Popular Roommates in Simply Exponential Time

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Abstract

We consider the popular matching problem in a graph $G = (V, E)$ on n vertices with strict preferences. A matching M is *popular* if there is no matching N in G such that vertices that prefer N to M outnumber those that prefer M to N. It is known that it is NP-hard to decide if G has a popular matching or not. There is no faster algorithm known for this problem than the brute force algorithm that could take $n!$ time. Here we show a simply exponential time algorithm for this problem, i.e., one that runs in $O^*(k^n)$ time, where k is a constant.

We use the recent breakthrough result on the maximum number of stable matchings possible in such instances to analyze our algorithm for the popular matching problem. We identify a natural (also, hard) subclass of popular matchings called *truly popular* matchings and show an $O^*(2^n)$ time algorithm for the truly popular matching problem.

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1 Introduction

Consider a matching problem in a graph $G = (V, E)$ on n vertices where each vertex has a strict ranking of its neighbors: such a graph is called a *roommates* instance. Matching M in G is *stable* if M has no blocking edge, i.e., an edge (u, v) such that both u and v prefer each other to their respective assignments in M. Stable matchings need not exist in G and a classical problem here is the *stable roommates* problem, i.e., does G admit a stable matching? There are several polynomial time algorithms [24, 30, 31] to solve this problem.

We consider a more relaxed notion of stability called *popularity*. A vertex u prefers matching M to matching N if either (i) u is matched in M and unmatched in N or (ii) u is matched in both M, N and prefers its partner in M to its partner in N. For any two matchings M_0 and M_1, let $\phi(M_0, M_1)$ be the number of vertices that prefer M_0 to M_1.

▶ **Definition 1.** *A matching M in $G = (V, E)$ is* popular *if $\phi(M, N) \geq \phi(N, M)$ for every matching N, i.e., $\Delta(N, M) \leq 0$ where $\Delta(N, M) = \phi(N, M) - \phi(M, N)$.*

In an election between M and N where vertices cast votes, $\phi(M, N)$ is the number of votes won by M and $\phi(N, M)$ is the number of votes won by N. By definition, a popular matching never loses an election to another matching; thus it is a weak *Condorcet winner* [5, 6] in the corresponding voting instance. Every stable matching in G is also popular [4, 17].

There are roommates instances with no stable matchings but with popular matchings, as shown in Fig. 1. Vertex a prefers b to c while b prefers c to a, and c prefers a to b. The last choice of a, b, c is d and d's preference is $a \succ b \succ c$. This instance has no stable matching, however it has 2 popular matchings $M_1 = \{(a, d), (b, c)\}$ and $M_2 = \{(a, c), (b, d)\}$.

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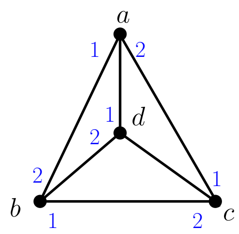

$a : b \succ c \succ d$

$b : c \succ a \succ d$

$c : a \succ b \succ d$

$d : a \succ b \succ c$

Figure 1 An instance with no stable matching, however it has two popular matchings. Numbers on edges indicate their preferences. The vertex d is the last choice of a, b, c and d's last choice is c.

Popular matchings need not always exist in a roommates instance. Consider the above instance without the vertex d. In any matching in the resulting instance, one of a, b, c (each is a top choice neighbor for some vertex) has to be left unmatched. Hence for any matching here, there is a more popular matching.

Popularity is a natural notion of "global stability" and popular matchings may exist in roommates instances with no stable matchings. The *popular roommates* problem is to decide if a given instance $G = (V, E)$ admits a popular matching or not. Unlike stable matchings, it is NP-hard to decide if a roommates instance admits a popular matching or not [13, 18]. There is no faster algorithm known for the popular roommates problem than the brute force algorithm that goes through all matchings in G and tests each for popularity. This algorithm could take $n!$ time. Can a faster algorithm be shown for the popular roommates problem?

1.1 Our results

Our main result is a simply exponential time algorithm for the popular roommates problem. Note that $O^*(k^n)$ denotes $O(k^n \cdot \mathsf{poly}(n))$.

▶ **Theorem 2.** *Given a roommates instance $G = (V, E)$ on n vertices with strict preferences, the popular roommates problem can be solved in $O^*(k^n)$ time, where k is a constant.*

When there is a cost function on the edge set, our algorithm also solves the min-cost popular matching problem. Regarding the constant k in the $O^*(k^n)$ running time, we show that $k \leq 3c$ where c is the constant involved in the recent breakthrough result [25] that showed an upper bound of c^n on the maximum number of stable matchings in a bipartite graph with n vertices on each side. It is known [25, 32] that $c_0 \leq c \leq 2^{17}$ where $c_0 \approx 2.28$.

We also identify a natural subclass of popular matchings called *truly popular* matchings; these are popular matchings that are also popular *fractional* matchings (defined in Section 2). The NP-hardness proof of the popular roommates problem [13] shows that the problem of deciding if a roommates instance admits a truly popular matching or not is NP-hard. We show an algorithm with running time $O^*(2^n)$ for the truly popular matching problem in a roommates instance G on n vertices.

▶ **Theorem 3.** *Given a roommates instance $G = (V, E)$ on n vertices with strict preferences, the problem of deciding whether G admits a truly popular matching or not can be solved in $O^*(2^n)$ time.*

1.2 Background and related results

The notion of popularity was proposed by Gärdenfors [17] in 1975 in bipartite graphs. Popular matchings always exist in bipartite graphs with strict preferences since stable matchings always exist here [16]. During the last 10-15 years, algorithms for popular matchings in bipartite graphs have been well-studied [1, 7, 8, 20, 22, 23, 26, 27, 28]: some of these results are in the domain of *one-sided* popularity, i.e., vertices on only one side of the bipartite graph have preferences.

In comparison, there are not many positive results for popular matchings in non-bipartite graphs. It was shown in [2] that given a matching M, it can be tested in polynomial time whether M is popular or not, even when there are ties in preference lists. It was shown in [21] that every roommates instance G admits a matching with *unpopularity factor* $O(\log n)$.

The popular roommates problem is NP-hard [13, 18]. In a complete graph on n vertices, this problem can be efficiently solved when n is odd, however it is NP-hard for even n [9]. The max-size popular matching problem is NP-hard even in instances with stable matchings (these are min-size popular matchings) [3]. The only known tractable subclasses of popular matchings are the class of stable matchings and the class of *strongly dominant* matchings [13] (a subclass of max-size popular matchings). When G has bounded treewidth, the min-cost popular matching problem can be solved in polynomial time [13].

There is a vast literature on fast exponential time algorithms for NP-hard problems and we refer to the book [15] on this subject. Fast exponential time algorithms for some hard problems in matchings under preferences are known: one such problem is the sex-equal stable marriage problem in bipartite graphs where the objective is to find a "fair" stable matching. When the length of preference lists of vertices on one side of the bipartite graph is bounded from above by a small value, a fast exponential time algorithm for finding a fair stable matching is known [29].

1.3 Our techniques

Let $G = (V, E)$ be the given roommates instance. It follows from LP-duality that every popular matching M in G has a *witness* to its popularity (Section 2 has these details). Witnesses have been used to show several results for popular matchings in bipartite graphs [13, 23, 27, 28]. Witnesses for popular matchings in non-bipartite graphs are more complicated than those in bipartite graphs. In non-bipartite graphs, witnesses have been used in [3, 9, 13] as *certificates* of popularity, i.e., to prove that certain matchings are popular.

In this paper we show a necessary condition for popularity in terms of witnesses. We then use this necessary condition to show a decomposition result for popular matchings: we show that every popular matching can be partitioned into a stable part and a *truly popular* part. Truly popular matchings are a new subclass of popular matchings introduced here and we characterize these matchings in terms of witnesses.

We use this characterization of truly popular matchings to show that every such matching can be realized as a stable matching in one of 2^n new roommates instances. In bipartite graphs, a mapping from a subset of max-size popular matchings to the set of stable matchings in a larger graph was shown in [8]. Our mapping from the set of truly popular matchings to the union of sets of stable matchings in 2^n graphs may be regarded as an extension of this. Our mapping is more complicated than the one in [8].

Organization of the paper. Section 2 discusses preliminaries. Witnesses for popular matchings and our main algorithmic result are in Section 3. Our fast exponential time algorithm for truly popular matchings is in Section 4.

2 Preliminaries

Our input is $G = (V, E)$ on n vertices and m edges where every vertex has a strict preference list ranking its neighbors. It would be convenient to regard every matching in G as a *perfect* matching, hence we augment G with self-loops so that every vertex is its own last choice neighbor. Thus any matching M in G becomes a perfect matching by including self-loops for vertices left unmatched.

Given a (perfect) matching M, consider the following edge weight function. For any edge (u, v) in E:

$$\text{let } \mathsf{wt}_M(u,v) \;=\; \begin{cases} 2 & \text{if } (u,v) \text{ is a blocking edge to } M \\ -2 & \text{if } u \text{ and } v \text{ prefer their respective partners in } M \text{ to each other} \\ 0 & \text{otherwise.} \end{cases}$$

For any edge (u, v), note that $\mathsf{wt}_M(u, v) = \mathsf{vote}_u(v, M(u)) + \mathsf{vote}_v(u, M(v))$, where for any pair of adjacent vertices u and v, $\mathsf{vote}_u(v, M(u))$ is u's vote for v versus $M(u)$: it is 1 if u prefers v to $M(u)$, it is -1 if u prefers $M(u)$ to v, and 0 otherwise, i.e., $v = M(u)$.

For any vertex u, define $\mathsf{wt}_M(u, u) = \mathsf{vote}_u(u, M(u))$ where $\mathsf{vote}_u(u, M(u)) = 0$ if the perfect matching M includes the self-loop (u, u), else $\mathsf{wt}_M(u, u) = -1$. For any perfect matching N, we have:

$$\mathsf{wt}_M(N) = \sum_{(u,v)\in N} \mathsf{wt}_M(u,v) = \sum_{u\in V} \mathsf{vote}_u(N(u), M(u)) = \phi(N,M) - \phi(M,N) = \Delta(N,M).$$

Matching M is popular if and only if $\Delta(N, M) = \mathsf{wt}_M(N) \leq 0$ for all matchings N, i.e., if and only if every perfect matching in G with edge weight function wt_M has weight at most 0. Since $\mathsf{wt}_M(M) = 0$, a max-weight perfect matching has weight exactly 0. The max-weight perfect matching LP in G is described below.

$$\text{maximize} \sum_{e\in E'} \mathsf{wt}_M(e)\cdot x_e \qquad \text{(LP1)}$$

subject to

$$\textstyle\sum_{e\in\delta'(u)} x_e \;=\; 1 \;\; \forall u \in V$$

$$\sum_{e\in E[B]} x_e \;\leq\; \lfloor |B|/2 \rfloor \;\; \forall B \in \Omega \quad \text{and} \quad x_e \;\geq\; 0 \;\; \forall e \in E'.$$

Here E' is the set of edges in the graph G augmented with self-loops and $\delta'(u) = \delta(u) \cup \{(u, u)\}$ is the set of edges incident to u. Also, Ω is the collection of all odd-sized sets $B \subseteq V$ with $|B| \geq 3$. Note that $E[B]$ is the set of edges in E with both endpoints in B and self-loops do not belong to $E[B]$. Consider LP2: this is the dual LP corresponding to LP1.

$$\text{minimize} \sum_{u\in V} \alpha_u \;+\; \sum_{B\in\Omega} \lfloor\, |B|/2 \,\rfloor \cdot z_B \qquad \text{(LP2)}$$

subject to

$$\alpha_u + \alpha_v + \sum_{\substack{B\in\Omega\\ u,v\in B}} z_B \;\geq\; \mathsf{wt}_M(u,v) \quad \forall\, (u,v) \in E$$

$$\alpha_u \;\geq\; \mathsf{wt}_M(u,u) \quad \forall\, u \in V \quad \text{and} \quad z_B \;\geq\; 0 \;\; \forall B \in \Omega.$$

Thus M is popular if and only if the optimal solution to LP2 is 0, i.e., if and only if there exists a feasible solution $(\vec{\alpha}, \vec{z})$ to LP2 such that $\sum_{u\in V} \alpha_u \;+\; \sum_{B\in\Omega} \lfloor\, |B|/2 \,\rfloor \cdot z_B = 0$.

▶ **Definition 4.** *For a popular matching M, an optimal solution $(\vec{\alpha}, \vec{z})$ to LP2 is called a* witness*.*

Popular fractional matchings. Recall that G is augmented with self-loops, so it has $m + n$ edges. A vector $\vec{p} \in \mathbb{R}^{m+n}_{\geq 0}$ such that $\sum_{e \in \delta'(u)} p_e = 1$ for all vertices u is a (perfect) fractional matching in G. The notion of popularity extends to fractional matchings as well. Here we compare an integral matching M with a fractional matching $\vec{p}$ as follows:

$$\Delta(\vec{p}, M) \;=\; \sum_{u \in V} \mathsf{vote}_u(\vec{p}, M) \;=\; \sum_{u \in V} \sum_{v \in \mathsf{Nbr}'(u)} p_{(u,v)} \cdot \mathsf{vote}_u(v, M(u)),$$

where $\mathsf{Nbr}'(u) = \mathsf{Nbr}(u) \cup \{u\}$. Note that $\mathsf{Nbr}(u)$ is the set of u's neighbors in the original graph G (without self-loops).

An integral matching M is a *popular fractional* matching if $\Delta(\vec{p}, M) \leq 0$ for all fractional matchings $\vec{p}$ in G. Every popular matching in G need not be a popular fractional matching. See the instance G in Fig. 2 where vertex preferences are indicated on edges.

Here a is the top choice of b, c, s while b and c are each other's second choices. Vertex a's preference order is $b \succ c \succ s$. Vertex q's order is $r \succ s$ and r's order is $s \succ q$, and s's order is $a \succ q \succ r$.

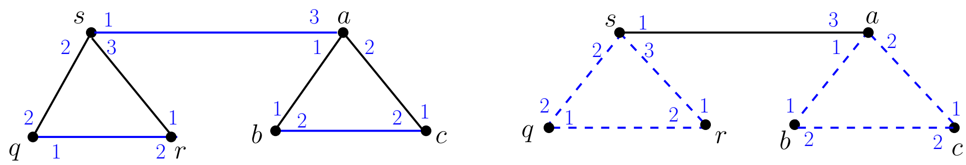

Figure 2 The half-integral matching on the right with a value of 1/2 on the dashed edges is more popular than $M = \{(a, s), (b, c), (q, r)\}$. Note that M is a popular matching.

▷ Claim 5. $M = \{(a, s), (b, c), (q, r)\}$ is popular in G (see Fig. 2), however M is not a popular fractional matching in G.

Proof. We prove the popularity of M via the witness $(\vec{\alpha}, \vec{z})$ where $\alpha_a = \alpha_r = 1$ and $\alpha_b = \alpha_c = \alpha_q = \alpha_s = -1$ along with $z_{\{a,b,c\}} = 2$ and $z_B = 0$ for all other odd sets B. It is easy to check that $(\vec{\alpha}, \vec{z})$ satisfies the constraints in LP2. Also $\sum_u \alpha_u \;+\; \sum_B \lfloor |B|/2 \rfloor z_B = 2 - 4 + 2 = 0$. Thus M is popular in G.

However M is not a popular fractional matching in G. We will show a more popular fractional matching. Consider the half-integral matching $\vec{p}$ indicated on the right in Fig. 2. So $p_e = 1/2$ for $e \in \{(a, b), (b, c), (c, a), (q, r), (r, s), (s, q)\}$. We have $\Delta(\vec{p}, M) = 5/2 - 3/2 = 1$ since $\vec{p}$ gets the vote of a and 1/2-votes of b, c, r while M gets the vote of s and 1/2-vote of q. Thus $\vec{p}$ defeats M, so M is not a popular fractional matching in G. ◁

Hence a popular matching may lose an election against a fractional matching. We introduce the following natural subclass of popular matchings.

▶ **Definition 6.** *A matching M in G is* truly popular *if M is a popular fractional matching.*

Thus M is a truly popular matching if $\Delta(\vec{p}, M) \leq 0$ for all fractional matchings $\vec{p}$. The NP-hardness proof of popular roommates problem in [13] implies that the problem of deciding if a roommates instance G admits a truly popular matching or not is also NP-hard.

Note that a roommates instance may admit popular matchings but *no* truly popular matching. For instance, $M = \{(a, s), (b, c), (q, r)\}$ is the only popular matching in the instance given in Fig. 2 and we know from Claim 5 that M is not truly popular.

3 An algorithm for the popular roommates problem

In this section we show that every popular matching admits a witness with certain structure. This will be used in a structural decomposition result and our algorithm for the popular roommates problem is based on this decomposition.

3.1 Popular matchings and witnesses

In this section we study witnesses for popular matchings. Our first result is the following.

▶ **Lemma 7.** *Let M be a popular matching in G. Then M has a witness $(\vec{\alpha}, \vec{z})$ such that $\vec{\alpha} \in \{0, \pm 1\}^n$ and $z_B \in \{0, 1, 2\}$ for all $B \in \Omega$.*

Proof. Let M be a popular matching in G. Consider LP1 from Section 2: this is the LP for max-weight perfect matching in the graph G augmented with self-loops and with edge weights given by wt_M. Since $\mathsf{wt}_M(M) = \Delta(M, M) = 0$, the characteristic vector of M is an optimal solution to LP1. The constraint system corresponding to LP1 is totally dual integral (TDI) [10]. Thus there is an optimal integral solution $(\vec{\alpha}, \vec{z})$ to the dual LP, i.e., LP2.

We have $\alpha_u \ge \mathsf{wt}_M(u, u) \ge -1$ for all vertices u. Moreover, if $(u, u) \in M$, this constraint is tight by complementary slackness: so $\alpha_u = \mathsf{wt}_M(u, u) = 0$ for such a vertex u. Similarly, for a vertex u matched to a non-trivial neighbor in M (say, $(u, v) \in M$), we have by complementary slackness:

$$\alpha_u + \alpha_v + \sum_{\substack{B \in \Omega \\ u, v \in B}} z_B \;=\; \mathsf{wt}_M(u, v) = 0. \tag{1}$$

Since $z_B \ge 0$ for all B, this means $\alpha_u + \alpha_v \le 0$, so $\alpha_u \le -\alpha_v \le 1$. Hence $\vec{\alpha} \in \{0, \pm 1\}^n$. Let $B \in \Omega$ be such that $z_B > 0$. Then complementary slackness on LP1 implies:

$$\sum_{e \in E[B]} x_e = \lfloor |B|/2 \rfloor. \tag{2}$$

Since $|B| \ge 3$, any $B \in \Omega$ with $z_B > 0$ has at least 1 matched edge in it. Let $(u, v) \in M \cap E[B]$. Then non-negativity of z_B-values and (1) imply that $z_B \le -(\alpha_u + \alpha_v) \le 2$. Hence $z_B \in \{0, 1, 2\}$ for every $B \in \Omega$. ◀

We now characterize truly popular matchings in terms of witnesses.

▶ **Theorem 8.** *A matching M is truly popular iff M has a witness $(\vec{\alpha}, \vec{z})$ such that $\vec{\alpha} \in \{0, \pm 1\}^n$ and $\vec{z} = \vec{0}$.*

Proof. We assume G is augmented with self-loops, so any fractional matching $\vec{p}$ becomes a perfect fractional matching by using self-loops. For any perfect fractional matching $\vec{p}$ in G: (recall that $E' = E \cup \{(u, u) : u \in V\}$)

$$\mathsf{wt}_M(\vec{p}) \;=\; \sum_{e \in E'} p_e \cdot \mathsf{wt}_M(e) \;=\; \sum_{u \in V} \sum_{v \in \mathsf{Nbr}'(u)} p_{(u,v)} \cdot \mathsf{vote}_u(v, M(u)) \;=\; \Delta(\vec{p}, M).$$

Thus M is a popular fractional matching if and only if $\mathsf{wt}_M(\vec{p}) = \Delta(\vec{p}, M) \le 0$ for all fractional matchings $\vec{p}$. Consider LP3 given below. LP3 is the max-weight perfect fractional matching LP in the graph G with edge weight function wt_M. LP4 is the dual of LP3.

Suppose M is a matching in G with a witness $(\vec{\alpha}, \vec{0})$ for some $\vec{\alpha} \in \{0, \pm 1\}^n$. So: (i) $\sum_u \alpha_u = 0$, (ii) $\alpha_v \ge \mathsf{wt}_M(v, v)\ \forall\, v \in V$, and (iii) $\alpha_u + \alpha_v \ge \mathsf{wt}_M(u, v)\ \forall\, (u, v) \in E$.

$$\max \sum_{e \in E'} \mathsf{wt}_M(e) \cdot x_e \qquad \text{(LP3)}$$
$$\text{s.t.} \quad \sum_{e \in \delta'(u)} x_e = 1 \quad \forall\, u \in V$$
$$x_e \geq 0 \quad \forall\, e \in E'.$$

$$\min \sum_{u \in V} \alpha_u \qquad \text{(LP4)}$$
$$\text{s.t.} \quad \alpha_u + \alpha_v \geq \mathsf{wt}_M(u,v) \quad \forall\, (u,v) \in E$$
$$\alpha_u \geq \mathsf{wt}_M(u,u) \quad \forall\, u \in V$$

It follows from properties (ii) and (iii) stated above that $\vec{\alpha}$ is a feasible solution to LP4. It follows from property (i) that the optimal value of LP4 is at most 0. Thus the optimal value of LP3 is at most 0. Since $\mathsf{wt}_M(M) = \Delta(M, M) = 0$, this means that M is an optimal solution to LP3. So $\mathsf{wt}_M(\vec{p}) \leq \mathsf{wt}_M(M) = 0$ for all fractional matchings $\vec{p}$. Thus $\Delta(\vec{p}, M) \leq 0$ for all fractional matchings $\vec{p}$, i.e., M is a popular fractional matching.

Conversely, suppose M is a truly popular matching in G. So M is a popular fractional matching in G. Hence $\Delta(\vec{p}, M) \leq 0$ for all perfect fractional matchings $\vec{p}$, thus $\mathsf{wt}_M(\vec{p}) = \Delta(\vec{p}, M) \leq 0$. Since $\mathsf{wt}_M(M) = 0$, this means M is an optimal solution to LP3.

▷ Claim 9. LP4 has an optimal solution that is integral.

The proof of Claim 9 is given below. Let $\vec{\alpha}$ be an optimal solution of LP4 that is integral. We have $\alpha_u \geq \mathsf{wt}_M(u,u)$ from the constraints. Since $\mathsf{wt}_M(u,u) \geq -1$, we have $\alpha_u \geq -1$ for all vertices u. It follows from complementary slackness conditions that $\alpha_u + \alpha_v = \mathsf{wt}_M(u,v) = 0$ for every edge $(u,v) \in M$. Since $\alpha_v \geq -1$, it follows that $\alpha_u \leq 1$.

It also follows from complementary slackness conditions that $\alpha_u = \mathsf{wt}_M(u,u) = 0$ for every vertex u matched in M along the self-loop (u,u). Thus M has a witness $(\vec{\alpha}, \vec{0})$ such that $\vec{\alpha} \in \{0, \pm 1\}^n$. ◀

Proof of Claim 9. Let $\vec{\alpha}$ be any extreme point of the feasible region of LP4. So we have $A\vec{\alpha} = b$ for some submatrix A of the constraint matrix of LP4. Some of the tight constraints are of the type $\alpha_u = \mathsf{wt}_M(u,u)$: this immediately implies α_u is either 0 or -1, i.e., these coordinates in $\vec{\alpha}$ are integral. Let us remove these constraints from $A\vec{\alpha} = b$, so we have $A'\vec{\alpha}' = b'$ where all the constraints are of the type $\alpha_u + \alpha_v = \mathsf{wt}_M(u,v)$ for $(u,v) \in E$. So $\vec{\alpha}' = A'^{-1} \cdot b'$.

It is easy to see that all entries in A'^{-1} are half-integral. This follows from the fact that the fractional matching polytope of G is half-integral: this is due to the integrality of the fractional matching polytope in bipartite graphs (Birkhoff-von Neumann theorem).

Since $\mathsf{wt}_M(e) \in \{0, \pm 2\}$ for every $e \in E$, every entry in b' is an even integer. Hence $\vec{\alpha}' = A'^{-1} \cdot b'$ is an integral vector. Thus $\vec{\alpha}$ is integral. ◁

Hence M is a truly popular matching if and only if M has a witness $(\vec{\alpha}, \vec{0})$ such that $\vec{\alpha} \in \{0, \pm 1\}^n$. For the sake of brevity, we will say $\vec{\alpha}$ is a witness of M.

3.2 A decomposition result for popular matchings

The following theorem shows that every popular matching in G can be partitioned into a stable part and a *truly popular* part. This decomposition resembles a result from [8] that shows that every popular matching M in a bipartite graph can be decomposed into a stable part and a *dominant*[1] part.

[1] A popular matching N is dominant if N is more popular than any larger matching.

▶ **Theorem 10.** *Let M be a popular matching in $G = (V, E)$. Then $M = M_0 \cup M_1$ such that*
1. *M_0 is stable in the subgraph induced on some subset $C \subseteq V$;*
2. *M_1 is truly popular in the subgraph induced on $V \setminus C$.*

Proof. We know from Lemma 7 that every integral witness $(\vec{\alpha}, \vec{z})$ of a popular matching M satisfies $\vec{\alpha} \in \{0, \pm 1\}^n$ and $z_B \in \{0, 1, 2\}$ for all $B \in \Omega$. Let $(\vec{\alpha}, \vec{z})$ be an integral witness of M such that the sets B with $z_B > 0$ form a laminar family $\mathcal{B}$. The primal-dual algorithm of Edmonds [12] shows that M has such a witness.

Let $B_1, \ldots, B_k$ be the maximal sets in $\mathcal{B}$. We know from (2) that each $B_i \in \mathcal{B}$ has $\lfloor |B_i|/2 \rfloor$ edges of M within it. For $1 \le i \le k$, let b_i be the lone vertex in B_i that is *not* matched to a vertex inside B_i. That is, every vertex in $B_i \setminus \{b_i\}$ is matched in M to another vertex in $B_i \setminus \{b_i\}$. Let C denote the vertex set $\cup_{i=1}^{k} (B_i \setminus \{b_i\})$.

Let M_0 be the matching M restricted to the subgraph induced on C and let M_1 be the matching M restricted to the subgraph induced on $V \setminus C$. Observe that $M = M_0 \cup M_1$. Claim 11 and Claim 12 show that M_0 and M_1 are what we seek.

▷ Claim 11. The matching M_0 is stable in the subgraph induced on C.

▷ Claim 12. The matching M_1 is truly popular in the subgraph induced on $V \setminus C$.

Claim 11 and Claim 12 are proved below. This finishes the proof of Theorem 10. ◀

Proof of Claim 11. We will prove the stability of M_0 by showing that no edge with both endpoints in C blocks M. Consider any edge (u, v) with $u, v \in C$. We know that $\alpha_u + \alpha_v + \sum_{\substack{B \in \mathcal{B} \\ u,v \in B}} z_B \ge \mathsf{wt}_M(u, v)$.

$\mathcal{B}$ is a laminar family. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the collection of sets with both u and v. We need to bound $\sum_{B \in \mathcal{B}'} z_B$. Let B' be the minimal set in $\mathcal{B}'$. It follows from (2) that the partner of at least one of u, v (say, u) is in B' and hence in every set in $\mathcal{B}'$. So we can use (1) for the pair $u, M(u)$ to bound $\sum_{B \in \mathcal{B}'} z_B$. Since $\alpha_u, \alpha_{M(u)} \ge -1$, we have $\sum_{B \in \mathcal{B}'} z_B \le 2$.

The definition of C implies that every vertex $x \in C$ is matched in M to another vertex $M(x)$ in C. Moreover there is some $B_i \in \mathcal{B}$ such that $x, M(x) \in B_i$. Thus $\sum_{B \in \mathcal{B}: x, M(x) \in B} z_B$ is at least 1 and so $\alpha_x + \alpha_{M(x)} \le -1$ by (1). Hence α_x is in $\{0, -1\}$ for every $x \in C$.

Suppose $\alpha_u = 0$. Then $\alpha_u + \alpha_{M(u)} + \sum_{B: u, M(u) \in B} z_B = \mathsf{wt}_M(u, M(u)) = 0$ along with $\alpha_u = 0$ and $\alpha_{M(u)} \ge -1$ implies that $\sum_{B: u, M(u) \in B} z_B \le 1$. Since this sum is integral and positive, it equals 1. So $\mathsf{wt}_M(u, v) \le 1$ in this case. Similarly, when $\alpha_u = -1$, $\mathsf{wt}_M(u, v) \le \alpha_u + \alpha_v + \sum_{B \in \mathcal{B}'} z_B \le -1 + 0 + 2 = 1$. Hence in both cases, $\mathsf{wt}_M(u, v) \le 0$ (since it is in $\{0, \pm 2\}$).

So there is no blocking edge to M with both endpoints in C. Thus M_0 is stable in the subgraph induced on C. ◁

Proof of Claim 12. Let $(\vec{\alpha}, \vec{z})$ be M's witness using which C was defined. We claim $(\vec{\alpha}, \vec{0})$ is a witness for M_1 in the subgraph induced on $V \setminus C$. So we need to show that $\sum_{u \in V \setminus C} \alpha_u = 0$ and $\alpha_u + \alpha_v \ge \mathsf{wt}_{M_1}(u, v) = \mathsf{wt}_M(u, v)$ for every edge (u, v) in this subgraph. We already know that $\alpha_u \ge \mathsf{wt}_{M_1}(u, u) = \mathsf{wt}_M(u, u)$ for all $u \in V$.

- We have $\alpha_u + \alpha_v + \sum_{B: u, v \in B} z_B \ge \mathsf{wt}_M(u, v)$ for every edge (u, v) in G. There is no $B \in \mathcal{B}$ that contains two vertices in $V \setminus C$. Thus $\sum_{B: u, v \in B} z_B = 0$ and so we have the desired constraint $\alpha_u + \alpha_v \ge \mathsf{wt}_M(u, v)$ for every edge (u, v) in this subgraph.
- For any vertex $u \in V \setminus C$ that is matched in M, its partner $M(u) = v$ is also in $V \setminus C$ and we have $\alpha_u + \alpha_v = \mathsf{wt}_M(u, v) = 0$ by complementary slackness (see (1)). For any vertex u matched in M along its self-loop, $\alpha_u = \mathsf{wt}_M(u, u) = 0$. Thus $\sum_{u \in V \setminus C} \alpha_u = 0$. ◁

The proof of Theorem 10 allows us to show a more structured partition of popular matchings as stated in Lemma 13 below. Call a truly popular matching M *special* if M admits a witness $\vec{\alpha} \in \{\pm 1\}^n$.

▶ **Lemma 13.** *Let M be a popular matching in $G = (V, E)$. Then $M = M'_0 \cup M'_1$ where M'_0 is a stable matching in the subgraph induced on some $U \subseteq V$ and M'_1 is a special truly popular matching in the subgraph induced on $V \setminus U$.*

Proof. We will use Theorem 10 here. Let $\mathcal{B} \subseteq \Omega$, $C \subseteq V$, and $\vec{\alpha} \in \{0, \pm 1\}^n$ be as defined in the proof of Theorem 10. Let $U = C \cup \{u \in V \setminus C : \alpha_u = 0\}$.

Let M'_0 be the matching M restricted to the subgraph induced on U. Since $U \supseteq C$, we have $M'_0 \supseteq M_0$, where M_0 was defined in Theorem 10. We claim M'_0 is stable in the subgraph induced on U. It follows from the proofs of Claim 11 and Claim 12 that there is no blocking edge (u, v) to M'_0 where both $u, v \in C$ or both $u, v \in U \setminus C$ (in this case $\alpha_u = \alpha_v = 0$). So what we need to show now is that there is no blocking edge (u, v) to M'_0 where $u \in C$ and $v \in U \setminus C$.

If there is no $B \in \mathcal{B}$ such that $u, v \in B$ then $\mathsf{wt}_M(u, v) \le \alpha_u + \alpha_v \le 0$. Suppose there is some $B \in \mathcal{B}$ with $u, v \in B$. It follows from (2) and the definition of C that u and its partner $M(u)$ are in B. We know from the proof of Claim 11 that either (i) $\alpha_u = -1$ or (ii) $\alpha_u = 0$ and $\sum_{B: u, M(u) \in B} z_B \le 1$. Since $\alpha_v = 0$, this means that $\alpha_u + \alpha_v + \sum_{B: u, v \in B} z_B \le 1$. So $\mathsf{wt}_M(u, v) \le 1$, i.e., $\mathsf{wt}_M(u, v) \le 0$ (since it is even). Thus (u, v) does *not* block M.

So M'_0 is stable in the subgraph induced on U. Let M'_1 be the matching M restricted to the subgraph induced on $V \setminus U$. It follows from the definition of U that M'_1 has a witness $\vec{\alpha}$ where $\alpha_u \in \{\pm 1\}$ for all $u \in V \setminus U$. Hence M'_1 is a special truly popular matching in the subgraph induced on $V \setminus U$. ◀

3.3 Our algorithm

We present our algorithm for the popular roommates problem. The input is $G = (V, E)$.

1. For each $U \subseteq V$ do:
   a. For each stable matching S in the subgraph induced on U do:
   b. For each special truly popular matching T in the subgraph induced on $V \setminus U$ do:
      - If $S \cup T$ is popular in G then return $S \cup T$.
2. Return "G has no popular matching".

A matching M can be tested for popularity via LP1 (see Section 2). There are also combinatorial algorithms [2, 22] to check if a given matching in a roommates instance is popular or not. Lemma 13 shows that every popular matching M admits a decomposition as $M = S \cup T$ where S is stable in some subgraph and T is a special truly popular matching in the remaining part of G. Thus if no matching of the form $S \cup T$ is popular then G has no popular matching. This proves the correctness of our algorithm.

Implementation. All stable matchings in the graph $G_U = (U, E')$ induced on U can be listed by enumerating all stable matchings in the bipartite graph $G'_U = (U' \cup U'', E'')$ [11] where $U' = \{u' : u \in U\}$ and $U'' = \{u'' : u \in U\}$; for every edge (u, v) in G_U, there are 2 edges (u', v'') and (v', u'') in G'_U. Preferences in G'_U are inherited from G_U. Every matching in the bipartite graph G'_U becomes a half-integral matching in the given graph G_U.

It is known how to enumerate all stable matchings in a bipartite graph in $O^*(s)$ time where s is the number of stable matchings in this bipartite graph [19]. It was recently shown [25] that the maximum number of stable matchings possible in a bipartite graph with n vertices on each side is c^n for some constant c. Thus in $O^*(c^n)$ time we can enumerate all stable matchings in a roommates instance on n vertices.

We bound the running time of our algorithm via the following bound on the number of "special truly popular" matchings present in a roommates instance. Here c is the constant from [25] that was used in the paragraph above.

▶ **Lemma 14.** *A roommates instance H on t vertices has at most $(2c)^t$ special truly popular matchings.*

The proof of Lemma 14 shows that every special truly popular matching in H can be realized as a stable matching in one of 2^t roommates instances, each on t vertices. This proof is given in Section 4.1.

Running time of our algorithm. The total number of candidate matchings tested by our algorithm is at most:

$$\sum_{i=0}^{n}\binom{n}{i}\cdot c^i\cdot(2c)^{n-i} \quad = \quad c^n\cdot\sum_{i=0}^{n}\binom{n}{i}2^{n-i} \quad = \quad (3c)^n.$$

In the summation above, c^i is the bound on the number of stable matchings in the subgraph G_U induced on U (where $|U| = i$) and the second term, which is $(2c)^{n-i}$, is the bound on the number of special truly popular matchings in the subgraph G_W induced on $W = V \setminus U$ (note that $|V \setminus U| = n - i$). This proves Theorem 2 stated in Section 1.

4 Truly popular matchings

In this section we use the characterization of truly popular matchings from Theorem 8 to show a fast exponential time algorithm for the problem of deciding if G admits a truly popular matching or not. Our algorithm goes through all $S \subseteq V$ and checks if there is a popular matching in G with a witness $\vec{\alpha}$ such that $\alpha_v = 0$ for all $v \in S$ and $\alpha_v \in \{\pm 1\}$ for all $v \in V \setminus S$. So the problem we look to efficiently solve is:

∗ given $S \subseteq V$, is there a truly popular matching in G with a witness $\vec{\alpha} \in \{0, \pm 1\}^n$ such that $\alpha_v = 0$ if and only if $v \in S$.

We will now show an efficient algorithm for the above problem. We solve this problem by posing it as a stable roommates problem with forbidden edges, which can be solved in linear time [14]. Given any subset $S \subseteq V$, we will construct a new roommates instance $G_S = (V_S, E_S)$ as follows. The vertex set $V_S = \{u_0 : u \in S\} \cup \{u_-, u_+, \ell(u) : u \in V \setminus S\}$.

The vertex $\ell(u)$ will be called a *dummy vertex* as its purpose is to ensure that only *one* of u_+, u_- can be matched to a non-dummy neighbor, i.e., an element in $\{v_+, v_0, v_- : v \in \mathsf{Nbr}(u)\}$. The edge set E_S consists of the following edges:

- For every $(u, v) \in E$ where $u, v \in S$: the edge $(u_0, v_0) \in E_S$.
- For every $(u, v) \in E$ where $u \in V \setminus S$ and $v \in S$: the edge $(u_+, v_0) \in E_S$.
- For every $(u, v) \in E$ where $u, v \in V \setminus S$: if u prefers v to every neighbor in S then $(u_-, v_+) \in E_S$.

Also, for every vertex $u \in V \setminus S$: the edges $(u_+, \ell(u))$ and $(u_-, \ell(u))$ are in E_S. The preference order of vertices in V_S is as follows.

1. For any dummy vertex $\ell(u)$: the order is $u_+ \succ u_-$.
2. For any subscript 0 vertex u_0: the order among its neighbors is as per u's original preference order in G. Suppose u's preference order in G is: $a \succ b \succ c \succ d$ where $a, c \in S$ and $b, d \in V \setminus S$, then u_0's neighbors in G_S are a_0, b_+, c_0, d_+ and u_0's preference order is: $a_0 \succ b_+ \succ c_0 \succ d_+$.

3. For any subscript $+$ vertex u_+: the order among its neighbors in G_S is as per u's preference order in G with $\ell(u)$ as its least preferred vertex.
4. For any subscript $-$ vertex u_-: the order among its neighbors is $\ell(u)$ as its top choice followed by its other neighbors in G_S as per u's preference order in G.

The following theorem shows the equivalence we need.

▶ **Theorem 15.** *The instance G admits a truly popular matching with a witness $\vec{\alpha}$ where $\alpha_u = 0$ for $u \in S$ and $\alpha_v \in \{\pm 1\}$ for $v \in V \setminus S$ iff G_S has a stable matching M_S with the following properties:*
1. *M_S avoids all edges between a subscript 0 vertex and a subscript $+$ vertex;*
2. *M_S matches all subscript $-$ vertices.*

Proof. Suppose G admits a truly popular matching T_S with such a witness $\vec{\alpha}$. We will show a desired stable matching M_S in G_S. For any vertex u, let $s_u = +/-/0$ corresponding to $\alpha_u = +1/-1/0$, respectively. For any vertex $u \in V \setminus S$, we have $\alpha_u \in \{\pm 1\}$ and so $s_u \in \{\pm\}$; if $s_u = +$ then let $t_u = -$, else let $t_u = +$.

Let $M_S = \{(u_{s_u}, v_{s_v}) : (u,v) \in T_S\} \cup \{(u_{t_u}, \ell(u)) : u \in V \setminus S\}$.

▷ Claim 16. $M_S \subseteq E_S$, i.e., for every (u,v) in T_S, the edge (u_{s_u}, v_{s_v}) is present in G_S.

Proof. Since T_S is truly popular, the characteristic vector of T_S is an optimal solution of LP3. We also know that $\vec{\alpha}$ is an optimal solution of LP4. It follows from complementary slackness conditions on LP3 and LP4 that for every edge $(u,v) \in T_S$, $\alpha_u + \alpha_v = \mathsf{wt}_{T_S}(u,v)$. Since $\mathsf{wt}_{T_S}(u,v) = 0$ for any edge $(u,v) \in T_S$, either $\alpha_u = \alpha_v = 0$ or $\{\alpha_u, \alpha_v\} = \{-1, 1\}$. So every edge in M_S that is not incident to any ℓ-vertex is of the type either (u_0, v_0) or (u_+, v_-).

For every edge (u,v) in G where $\alpha_u = \alpha_v = 0$, the edge (u_0, v_0) is in G_S. Consider an edge (u,v) in T_S where $\alpha_u = -1$. We need to show that (u_-, v_+) is in G_S. Since $\vec{\alpha}$ is a witness of T_S, we have $\mathsf{wt}_{T_S}(u,r) \le \alpha_u + \alpha_r = -1 + 0 = -1$ for every neighbor $r \in S$. Since $\mathsf{wt}_{T_S}(e) \in \{0, \pm 2\}$ for all $e \in E$, this means $\mathsf{wt}_{T_S}(u,r) = -2$, i.e., u prefers its partner in T_S (this is v) to r. Since this constraint holds for every $r \in S \cap \mathsf{Nbr}(u)$, it follows from the definition of E_S that $(u_-, v_+) \in E_S$. ◁

We next show that M_S obeys properties (1) and (2) given in the lemma statement.

(1) Since every edge in M_S that is not incident to any ℓ-vertex is of the type either (u_+, v_-) or (u_0, v_0), M_S avoids all edges between a subscript 0 vertex and a subscript $+$ vertex.

(2) For any vertex u unmatched in T_S, we have (by complementary slackness) $\alpha_u = \mathsf{wt}_{T_S}(u,u) = 0$, i.e., $u \in S$. Thus for every $u \in V \setminus S$, we have $(u,v) \in T_S$ for some $v \in \mathsf{Nbr}(u)$; if $\alpha_u = -1$ then $(u_-, v_+) \in M_S$ else $(u_-, \ell(u)) \in M_S$. Thus all vertices in $\{u_- : u \in V \setminus S\}$ are matched in M_S.

In order to show M_S is a desired stable matching in G_S, we need to show this claim.

▷ Claim 17. M_S is a stable matching in G_S.

Proof. By the definition of M_S, the vertices $\ell(u)$ for all $u \in V \setminus S$ are matched in M_S. Thus for any $u \in V \setminus S$, all of $u_+, u_-, \ell(u)$ are matched in M_S, so neither $(u_+, \ell(u))$ nor $(u_-, \ell(u))$ blocks M_S. Other than edges incident to dummy vertices, the graph G_S consists of edges of the type $(u_+, v_-), (u_0, v_0), (u_+, v_0)$, i.e., $\{\alpha_u, \alpha_v\}$ is one of $\{1, -1\}, \{0, 0\}, \{1, 0\}$.

So for every $(u,v) \in E$ such that (u_{s_u}, v_{s_v}) is in G_S, we have $\alpha_u + \alpha_v \le 1$, i.e., $\mathsf{wt}_{T_S}(u,v) \le 1$ which means $\mathsf{wt}_{T_S}(u,v) \le 0$. The constraint $\mathsf{wt}_{T_S}(u,v) \le 0$ implies one of the 3 possibilities: (i) $(u,v) \in T_S$, (ii) u prefers $T_S(u)$ to v, (iii) v prefers $T_S(v)$ to u. In case (i), we have $(u_{s_u}, v_{s_v}) \in M_S$ and in cases (ii) and (iii), one of u_{s_u}, v_{s_v} is matched in M_S to a more preferred neighbor in G_S. Thus M_S is a stable matching in G_S. ◁

Conversely, suppose G_S admits such a stable matching M_S. We will show a truly popular matching T_S in G with a desired witness $\vec{\alpha}$. The matching T_S is easy to define:

$$T_S = \{(u,v) : (u_0, v_0) \in M_S \text{ or } (u_+, v_-) \in M_S\}.$$

We now need to show that T_S is a truly popular matching in G. For this, we will show a witness $\vec{\alpha} \in \{0, \pm 1\}^n$. Define $\alpha_u = 0$ for all $u \in S$. We will now define α_u for each $u \in V \setminus S$.

For each $u \in V$, note that $\ell(u)$ is top choice for u_-: hence $\ell(u)$ always has to be matched in any stable matching in G_S. For each $u \in V \setminus S$:

$$\text{let } \alpha_u = \begin{cases} -1 & \text{if } (u_+, \ell(u)) \in M_S \\ 1 & \text{if } (u_-, \ell(u)) \in M_S. \end{cases}$$

Observe that all edges in M_S not involving any ℓ-vertex are of the form either (u_+, v_-) or (u_0, v_0). This is because M_S avoids all edges of the type (u_+, v_0) by property (1) of a desired stable matching. Thus $\alpha_u + \alpha_v = 0$ for all $(u,v) \in T_S$.

▷ Claim 18. For any vertex u left unmatched in T_S, we have $u \in S$, i.e., $\alpha_u = 0$.

Proof. Every vertex of the form u_+ (being the top choice vertex of $\ell(u)$) has to be matched in every stable matching in G_S; also, all vertices in $\{u_- : u \in V \setminus S\}$ are matched in M_S by property (2). Hence M_S matches u_+, u_- for all $u \in V \setminus S$; thus one of u_+, u_- has to be matched to a non-dummy neighbor, i.e., a vertex other than $\ell(u)$. Hence for any vertex u left unmatched in T_S, we have $u \in S$. ◁

We have $\sum_{u \in V} \alpha_u = \sum_{(u,v) \in T_S} (\alpha_u + \alpha_v)$ from Claim 18 and by definition, $\alpha_u + \alpha_v = 0$ for each $(u,v) \in T_S$. Hence $\sum_{u \in V} \alpha_u = 0$. Every vertex in $V \setminus S$ is matched in T_S (by Claim 18) and so we have $\alpha_u \ge -1 = \mathsf{wt}_{T_S}(u,u)$ for $u \in V \setminus S$. For any vertex $u \in S$, we have $\alpha_u = 0 \ge \mathsf{wt}_{T_S}(u,u)$. Thus $\alpha_u \ge \mathsf{wt}_{T_S}(u,u)$ for every vertex u.

It can also be shown that $\alpha_u + \alpha_v \ge \mathsf{wt}_{T_S}(u,v)$ for every edge (u,v) in G. Thus T_S is a truly popular matching in G and the theorem follows. ◀

All stable matchings in a roommates instance match the same subset of vertices [19]. Call these vertices *stable*. Our algorithm for deciding if G admits a truly popular matching (and returning one, if so) is as follows:

1. For each set $S \subseteq V$ do:
   - Build the graph G_S and check if (i) all subscript $-$ vertices are stable in G_S and (ii) G_S admits a stable matching M_S that satisfies property 1 given in Theorem 15; if so, then return the corresponding matching T_S in G.
2. Return "no".

If our algorithm returns a matching T_S in Step 1, then T_S is truly popular (by Theorem 15). Suppose the algorithm reaches Step 2: so there is no $S \subseteq V$ such that G_S admits a stable matching that satisfies property 1. Then G has no truly popular matching (by Theorem 15). Thus the correctness of our algorithm follows from Theorem 15.

Step 1, part (i) is implemented by running a stable matching algorithm (say, [24]) in G_S. Step 1, part (ii) is implemented by running the algorithm for finding a stable matching in a roommates instance with forbidden edges [14]. Since there are 2^n sets $S \subseteq V$, the running time of our algorithm is $O^*(2^n)$. Thus we have shown Theorem 3 stated in Section 1.

4.1 Proof of Lemma 14

We bound the number of special truly popular matchings in a graph H by bounding the number of stable matchings in some related graphs that we construct below. Let S_α be the set of special truly popular matchings in H with a specific witness $\vec{\alpha} \in \{\pm 1\}^t$, where t is the number of vertices in H. Define $\sigma \in \{\pm\}^t$ as follows: $\sigma_u = \mathsf{sign}(\alpha_u)$ for all vertices u in H where $\mathsf{sign}(\alpha_u) = +$ if $\alpha_u = 1$, else $\mathsf{sign}(\alpha_u) = -$.

Corresponding to $\sigma \in \{\pm\}^t$, we build the graph H_σ as follows. The vertex set of H_σ is $\{u_{\sigma_u} : u \text{ is a vertex in } H\}$. For each edge (u, v) in H where $\sigma_u = -$ and $\sigma_v = +$ do:

- if u prefers v to all its neighbors w in H with $\sigma_w = -$ then add the edge (u_-, v_+) to H_σ.

For any vertex u_{σ_u} in H_σ: u_{σ_u}'s preference order of neighbors in H_σ is as per u's preference order in H. Note that for any neighbor v_{σ_v} of u_{σ_u} in H_σ, we have $\sigma_v = +$ if $\sigma_u = -$ and vice-versa. This is because the edge set of H_σ consists only of edges of the type (a_-, b_+).

For each $M \in S_\alpha$, define $f_\alpha(M) = \{(u_{\sigma_u}, v_{\sigma_v}) : (u, v) \in M\}$. We show in Claim 19 below that for every $(u, v) \in M$, the edge $(u_{\sigma_u}, v_{\sigma_v})$ is in H_σ. Thus $f_\alpha(M)$ is a matching in H_σ. Moreover, $f_\alpha(M)$ is a stable matching in H_σ (see Claim 20). Note that f_α is one-to-one. Hence the total number of special truly popular matchings in H is at most the the maximum number of stable matchings in H_σ summed up over all $\sigma \in \{\pm\}^t$, or equivalently, over all $\vec{\alpha} \in \{\pm 1\}^t$. This sum is at most $c^t \cdot 2^t = (2c)^t$. ◀

▷ Claim 19. For every $(u, v) \in M$, the edge $(u_{\sigma_u}, v_{\sigma_v})$ is in H_σ.

Proof. We have $\alpha_u + \alpha_v = \mathsf{wt}_M(u, v) = 0$ (by complementary slackness) and so $\{\alpha_u, \alpha_v\} = \{-1, 1\}$. Assume without loss of generality that $\alpha_u = -1$ and $\alpha_v = 1$. So $\sigma_u = -$ and $\sigma_v = +$. For any neighbor w of u with $\sigma_w = -$, we have $\mathsf{wt}_M(u, w) \le \alpha_u + \alpha_w = -1 - 1 = -2$, i.e., both u and w prefer their partners in M to each other. Thus u prefers v to all its neighbors w in H with $\sigma_w = -$. Hence (u_-, v_+) is in H_σ. ◁

▷ Claim 20. $f_\alpha(M)$ is a stable matching in H_σ.

Proof. Every edge in H_σ is of the form (a_-, b_+) for some adjacent pair of vertices a, b in H and $\alpha_a = -1, \alpha_b = 1$. Since $\vec{\alpha}$ is a witness of M, we have $\mathsf{wt}_M(a, b) \le \alpha_a + \alpha_b = 0$. Thus either $(a_-, b_+) \in f_\alpha(M)$ or at least one of a, b is matched in M to a more preferred neighbor. So (a_-, b_+) does not block $f_\alpha(M)$. Thus $f_\alpha(M)$ has no blocking edge in H_σ. ◁

References

1 D. J. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. Popular Matchings. *SIAM Journal on Computing*, 37(4):1030–1045, 2007.
2 P. Biró, R. W. Irving, and D. F. Manlove. Popular matchings in the marriage and roommates problems. In *Proceedings of the 7th International Conference on Algorithms and Complexity (CIAC)*, pages 97–108, 2010.
3 F. Brandl and T. Kavitha. Two Problems in Max-Size Popular Matchings. *Algorithmica*, 81(7):2738–2764, 2019.
4 K.S. Chung. On the Existence of Stable Roommate Matchings. *Games and Economic Behavior*, 33(2):206–230, 2000.

5 M.-J.-A.-N. de C. (Marquis de) Condorcet. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.* L'Imprimerie Royale, 1785.
6 Condorcet method. `https://en.wikipedia.org/wiki/Condorcet_method`.
7 Á. Cseh, C.-C. Huang, and T. Kavitha. Popular matchings with two-sided preferences and one-sided ties. *SIAM Journal on Discrete Mathematics*, 31(4):2348–2377, 2017.
8 Á. Cseh and T. Kavitha. Popular edges and dominant matchings. *Mathematical Programming*, 172(1):209–229, 2018.
9 Á. Cseh and T. Kavitha. Popular Matchings in Complete Graphs. In *Proceedings of the 38th Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 17:1–17:14, 2018.
10 W. H. Cunningham and A. B. Marsh. A primal algorithm for optimal matching. *Mathematical Programming*, 8:50–72, 1978.
11 B. C. Dean and S. Munshi. Faster algorithms for stable allocation problems. *Algorithmica*, 58(1):59–81, 2010.
12 J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards B*, 69B:125–130, 1965.
13 Y. Faenza, T. Kavitha, V. Powers, and X. Zhang. Popular Matchings and Limits to Tractability. In *Proceedings of the 30th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2790–2809, 2019.
14 T. Fleiner, R. W. Irving, and D. F. Manlove. Efficient algorithms for generalised stable marriage and roommates problems. *Theoretical Computer Scienc*, 381:162–176, 2007.
15 F. V. Fomin and D. Kratsch. *Exact exponential algorithms.* Springer-Verlag New York, Inc., New York, 2010.
16 D. Gale and L.S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.
17 P. Gärdenfors. Match making: assignments based on bilateral preferences. *Behavioural Science*, 20(3):166–173, 1975.
18 S. Gupta, P. Misra, S. Saurabh, and M. Zehavi. Popular matching in roommates setting is NP-hard. In *Proceedings of the 30th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2810–2822, 2019.
19 D. Gusfield and R. W. Irving. *The Stable Marriage Problem: Structure and Algorithms.* MIT Press, Boston, MA, 1989.
20 M. Hirakawa, Y. Yamauchi, S. Kijima, and M. Yamashita. *On The Structure of Popular Matchings in The Stable Marriage Problem - Who Can Join a Popular Matching?* In the 3rd International Workshop on Matching Under Preferences (MATCH-UP), 2015.
21 C.-C. Huang and T. Kavitha. Near-popular matchings in the Roommates problem. *SIAM Journal on Discrete Mathematics*, 27(1):43–62, 2013.
22 C.-C. Huang and T. Kavitha. Popular matchings in the stable marriage problem. *Information and Computation*, 222:180–194, 2013.
23 C.-C. Huang and T. Kavitha. Popularity, mixed matchings, and self-duality. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2294–2310, 2017.
24 R.W. Irving. An efficient algorithm for the stable roommates problem. *Journal of Algorithms*, 6:577–595, 1985.
25 A. R. Karlin, S. Oveis Gharan, and R. Weber. A simply exponential upper bound on the maximum number of stable matchings. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 920–925, 2018.
26 T. Kavitha. A size-popularity tradeoff in the stable marriage problem. *SIAM Journal on Computing*, 43(1):52–71, 2014.
27 T. Kavitha. Popular half-integral matchings. In *Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 22.1–22.13, 2016.

28 T. Kavitha, J. Mestre, and M. Nasre. Popular Mixed Matchings. *Theoretical Computer Science*, 412(24):2679–2690, 2011.

29 E. McDermid and R. W. Irving. Sex-equal stable matchings: Complexity and exact algorithms. *Algorithmica*, 68:545–570, 2014.

30 A. Subramanian. A New Approach to Stable Matching Problems. *SIAM Journal on Computing*, 23(4):671–700, 1994.

31 C.-P. Teo and J. Sethuraman. The geometry of fractional stable matchings and its applications. *Mathematics of Operations Research*, 23(4):874–891, 1998.

32 E. G. Thurber. Concerning the maximum number of stable matchings in the stable marriage problem. *Discrete Mathematics*, 248(1-3):195–219, 2002.