI, which is a direct consequence of Theorem 2:

▶ **Corollary 3.** *MAX SMTI-1TM admits a woman-strategy-proof 2-approximate-stable mechanism, but no woman-strategy-proof* $(2-\epsilon)$*-approximate-stable mechanism for any positive ϵ.*

For man-strategy-proofness, we can reduce the approximation ratio to 1.5, which is the main result of this paper.

▶ **Theorem 4.** *MAX SMTI-1TM admits a man-strategy-proof 1.5-approximate-stable mechanism, but no man-strategy-proof* $(1.5-\epsilon)$*-approximate-stable mechanism for any positive ϵ.*

We remark that no assumptions on running times are made for our negative results, while algorithms in our positive results run in linear time. Note also that the current best polynomial-time approximation algorithms for MAX SMTI and MAX SMTI-1TM have the approximation ratios better than those in our negative results (Theorems 2 and 4). Hence our results provide gaps between polynomial-time computation and strategy-proof computation.

Coalition. In the above discussion, man-strategy-proofness (woman-strategy-proofness) is defined in terms of a manipulation of a preference list by one man (woman). We can extend this notion to a *coalition* of men (or women) as follows; a coalition C of men has a successful strategy if there is a way of falsifying preference lists of members of C which improves the outcome of *every* member of C. It is known that MGS is strategy-proof against a coalition of men in this sense (Theorem 1.7.1 of [7]), and this strategy-proofness holds also in the stable marriage with incomplete lists (SMI) (page 57 of [7]). Since all our strategy-proofness results (Lemmas 5 and 11) are attributed to strategy-proofness of MGS in SMI, we can easily modify the proofs so that Theorem 2, Corollary 3, and Theorem 4 hold for strategy-proofness against coalitions.

Many-to-One Setting. Clearly, the negative parts of Theorem 2, Corollary 3, and Theorem 4 hold for a many-to-one extension of MAX SMTI, denoted *MAX HRT*. Also, we can show that man-strategy-proofness in Theorems 2 and 4 carry over to resident-strategy-proofness in MAX HRT by cloning hospitals (see e.g., page 283 of [9] for cloning). By contrast, woman-strategy-proofness in Theorem 2 and Corollary 3 do not hold for hospital-strategy-proofness in MAX HRT; there is no hospital-strategy-proof stable mechanism even without ties (see Sec. 1.7.3 of [7]).

Overview of Techniques. Since MGS is a man-strategy-proof stable mechanism for SM, such types of algorithms are good candidates for proving the positive part of Theorem 4. Existing 1.5-approximation algorithms for MAX SMTI for one-sided ties are of GS-type, but in these algorithms, proposals are made from the side with no ties (women, in our case), so we cannot use them for our purpose. As mentioned above, there are 1.5-approximation algorithms for the general MAX SMTI [16, 17, 11], which are fortunately of GS-type and can handle proposals from the side with ties (men, in our case). Hence one may expect that these algorithms will work. However, it is not the case. The main reason is as follows: Suppose that some man m is going to propose to a woman, and the head of m's current list is a tie, which is a mixture of unmatched and matched women. In this case, m's proposal will be sent to an unmatched woman, say w. Suppose that, just one step before, another man m' has proposed to w'. Then if m' moves w to the position just before w', he can make w already matched when m is about to propose to her, and as a result of this, m does not propose to w but to another unmatched woman. In this way, a man can change another man's proposal order, which destroys the strategy-proofness (see Appendix B for more details). To overcome it, we modify Király's 1.5-approximation algorithm [11] (or more precisely, the algorithm M-KNA given in Appendix B) to be *robust* in the sense that a man's proposal order is not affected by other men's preference lists.

Ties *or* Incomplete Lists. When only ties are present (SMT) or only incomplete lists are present (SMI), all the stable matchings of one instance have the same cardinality. The former is due to the fact that any stable matching is a perfect matching, and the latter is due to the Rural Hospitals theorem [6, 19, 20]. Hence approximability is not an important issue in these cases. As for strategy-proofness, since SMT and SMI are generalizations of SM, Roth's impossibility theorem holds and no strategy-proof stable mechanism exists. Existence of one-sided strategy-proofness for SMI is already known as we have mentioned in "Coalition" part above, and that for SMT follows directly from Theorem 2.

1.4 Related Work

There are some literature studying trade-offs between approximability and strategy-proofness. Krysta et al. [12] consider to approximate the size of a Pareto optimal matching in the House Allocation problem, where preference lists may include ties. They give upper and lower bounds on the approximation ratio of randomized strategy-proof mechanisms for computing a Pareto optimal matching. Dughmi and Ghosh [3] study the generalized assignment problem (GAP) and its variants. Their objective is to maximize the sum of the values of the assigned jobs. They present a strategy-proof $O(\log n)$-approximate mechanism for the GAP, where n represents the number of jobs.

The following papers discuss strategy-proofness in the stable matching problem with indifference. Erdil and Ergin [4] consider the Hospitals/Residents problem where only hospitals' preference lists may have ties. They consider the algorithm that first breaks ties according to a tie-breaking rule τ and then applies the resident-oriented GS algorithm (let us call this algorithm GS^{τ}). They give an instance and a tie-breaking rule τ such that GS^{τ} does not produce a resident-optimal stable matching. They also show that seeking for a resident-optimal stable matching loses strategy-proofness, that is, no deterministic resident-optimal stable mechanism can be resident-strategy-proof. Abdulkadiroğlu et al. [1] give an evidence to support GS^{τ}. They show that for any tie-breaking rule τ, no resident-strategy-proof mechanism dominates GS^{τ} (with respect to residents).

2 Results for MAX SMTI

In this section, we give a proof of Theorem 2. We start with the positive part:

▶ **Lemma 5.** *MAX SMTI admits both a man-strategy-proof* 2*-approximate-stable mechanism and a woman-strategy-proof* 2*-approximate-stable mechanism.*

Proof. Consider a mechanism S^* that is described by the following algorithm. Given a MAX SMTI instance I, S^* first breaks each tie so that persons in a tie are ordered increasingly in their indices, that is, if q_i and q_j are in the same tie of p's list, then after the tie break $q_i \succ_p q_j$ holds if and only if $i < j$. Let I' be the resulting instance. Its preference lists are incomplete but do not include ties; such an instance is called an *SMI instance*. It then applies MGS modified for SMI [7] to I' and obtains a stable matching M for I'. It is easy to see that M is stable for I. Also it is well-known that in MAX SMTI, any stable matching is a 2-approximate solution [15]. Hence S^* is a 2-approximate-stable mechanism.

We then show that S^* is a man-strategy-proof mechanism. Suppose not. Then there is a MAX SMTI instance I and a man m who has a successful strategy in I. Let J be a MAX SMTI instance in which only m's preference list differs from I, and by using it m obtains a better outcome. Let M_I and M_J be the outputs of S^* on I and J, respectively. Then m prefers M_J to M_I, that is, either (i) $M_J(m) \succ_m M_I(m)$ with respect to m's true preference list in I, or (ii) m is single in M_I and matched in M_J, and $M_J(m)$ is acceptable to m in I. Let I' and J', respectively, be the SMI-instances constructed from I and J by breaking ties in the above mentioned manner. Then M_I and M_J are, respectively, the results of MGS applied to I' and J'. Since I' is the result of tie-breaking of I and m prefers M_J to M_I in I, m prefers M_J to M_I in I'. Note that, due to the tie-breaking rule, the preference lists of people except for m are same in I' and J'. This means that when MGS is used in SMI, m can have a successful strategy in I' (i.e., to change his list to that of J'), contradicting man-strategy-proofness of MGS for SMI (page 57 of [7]).

If we exchange the roles of men and women in S^*, we obtain a woman-strategy-proof 2-approximate-stable mechanism. ◀

We then show the negative part. We remark that ϵ is not necessarily a constant.

▶ **Lemma 6.** *(1) For any positive* ϵ*, there is no man-strategy-proof* $(2-\epsilon)$*-approximate-stable mechanism for MAX SMTI, even if ties appear in only women's preference lists. (2) For any positive* ϵ*, there is no woman-strategy-proof* $(2-\epsilon)$*-approximate-stable mechanism for MAX SMTI, even if ties appear in only men's preference lists.*

Proof. (1) Consider the instance I_1 given in Fig. 1, where m_3's preference list is empty. It is straightforward to verify that I_1 has two stable matchings $M_1 = \{(m_1, w_1), (m_2, w_2)\}$ and $M_2 = \{(m_1, w_2), (m_2, w_3)\}$, both of which are of maximum size.

m_1:	w_2	w_1		w_1:	m_1	
m_2:	w_2	w_3		w_2:	$(m_1$	$m_2)$
m_3:				w_3:	m_2	

Figure 1 A MAX SMTI instance I_1.

Let S be an arbitrary $(2-\epsilon)$-approximate-stable mechanism for MAX SMTI. Since S is a stable mechanism, it must output either M_1 or M_2 on I_1. First suppose that it outputs M_1. Let I'_1 be the instance obtained from I_1 by deleting w_1 from m_1's preference list. Then since

M_2 is still a stable matching for I'_1 and S is a $(2-\epsilon)$-approximate-stable mechanism, S must output a stable matching of size 2. But since M_2 is now the only stable matching of size 2, S outputs M_2 on I'_1. Thus m_1 can obtain a better partner by manipulating his preference list. On the other hand, suppose that S outputs M_2 on I_1. Then let I''_1 be the instance obtained from I_1 by deleting w_3 from m_2's preference list. By a similar argument, S must output M_1 on I''_1 and hence m_2 can obtain a better partner by manipulation. We have shown that, for any $(2-\epsilon)$-approximate-stable mechanism S, some man has a successful strategy in I_1 and hence S is not a man-strategy-proof mechanism.

(2) We use the instance I_2 given in Fig. 2, which is symmetric to I_1. By the same argument as above, we can show that for any $(2-\epsilon)$-approximate-stable mechanism S, some woman has a successful strategy in I_2 and hence S is not a woman-strategy-proof mechanism.

m_1:	w_1		w_1:	m_2	m_1
m_2:	$(w_1$	$w_2)$	w_2:	m_2	m_3
m_3:	w_2		w_3:		

Figure 2 A MAX SMTI instance I_2. ◀

3 Results for MAX SMTI-1TM

Recall that MAX SMTI-1TM is a restriction of MAX SMTI where ties can appear in men's preference lists only. Then Corollary 3 is immediate from Lemma 5 and Lemma 6(2).

We then move to man-strategy-proofness and give a proof for Theorem 4. We start with the negative part:

▶ **Lemma 7.** *For any positive ϵ, there is no man-strategy-proof $(1.5-\epsilon)$-approximate-stable mechanism for MAX SMTI-1TM.*

Proof. The proof goes like that of Lemma 6. Consider the instance I_3 in Fig. 3. I_3 has four matchings of size 3, namely, $M_3 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$, $M_4 = \{(m_1, w_1), (m_2, w_2), (m_3, w_4)\}$, $M_5 = \{(m_1, w_1), (m_2, w_3), (m_3, w_4)\}$, and $M_6 = \{(m_1, w_2), (m_2, w_3), (m_3, w_4)\}$. Among them, M_3 and M_6 are stable (M_4 is blocked by (m_3, w_3) and M_5 is blocked by (m_1, w_2)). Hence any $(1.5-\epsilon)$-approximate-stable mechanism outputs either M_3 or M_6, since a stable matching of size 2 is not a $(1.5-\epsilon)$-approximate solution.

m_1:	w_2	w_1	w_1:	m_1	
m_2:	$(w_2$	$w_3)$	w_2:	m_2	m_1
m_3:	w_3	w_4	w_3:	m_2	m_3
m_4:			w_4:	m_3	

Figure 3 A MAX SMTI-1TM instance I_3.

Consider an arbitrary $(1.5-\epsilon)$-approximate-stable mechanism S for MAX SMTI-1TM, and suppose that S outputs M_3 on I_3. Then if m_1 deletes w_1 from the list, M_6 is the unique maximum stable matching (of size 3); hence S must output M_6 and so m_1 can obtain a better partner w_2. Similarly, if S outputs M_6 on I_3, m_3 can force S to output M_3 by deleting w_4 from the list. In either case, some man has a successful strategy in I_3 and hence S is not a man-strategy-proof mechanism. ◀

Finally, we give a proof for the positive part, which is the main result of this paper.

▶ **Lemma 8.** *There exists a man-strategy-proof* 1.5*-approximate-stable mechanism for MAX SMTI-1TM.*

Proof. We give Algorithm 1 and show that it is a man-strategy-proof 1.5-approximate-stable mechanism by three subsequent lemmas (Lemmas 9–11). Algorithm 1 first translates an SMTI-1TM instance I to an SMI instance I' using Algorithm 2, then applies MGS to I' and obtains a matching M', and finally constructs a matching M of I from M'. The new instance I' contains $2n$ men a_i and b_j $(1 \le i \le n,\ 1 \le j \le n)$ and $2n$ women s_j and t_j $(1 \le j \le n)$ (lines 2 and 3 of Algorithm 2). It is important to note that a man a_i corresponds to a man m_i of I, while a man b_j and two women s_j and t_j correspond to a woman w_j of I. As will be seen later, b_j is definitely matched with s_j or t_j in M', and the other woman (i.e., either s_j or t_j who is not matched with b_j) plays a role of woman w_j of I: If she is single in M', then w_j is single in M. If she is matched with a_i in M', then w_j is matched with m_i in M.

Algorithm 1 An algorithm for MAX SMTI-1TM.

Input: An instance I for MAX SMTI-1TM.
Output: A matching M for I.
1: Construct an SMI instance I' from I using Algorithm 2.
2: Apply MGS to I' and obtain a matching M'.
3: Let $M := \{(m_i, w_j) \mid (a_i, s_j) \in M' \vee (a_i, t_j) \in M'\}$ and output M.

Algorithm 2 Translating instances.

Input: An instance I for MAX SMTI-1TM.
Output: An instance I' for SMI.
1: Let X and Y be the sets of men and women of I, respectively.
2: Let $X' := \{a_i \mid m_i \in X\} \cup \{b_j \mid w_j \in Y\}$ be the set of men of I'.
3: Let $Y' := \{s_j \mid w_j \in Y\} \cup \{t_j \mid w_j \in Y\}$ be the set of women of I'.
4: Each a_i's list is constructed as follows: Consider a tie $(w_{j_1}\ w_{j_2}\ \cdots\ w_{j_k})$ in m_i's list in I. We assume without loss of generality that $j_1 < j_2 < \cdots < j_k$. (If not, just arrange the order, which does not change the instance.) Replace each tie $(w_{j_1}\ w_{j_2}\ \cdots\ w_{j_k})$ by a strict order of $2k$ women $t_{j_1}\ t_{j_2}\ \cdots\ t_{j_k}\ s_{j_1}\ s_{j_2}\ \cdots\ s_{j_k}$. A woman who is not included in a tie is considered as a tie of length one.
5: Each b_j's list is defined as "$b_j : s_j\ \ t_j$".
6: For each j, let $P(w_j)$ be the list of w_j in I, and $Q(w_j)$ be the list obtained from $P(w_j)$ by replacing each man m_i by a_i. Then s_j and t_j's lists are defined as follows:

$$\begin{array}{lll} s_j : & Q(w_j) & b_j \\ t_j : & b_j & Q(w_j) \end{array}$$

We briefly give a high-level idea behind Algorithm 1. Consider an application of MGS to I' at line 2. Since men's proposal order does not affect the outcome, it is convenient to first let b_j propose to his first choice woman s_j for each j. At this moment, there are n pairs (b_j, s_j) $(1 \le j \le n)$. We regard this as an initial state, and as long as (b_j, s_j) is a pair, t_j acts as w_j. At some point, if s_j receives a proposal from some man a_i for the first time, s_j rejects b_j and b_j then proposes to his second choice woman t_j, which is accepted. We regard this as a change of the state, and the role of w_j is taken over to s_j. Once this happens, (b_j, t_j) remains a pair till the end of the algorithm. Recall that at line 4 of Algorithm 2, each man makes two copies of each tie. This is regarded as allowing a man to propose to woman w_j twice, first to t_j and second to s_j.

With these observations in mind, we can see that MGS for I' simulates the following GS-type algorithm for the original MAX SMTI instance I.

- Each free man proposes to a woman from the top of the list. When he encounters a tie T, he proposes to the women in T in a predetermined order (i.e., smaller index first). If he is rejected by all of them, he starts the second sequence of proposals to the women in T in the same order. If he is rejected by all the women in T again, then he proceeds to the next tie.
- Each woman's acceptance/rejection policy is as follows: If two proposals are first proposals, she respects her preference list. Similarly, if both are second proposals, she respects her preference list. If one is a first proposal and the other is a second proposal, she always chooses the second proposal (regardless of her list). Hence, once a woman receives a second proposal of some man, she never accepts a first proposal thereafter.

This algorithm achieves an approximation ratio of 1.5 for MAX SMTI, although we do not prove it here. A beneficial point of this algorithm is that a man's proposal order is predetermined and is not affected by other persons' states. As we explained in Sec. 1.3, absence of this property prevented existing algorithms from being man-strategy-proof.

The reason why we do not use this algorithm directly but translate it to an algorithm using MGS for SMI is to make the proof of man-strategy-proofness simpler; this translation allows us to attribute man-strategy-proofness of Algorithm 1 to that of MGS for SMI, as we did in the proof of Lemma 5.

Now we start formal proofs for the correctness.

▶ **Lemma 9.** *Algorithm 1 always outputs a stable matching.*

Proof. Let M be the output of Algorithm 1 and M' be the matching obtained at line 2 of Algorithm 1. We first show that M is a matching. Since M' is a matching, a_i appears at most once in M', so m_i appears at most once in M. Observe that b_j is matched in M', as otherwise (b_j, t_j) blocks M', contradicting the stability of M' in I'. Hence at most one of s_j and t_j can be matched with a_i for some i, which implies that w_j appears at most once in M. Thus M is a matching.

We then show the stability of M. Since M' is the output of MGS, it is stable in I'. Now suppose that M is unstable in I and there is a blocking pair (m_i, w_j) for M. There are four cases:

Case (i): both m_i and w_j are single. Since m_i is single in M, line 3 of Algorithm 1 implies that a_i is single in M'. Since w_j is single in M, s_j is not matched in M' with anyone in $Q(w_j)$, i.e., s_j is single or matched with b_j. Note that (a_i, s_j) is a mutually acceptable pair because (m_i, w_j) is a blocking pair, and $a_i \succ_{s_j} b_j$ in I' by construction. Thus (a_i, s_j) blocks M', a contradiction.

Case (ii): $w_j \succ_{m_i} M(m_i)$ and w_j is single. Let $M(m_i) = w_k$. Then, by construction of M, $M'(a_i)$ is either s_k or t_k. By construction of I', $w_j \succ_{m_i} w_k$ implies both $s_j \succ_{a_i} s_k$ and $s_j \succ_{a_i} t_k$, and in either case we have that $s_j \succ_{a_i} M'(a_i)$ in I'. Since w_j is single in M, by the same argument as Case (i), s_j is either single or matched with b_j in M'. Hence (a_i, s_j) blocks M'.

Case (iii): m_i is single and $m_i \succ_{w_j} M(w_j)$. Since m_i is single in M, a_i is single in M' by the same argument as Case (i). Let $M(w_j) = m_k$. Then, by construction of M, either s_j or t_j is matched with a_k, and the other is matched with b_j since b_j can never be single as we have seen in an earlier stage of this proof. In particular, $M'(s_j)$ is either a_k or b_j. Note that $m_i \succ_{w_j} m_k$ in $P(w_j)$ implies $a_i \succ_{s_j} a_k$ in $Q(w_j)$, so in either case $a_i \succ_{s_j} M'(s_j)$ in I' due to the construction of s_j's list. Therefore (a_i, s_j) blocks M'.

Case (iv): $w_j \succ_{m_i} M(m_i)$ and $m_i \succ_{w_j} M(w_j)$. By the same argument as Case (ii), we have that $s_j \succ_{a_i} M'(a_i)$ in I'. By the same argument as Case (iii), we have that $a_i \succ_{s_j} M'(s_j)$ in I'. Hence (a_i, s_j) blocks M'. ◀

▶ **Lemma 10.** *Algorithm 1 always outputs a 1.5-approximate solution.*

Proof. Let I be an input, M_{opt} be a maximum stable matching for I, and M be the output of Algorithm 1. We show that $\frac{|M_{opt}|}{|M|} \leq 1.5$. Let $G = (X \cup Y, E)$ be a bipartite (multi-)graph with vertex bipartition X and Y, where X corresponds to men and Y corresponds to women of I. The edge set E is a union of M and M_{opt}, that is, $(m_i, w_j) \in E$ if and only if (m_i, w_j) is a pair in M or M_{opt}. If (m_i, w_j) is a pair in both M and M_{opt}, then E contains two edges (m_i, w_j), which constitute a "cycle" of length two. An edge in E corresponding to M (resp. M_{opt}) is called an M-edge (resp. M_{opt}-edge). Since the degree of each vertex of G is at most 2, each connected component of G is an isolated vertex, a cycle, or a path.

It is easy to see that G does not contain a single M_{opt}-edge as a connected component, since if such an edge (m_i, w_j) exists, then (m_i, w_j) is a blocking pair for M, contradicting the stability of M. In the following, we show that G does not contain, as a connected component, a path of length three $m_i - w_j - m_k - w_\ell$ such that (m_i, w_j) and (m_k, w_ℓ) are M_{opt}-edges and (m_k, w_j) is an M-edge. If this is true, then for any connected component C of G, the number of M-edges in C is at least two-thirds of the number of M_{opt}-edges in C, implying $\frac{|M_{opt}|}{|M|} \leq 1.5$.

Suppose that such a path exists. Note that m_i and w_ℓ are single in M. If $m_i \succ_{w_j} m_k$, then (m_i, w_j) blocks M. Since women's preference lists do not contain ties, we have that $m_k \succ_{w_j} m_i$. If $w_\ell \succ_{m_k} w_j$, then (m_k, w_ℓ) blocks M. If $w_j \succ_{m_k} w_\ell$, then (m_k, w_j) blocks M_{opt}. Hence w_j and w_ℓ are tied in m_k's list. Then by construction of I', (i) $t_\ell \succ_{a_k} s_j$. (Hereafter, referring to Fig. 4 would be helpful. Here, the order of t_j and t_ℓ in a_k's list is uncertain, i.e., it may be the opposite, but this order is not important in the rest of the proof.) Since w_ℓ is single in M, either s_ℓ or t_ℓ is single in M'. If s_ℓ is single in M', then (b_ℓ, s_ℓ) blocks M', a contradiction. Hence (ii) t_ℓ is single in M'. Since $M(m_k) = w_j$, either $M'(a_k) = s_j$ or $M'(a_k) = t_j$ holds. In the former case, (i) and (ii) above imply that (a_k, t_ℓ) blocks M', so assume the latter, i.e., $M'(a_k) = t_j$. Recall from the proof of Lemma 9 that either s_j or t_j is matched with b_j in M', so $M'(s_j) = b_j$. Since (m_i, w_j) is an acceptable pair in I, we have that $a_i \succ_{s_j} b_j$ due to the construction of s_j's list. Since m_i is single in M, a_i is single in M'. Hence (a_i, s_j) blocks M', a contradiction. ◀

a_i: $\cdots$ s_j $\cdots$
b_i: s_i t_i

a_k: $\cdots$ t_j $\cdots$ t_ℓ $\cdots$ s_j $\cdots$
b_k: s_k t_k

a_ℓ: $\cdots$
b_ℓ: s_ℓ t_ℓ

s_j: $\cdots$ a_i $\cdots$ b_j
t_j: b_j $\cdots$ a_k $\cdots$

s_ℓ: $\cdots$ b_ℓ
t_ℓ: b_ℓ $\cdots$

Figure 4 A part of the preference lists of I'.

▶ **Lemma 11.** *Algorithm 1 is a man-strategy-proof mechanism.*

Proof. The proof is similar to that of Lemma 5. Suppose that Algorithm 1 is not a man-strategy-proof mechanism. Then there are MAX SMTI-1TM instances I and J and a man m_i having the following properties: I and J differ in only m_i's preference list, and m_i prefers

M_J to M_I, where M_I and M_J are the outputs of Algorithm 1 for I and J, respectively. Then either (i) $M_J(m_i) \succ_{m_i} M_I(m_i)$ in I, or (ii) m_i is single in M_I and $M_J(m_i)$ is acceptable to m_i in I.

Let I' and J' be the SMI-instances constructed by Algorithm 2. Since I and J differ in only m_i's preference list, I' and J' differ in only a_i's preference list. Let $M_{I'}$ and $M_{J'}$, respectively, be the outputs of MGS applied to I' and J'. In case of (i), we have that $M_{J'}(a_i) \succ_{a_i} M_{I'}(a_i)$ in I', due to line 4 of Algorithm 2 and line 3 of Algorithm 1. In case of (ii), a_i is single in $M_{I'}$ because m_i is single in M_I, and $M_{J'}(a_i)$ is acceptable to a_i in I' because $M_J(m_i)$ is acceptable to m_i in I. This implies that a_i has a successful strategy in I', contradicting man-strategy-proofness of MGS for SMI [7]. ◀

By Lemmas 9, 10, and 11, we can conclude that Algorithm 1 is a man-strategy-proof 1.5-approximate-stable mechanism for MAX SMTI-1TM. ◀

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A The Man-Oriented Gale-Shapley Algorithm

During the course of the algorithm, each person takes one of two states "free" and "engaged". At the beginning, everyone is free and the matching M is initialized to the empty set. At one step of the algorithm, an arbitrary free man m proposes to the top woman w in his current list. If w is free, then m and w are provisionally matched and (m, w) is added to M. If w is engaged and matched with m', then w compares m and m', takes the preferred one, and rejects the other. The rejected man deletes w from the list and becomes (or remains) free. When there is no free man, the matching M is output. The pseudo-code is given in Algorithm 3.

B Non-Strategy-Proofness of Existing 1.5-approximation Algorithms for MAX SMTI-1TM

Király [11] presented a 1.5-approximation algorithm for general MAX SMTI (i.e., ties can appear on both sides), which is named "New Algorithm". We modify it in the following two respects.

1. Men's proposals do not get into the second round.
2. When there is arbitrarity, the person with the smallest index is prioritized.

Ideas behind these modifications are as follows: For item 1, since there is no ties in women's preference lists, executing the second round does not change the result. The role of item 2 is to make the algorithm deterministic, so that the output is a function of an input (as we did in the proof of Lemma 5). For completeness, we give a pseudo-code of the algorithm, denoted M-KNA to stand for "Modified Király's New Algorithm", in Algorithm 4.

Each person takes one of three states, "free", "engaged", and "semi-engaged". Initially, all the persons are free. At lines 5, 10, and 14, man m proposes to woman w. Basically, the procedure is exactly the same as that of MGS. If w is free, then we let $M := M \cup \{(m, w)\}$

Algorithm 3 The man-oriented Gale-Shapley algorithm.

Let $M := \emptyset$ and all people be free.
while there is a free man whose preference list is non-empty **do**
  Let m be any free man.
  Let w be the woman at the top of m's current list.
  if w is free **then**
    Let $M := M \cup \{(m, w)\}$, and m and w be engaged.
  end if
  if w is engaged **then**
    Let m' be w's partner.
    if w prefers m' to m **then**
      Delete w from m's list.
    else
      Let $M := M \cup \{(m, w)\} \setminus \{(m', w)\}$.
      Let m' be free and m be engaged.
      Delete w from m''s list.
    end if
  end if
end while
Output M.

and both m and w be engaged (we say w *accepts* m). If w is engaged to m' (i.e., $(m', w) \in M$) and if $m \succ_w m'$, then we let $M := M \cup \{(m, w)\} \setminus \{(m', w)\}$, m be engaged, and m' be free. We also delete w from m''s preference list (we say w *accepts* m and *rejects* m'). If w is engaged to m' and $m' \succ_w m$, then we delete w from m's preference list (we say w *rejects* m).

There is an exception in the acceptance/rejection rule of a woman, when she receives the first and second proposals. This is actually the key for guaranteeing 1.5-approximation, but this rule is not used in the subsequent counter-example so we omit it here. Readers may consult to the original paper [11] for the full description of the algorithm.

It is already proved that the (original) Király's algorithm always outputs a stable matching which is a 1.5-approximate solution, and it is not hard to see that the same results hold for the above M-KNA for MAX SMTI-1TM. However, as the example in Figures 5 and 6 shows, it is not a man-strategy-proof mechanism.

m_1:	w_2	w_1	w_1:	m_2	m_4	m_1
m_2:	$(w_1$	$w_3)$	w_2:	m_4	m_1	
m_3:	w_3		w_3:	m_2	m_3	
m_4:	w_1	w_2	w_4:			

Figure 5 A counter-example (true lists).

m_1:	w_1	w_2	w_1:	m_2	m_4	m_1
m_2:	$(w_1$	$w_3)$	w_2:	m_4	m_1	
m_3:	w_3		w_3:	m_2	m_3	
m_4:	w_1	w_2	w_4:			

Figure 6 A counter-example (manipulated by m_1).

Algorithm 4 Modified Király's New Algorithm (M-KNA) [11].

Let $M := \emptyset$ and all people be free.
while there is a free man whose preference list is non-empty **do**
  Among those men, let m be the one with the smallest index.
  if the top of m's current preference list consists of only one woman w **then**
    Let m propose to w.
  end if
  if the top of m's current preference list is a tie **then**
    if all the women in the tie are engaged **then**
      Among those women, let w be the one with the smallest index.
      Let m propose to w.
    end if
    if there is a free woman in the tie **then**
      Among those free women, let w be the one with the smallest index.
      Let m propose to w.
    end if
  end if
end while
Output M.

If M-KNA is applied to the true preference lists in Figure 5, the obtained matching is $\{(m_2, w_1), (m_3, w_3), (m_4, w_2)\}$. Suppose that m_1 flips the order of w_1 and w_2 (Figure 6). This time, M-KNA outputs $\{(m_1, w_2), (m_2, w_3), (m_4, w_1)\}$ and m_1 successfully obtains a partner w_2. By proposing to w_1 first, m_1 is able to let m_2 propose to w_3. This allows m_4 to obtain w_1, which prevents m_4 from proposing to w_2. This eventually makes it possible for m_1 to obtain w_2.

We finally remark that the same example shows that the other two 1.5-approximation algorithms [16, 17] (with the tie-breaking rule 2 above) are not man-strategy-proof mechanisms either.