No-Dimensional Tverberg Theorems and Algorithms

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Abstract

Tverberg's theorem states that for any $k \geq 2$ and any set $P \subset \mathbb{R}^d$ of at least $(d+1)(k-1)+1$ points, we can partition P into k subsets whose convex hulls have a non-empty intersection. The associated search problem lies in the complexity class PPAD $\cap$ PLS, but no hardness results are known. In the *colorful* Tverberg theorem, the points in P have colors, and under certain conditions, P can be partitioned into *colorful* sets, in which each color appears exactly once and whose convex hulls intersect. To date, the complexity of the associated search problem is unresolved. Recently, Adiprasito, Bárány, and Mustafa [SODA 2019] gave a *no-dimensional* Tverberg theorem, in which the convex hulls may intersect in an *approximate* fashion. This relaxes the requirement on the cardinality of P. The argument is constructive, but does not result in a polynomial-time algorithm.

We present a deterministic algorithm that finds for any n-point set $P \subset \mathbb{R}^d$ and any $k \in \{2, \ldots, n\}$ in $O(nd\lceil \log k \rceil)$ time a k-partition of P such that there is a ball of radius $O\left(\frac{k}{\sqrt{n}}\mathrm{diam(P)}\right)$ that intersects the convex hull of each set. Given that this problem is not known to be solvable exactly in polynomial time, and that there are no approximation algorithms that are truly polynomial in any dimension, our result provides a remarkably efficient and simple new notion of approximation.

Our main contribution is to generalize Sarkaria's method [Israel Journal Math., 1992] to reduce the Tverberg problem to the Colorful Carathéodory problem (in the simplified tensor product interpretation of Bárány and Onn) and to apply it algorithmically. It turns out that this not only leads to an alternative algorithmic proof of a no-dimensional Tverberg theorem, but it also generalizes to other settings such as the colorful variant of the problem.

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1 Introduction

In 1921, Radon [19] proved a seminal theorem in convex geometry: given a set P of at least $d+2$ points in $\mathbb{R}^d$, one can always split P into two non-empty sets whose convex hulls intersect. In 1966, Tverberg [25] generalized Radon's theorem to allow for more sets in the partition. Specifically, he showed that any point set $P \subset \mathbb{R}^d$ of cardinality at least $(d+1)(k-1)+1$ can be split into k sets $T_1, \dots, T_k \subset P$ whose convex hulls have a non-empty intersection, i.e., $\text{conv}(T_1) \cap \dots \cap \text{conv}(T_k) \neq \emptyset$, where $\text{conv}(\cdot)$ denotes the convex hull.

By now, several alternative proofs of Tverberg's theorem are known, e.g., [3, 6, 15, 20, 21, 26, 27]. Perhaps the most elegant proof is due to Sarkaria [21], with simplifications by Bárány and Onn [6] and by Aroch et al. [3]. In this paper, all further references to *Sarkaria's method* refer to the simplified version. This proof proceeds by a reduction to the *Colorful Carathéodory Theorem*, another celebrated result in convex geometry: given $r \geq d+1$ point sets $P_1, \dots, P_r \subset \mathbb{R}^d$ that have a common point y in their convex hulls $\text{conv}(P_1), \dots, \text{conv}(P_r)$, there is a *traversal* $x_1 \in P_1, \dots, x_r \in P_r$, such that $\text{conv}(\{x_1, \dots, x_r\})$ contains y. Sarkaria's proof [21] uses a Tensor product to lift the original points of the Tverberg instance into higher dimensions, and then uses the Colorful Carathéodory traversal to obtain a Tverberg partition for the original point set.

From a computational point of view, a Radon partition is easy to find by solving $d+1$ linear equations. On the other hand, finding Tverberg partitions is not straightforward. Since a Tverberg partition must exist if P is large enough, finding such a partition is a total search problem. In fact, the problem of computing a Colorful Carathéodory traversal lies in the complexity class PPAD $\cap$ PLS [14, 16], but no better upper bound is known. Since Sarkaria's proof gives a polynomial-time reduction from the problem of finding a Tverberg partition to the problem of finding a colorful traversal, the same complexity applies to Tverberg partitions. Again, as of now we do not know better upper bounds for the general problem. Miller and Sheehy [15] and Mulzer and Werner [17] provided algorithms for finding *approximate* Tverberg partitions, computing a partition into fewer sets than is guaranteed by Tverberg's theorem in time that is linear in n, but quasi-polynomial in the dimension. These algorithms were motivated by applications in mesh generation and statistics that require finding a point that lies "deep" in P. A point in the common intersection of the convex hulls of a Tverberg partition has this property, with the partition serving as a certificate of depth.

Tverberg's theorem also admits a colorful variant, first conjectured by Bárány and Larman [5]. The setup consists of $d+1$ point sets $P_1, \dots, P_{d+1} \subset \mathbb{R}^d$, each set interpreted as a different color and having size t. For a given k, the goal is to find k pairwise-disjoint *colorful* sets (i.e., each set contains at most one point from each P_i) $A_1, \dots, A_k$ such that $\cap_{i=1}^k \text{conv}(A_i) \neq \emptyset$. The problem is to determine the optimal value for t such that such a colorful partition always exists. Bárány and Larman [5] conjectured that $t = k$ suffices and they proved the conjecture for $d = 2$ and arbitrary k, and for $k = 2$ and arbitrary d. The first result for the general case was given by Živaljević and Vrećica [29] through topological arguments. Using another topological argument, Blagojevič, Matschke, and Ziegler [7] showed that (i) if $k+1$ is prime, then $t = k$; and (ii) if $k+1$ is not prime, then $k \leq t \leq 2k-2$. These are the best known bounds for arbitrary k. Later Matoušek, Tancer, and Wagner [13] gave a geometric proof that is inspired by the proof of Blagojevič, Matschke, and Ziegler [7].

More recently, Soberón [22] showed that if more color classes are available, then the conjecture holds for any k. More precisely, for $P_1, \dots, P_n \subset \mathbb{R}^d$ with $n = (k-1)d+1$, each of size k, there exist k colorful sets whose convex hulls intersect. Moreover, there is a point in the common intersection so that the coefficients of its convex combination are the same for each colorful set in the partition. The proof uses Sarkaria's tensor product construction.

Recently Adiprasito, Bárány, and Mustafa [1] established a relaxed version of the Colorful Carathéodory Theorem and some of its descendants [4]. For the Colorful Carathéodory theorem, this allows for a (relaxed) traversal of arbitrary size, with a guarantee that the convex hull of the traversal is close to the common point y. For the Colorful Tverberg problem, they prove a version of the conjecture where the convex hulls of the colorful sets intersect approximately. This also gives a relaxation for Tverberg's theorem [25] that allows arbitrary-sized partitions, again with an approximation notion of intersection. Adiprasito et al. refer to these results as *no-dimensional* versions of the respective classic theorems, since the dependence on the ambient dimension is relaxed. The proofs use averaging arguments. The argument for the no-dimensional Colorful Carathéodory also gives an efficient algorithm to find a suitable traversal. However, the arguments for the no-dimensional Tverberg results do not give a polynomial-time algorithm for finding the partitions.

Our contributions. We prove no-dimensional variants of the Tverberg theorem and its colorful counterpart that allow for efficient algorithms. Our proofs are inspired by Sarkaria's method [21] and the averaging technique by Adiprasito, Bárány, and Mustafa [1]. For the colorful version, we additionally make use of ideas of Soberón [22]. Furthermore, we also give a no-dimensional generalized ham-sandwich theorem that interpolates [28] between the centerpoint theorem and the ham-sandwich theorem [24], again with an efficient algorithm.

Algorithmically, Tverberg's theorem is useful for finding centerpoints of high-dimensional point sets, which in turn has applications in statistics and mesh generation [15]. In fact, most algorithms for finding centerpoints are Monte-Carlo, returning some point p and a probabilistic guarantee that p is indeed a centerpoint [9, 11]. However, this is coNP-hard to verify. On the other hand, a (possibly approximate) Tverberg partition immediately gives a certificate of depth [15,17]. Unfortunately, there are no polynomial-time algorithms for finding optimal Tverberg partitions, and the approximation algorithms are not truly polynomial in the dimension. In this context, our result provides a fresh notion of approximation that also leads to very fast polynomial-time algorithms.

Furthermore, the Tverberg problem is intriguing from a complexity theoretic point of view, because it constitutes a total search problem that is not known to be solvable in polynomial time, but which is also unlikely to be NP-hard. So far, such problems have mostly been studied in the context of algorithmic game theory [18], and only very recently a similar line of investigation has been launched for problems in high-dimensional discrete geometry [10, 12, 14, 16]. Thus, we show that the *no-dimensional* variant of Tverberg's theorem is easy from this point of view. Our main results are as follows:

- Sarkaria's method uses a specific set of k vectors in $\mathbb{R}^{k-1}$ to lift the points in the Tverberg instance to a Colorful Carathéodory instance. We refine this method to vectors that are defined with the help of a given graph. The choice of this graph is important in proving good bounds for the partition and in the algorithm. We believe that this generalization is of independent interest and may prove useful in other scenarios that rely on the tensor product construction.
- We prove an efficient no-dimensional Tverberg result:

  ▶ **Theorem 1.1** (efficient no-dimensional Tverberg theorem)**.** *Let $P \subset \mathbb{R}^d$ be a set of n points, and let $k \in \{2, \ldots, n\}$ be an integer. Let $\mathbb{D}(\cdot)$ denote the diameter.*
  - *For any choice of positive integers $r_1, \ldots, r_k$ that satisfy $\sum_{i=1}^{k} r_i = n$, there is a partition $T_1, \ldots, T_k$ of P with $|T_1| = r_1, |T_2| = r_2, \ldots, |T_k| = r_k$, and a ball B of radius $\frac{n\,\mathbb{D}(P)}{\min_i r_i}\sqrt{\frac{10\lceil \log_4 k\rceil}{n-1}} = O\left(\frac{\sqrt{n \log k}}{\min_i r_i}\,\mathbb{D}(P)\right)$ that intersects the convex hull of each T_i.*
  - *The bound is better for the case $n = rk$ and $r_1 = \cdots = r_k = r$. There exists a partition $T_1, \ldots, T_k$ of P with $|T_1| = \cdots = |T_k| = r$ and a d-dimensional ball of radius $\sqrt{\frac{k(k-1)}{n-1}}\mathbb{D}(P) = O\left(\frac{k}{\sqrt{n}}\mathbb{D}(P)\right)$ that intersects the convex hull of each T_i.*

  In either case, we can compute the partition in deterministic time $O(nd\lceil \log k\rceil)$.

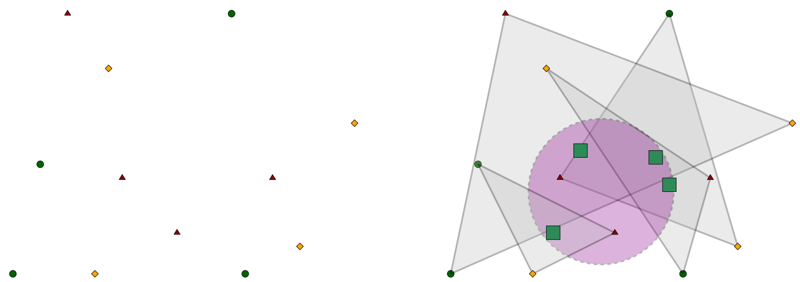


Figure 1 Left: a point set on three colors and four points of each color. Right: a colorful partition with a ball containing the centroids (squares) of the sets of the partition.

- and a colorful counterpart (for a simple example, see Figure 1):

  ▶ **Theorem 1.2** (efficient no-dimensional Colorful Tverberg)**.** *Let $P_1, \dots, P_n \subset \mathbb{R}^d$ be n point sets, each of size k, with k being a positive integer, so that the total number of points is $N = nk$. Then, there are k pairwise-disjoint colorful sets $A_1, \dots, A_k$ and a ball of radius $\sqrt{\frac{2k(k-1)}{N}} \max_i \mathbb{D}(P_i) = O\left(\frac{k}{\sqrt{N}} \max_i \mathbb{D}(P_i)\right)$ that intersects $\operatorname{conv}(A_i)$ for each $i \in [k]$. We can find the A_is in deterministic time $O(Ndk)$.*

- For any sets $P, x \subset \mathbb{R}^d$, the *depth* of x with respect to P is the largest positive integer k such that every half-space that contains x also contains at least k points of P.

  ▶ **Theorem 1.3** (no-dimensional Generalized Ham-Sandwich Theorem)**.** *Let $P_1, \dots, P_k$ be $k \leq d$ finite point sets in $\mathbb{R}^d$. Then there is a $(d-k+1)$-dimensional ball B and $k-1$ lines $\ell_1, \dots, \ell_{k-1}$ such that the d-dimensional Cartesian product $B \times \ell_{k-1} \times \ell_{k-2} \times \cdots \times \ell_1$ has depth at least $\left\lceil \frac{|P_i|}{m_i} \right\rceil$ with respect to P_i, for $i \in [k]$. Here, $\{2 \leq m_i \leq |P_i|, i \in [k]\}$ is any set of chosen integer parameters. The ball B has radius $(2+2\sqrt{2}) \max_i \frac{\mathbb{D}(P_i)}{\sqrt{m_i}}$ and the lines $\ell_1, \dots, \ell_{k-1}$ can be determined in $O(dk^2 + \sum_i |P_i|d)$ time.*

The colorful Tverberg result is similar in spirit to the regular version, but from a computational viewpoint, it does not make sense to use the colorful algorithm to solve the regular Tverberg problem. Due to space constraints, the colorful Tverberg and the Generalized Ham-Sandwich results have been deferred to an extended version in [8].

Compared to the results of Adiprasito et al. [1], our radius bounds are slightly worse. More precisely, they show that both in the colorful and the non-colorful case, there is a ball of radius $O\left(\sqrt{\frac{k}{n}} \mathbb{D}(P)\right)$ that intersects the convex hulls of the sets of the partition. They also show this bound is close to optimal. In contrast, our result is off by a factor of $O(\sqrt{k})$, but derandomizing the proof of Adiprasito et al. [1] gives only a brute-force $2^{O(n)}$-time algorithm. In contrast, our approach gives almost linear time algorithms for both cases, with a linear dependence on the dimension.

Techniques. Adiprasito et al. first prove the colorful no-dimensional Tverberg theorem using an averaging argument over an exponential number of possible partitions. Then, they specialize their result for the non-colorful case, obtaining a bound that is asymptotically optimal. Unfortunately, it is not clear how to derandomize the averaging argument efficiently. The method of conditional expectations applied to their averaging argument leads to a runtime of $2^{O(n)}$. To get around this, we follow an alternate approach towards both versions of the Tverberg theorem. Instead of a direct averaging argument, we use a reduction to the Colorful Carathéodory theorem that is inspired by Sarkaria's proof, with some additional twists. We will see that this reduction also works in the no-dimensional setting, i.e., by a reduction to the no-dimensional Colorful Carathéodory theorem of Adiprasito et al., we obtain a no-dimensional Tverberg theorem, with slightly weaker radius bounds, as stated above. This approach has the advantage that their Colorful Carathéodory theorem is based on an averaging argument that permits an efficient derandomization using the method of conditional expectations [2]. In fact, we will see that the special structure of the no-dimensional Colorful Carathéodory instance that we create allows for a very fast evaluation of the conditional expectations, as we fix the next part of the solution. This results in an algorithm whose running time is $O(nd\lceil\log k\rceil)$ instead of $O(ndk)$, as given by a naive application of the method. With a few interesting modifications, this idea also works in the colorful setting. This seems to be the first instance of using Sarkaria's method with special lifting vectors, and we hope that this will prove useful for further studies on Tverberg's theorem and related problems.

Outline of the paper. We describe our extension of Sarkaria's technique in Section 2 and then use it in combination with a result from Section 3 to prove the no-dimensional Tverberg result. In Section 3, we expand upon the details of an averaging argument that is useful for the Tverberg result. Section 4 is devoted to the algorithm for computing the Tverberg partition. We conclude in Section 5 with some observations and open questions. The results for the colorful setting and the generalized ham-sandwich theorem are presented in the extended version [8].

2 Tensor product and no-dimensional Tverberg Theorem

In this section, we prove a no-dimensional Tverberg result. Let $\mathbb{D}(\cdot)$ denote the diameter of any point set in $\mathbb{R}^d$. Let $P \subset \mathbb{R}^d$ be our given set of n points. We assume for simplicity that the centroid of P, that we denote by $c(P)$, coincides with the origin $\mathbf{0}$, so that $\sum_{x\in P} x = \mathbf{0}$. For ease of presentation, we denote the origin by $\mathbf{0}$ in all dimensions, as long as there is no danger of confusion. Also, we write $\langle\cdot,\cdot\rangle$ for the usual scalar product between two vectors in the appropriate dimension, and $[n]$ for the set $\{1,\dots,n\}$.

Tensor product. Let $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ and $y = (y_1,\dots,y_m) \in \mathbb{R}^m$ be any two vectors. The *tensor product* $\otimes$ is the operation that takes x and y to the dm-dimensional vector $x \otimes y$ whose ij-th component is $x_i y_j$. Easy calculations show that for any $x, x' \in \mathbb{R}^d, y, y' \in \mathbb{R}^m$, the operator $\otimes$ satisfies: (i) $x\otimes y + x'\otimes y = (x+x')\otimes y$; (ii) $x\otimes y + x\otimes y' = x\otimes(y+y')$; and (iii) $\langle x\otimes y, x'\otimes y'\rangle = \langle x,x'\rangle\langle y,y'\rangle$. By (iii), the L_2-norm $\|x\otimes y\|$ of the tensor product $x\otimes y$ is exactly $\|x\|\|y\|$. For any set of vectors $X = \{x_1, x_2, \dots\}$ in $\mathbb{R}^d$ and any m-dimensional vector $q \in \mathbb{R}^m$, we denote by $X\otimes q$ the set of tensor products $\{x_1\otimes q, x_2\otimes q, \dots\} \subset \mathbb{R}^{dm}$. Throughout this paper, all distances will be in the L_2-norm.

A set of lifting vectors. We generalize the tensor construction that was used by Sarkaria to prove the Tverberg theorem [21]. For this, we provide a way to construct a set of k vectors $\{q_1,\dots,q_k\}$ that we use to create tensor products. The motivation behind the precise choice of these vectors will be explained a little later in this section. Let $\mathcal{G}$ be an (undirected) simple, connected graph of k nodes. Let $\|\mathcal{G}\|$ denote the number of edges in $\mathcal{G}$, $\Delta(\mathcal{G})$ denote the maximum degree of any node in $\mathcal{G}$, and $\operatorname{diam}(\mathcal{G})$ denote the diameter of $\mathcal{G}$, i.e., the maximum length of a shortest path between a pair of vertices in $\mathcal{G}$.

We orient the edges of $\mathcal{G}$ in an arbitrary manner to obtain a directed graph. We use this directed version of $\mathcal{G}$ to define a set of k vectors $\{q_1,\dots,q_k\}$ in $\|\mathcal{G}\|$ dimensions. This is done as follows: each vector q_i corresponds to a unique node v_i of $\mathcal{G}$. Each coordinate position of the vectors corresponds to a unique edge of $\mathcal{G}$. If v_iv_j is a directed edge of $\mathcal{G}$, then q_i contains a 1 and q_j contains a -1 in the corresponding coordinate position. The remaining co-ordinates are zero. That means, the vectors $\{q_1, \dots, q_k\}$ are in $\mathbb{R}^{\|\mathcal{G}\|}$. Also, $\sum_{i=1}^k q_i = \mathbf{0}$. It can be verified that this is the unique linear dependence (up to scaling) between the vectors for any choice of edge orientations of $\mathcal{G}$. This means that the rank of the matrix with the q_is as the rows is $k-1$. It can be verified that:

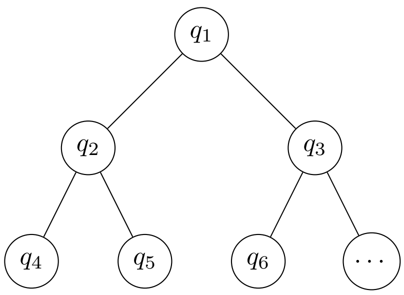
▷ Claim 2.1. For each vertex v_i, the squared norm $\|q_i\|^2$ is the degree of v_i. For $i \neq j$, the dot product $\langle q_i, q_j\rangle$ is -1 if v_iv_j is an edge in $\mathcal{G}$, and 0 otherwise.

An immediate application of Claim 2.1 and property (iii) of the tensor product is that for any set of k vectors $\{u_1, \dots, u_k\}$, each of the same dimension, the following relation holds:

$$\begin{aligned}
\left\|\sum_{i=1}^k u_i \otimes q_i\right\|^2 &= \sum_{i=1}^k\sum_{j=1}^k \langle u_i \otimes q_i, u_j \otimes q_j\rangle = \sum_{i=1}^k\sum_{j=1}^k \langle u_i, u_j\rangle\langle q_i, q_j\rangle \\
&= \sum_{i=1}^k \langle u_i, u_i\rangle\langle q_i, q_i\rangle + 2\sum_{1\le i<j\le k}\langle u_i, u_j\rangle\langle q_i, q_j\rangle = \sum_{i=1}^k \|u_i\|^2\|q_i\|^2 - 2\sum_{v_iv_j\in E}\langle u_i, u_j\rangle \\
&= \sum_{v_iv_j\in E}\|u_i - u_j\|^2, \qquad (1)
\end{aligned}$$

where E is the set of edges of $\mathcal{G}$.[1]

As an example, such a set of vectors can be formed by taking $\mathcal{G}$ as a balanced binary tree with k nodes, and orienting the edges away from the root. Let q_1 correspond to the root. A simple instance of the vectors is shown below:



The vectors in the figure above can be represented as the matrix

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ & & & \dots & & & & & \end{pmatrix}$$

where the i-th row of the matrix corresponds to vector q_i. As $\|\mathcal{G}\| = k-1$, each vector is in $\mathbb{R}^{k-1}$. The norm $\|q_i\|$ is one of $\sqrt{2}$, $\sqrt{3}$, or 1, depending on whether v_i is the root, an internal node with two children, or a leaf, respectively. The height of $\mathcal{G}$ is $\lceil \log k\rceil$ and the maximum degree is $\Delta(\mathcal{G}) = 3$.

[1] We note that this identity is very similar to the Laplacian quadratic form that is used in spectral graph theory; see, e.g., the lecture notes by Spielman [23] for more information.

Lifting the point set. Let $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$. Our goal is to find a (relaxed) Tverberg partition of P into k sets. For this, we first pick a graph $\mathcal{G}$ with k vertices, as in the previous paragraph, and we derive a set of k lifting vectors $\{q_1, \dots, q_k\}$ from $\mathcal{G}$. Then, we lift each point of P to a set of vectors in $d\|\mathcal{G}\|$ dimensions, by taking tensor products with the vectors $\{q_1, \dots, q_k\}$. More precisely, for $a \in [n]$ and $j = 1, \dots, k$, let $p_{a,j} = p_a \otimes q_j \in \mathbb{R}^{d\|\mathcal{G}\|}$. For $a \in [n]$, we let $P_a = \{p_{a,1}, \dots, p_{a,k}\}$ be the lifted points obtained from p_a. We have, $\|p_{a,j}\| = \|q_j\|\|p_a\| \le \sqrt{\Delta(\mathcal{G})}\|p_a\|$. By the bi-linear properties of the tensor product, we have

$$c(P_a) = \frac{1}{k}\sum_{j=1}^{k}(p_a \otimes q_j) = \frac{1}{k}\left(p_a \otimes \left(\sum_{j=1}^{k} q_j\right)\right) = \frac{1}{k}(p_a \otimes \mathbf{0}) = \mathbf{0},$$

so the centroid $c(P_a)$ coincides with the origin, for $a \in [n]$.

The next lemma contains the technical core of our argument. It shows how to use the lifted point sets to derive a useful partition of P into k subsets of prescribed sizes. We defer its proof to Section 3.

▶ **Lemma 2.2.** *Let $P = \{p_1, \dots, p_n\}$ be a set of n points in $\mathbb{R}^d$ satisfying $\sum_{i=1}^{p} p_i = \mathbf{0}$. Let $P_1, \dots, P_n$ denote the point sets obtained by lifting each $p \in P$ using the vectors $\{q_1, \dots, q_k\}$.*

For any choice of positive integers $r_1, \dots, r_k$ that satisfy $\sum_{i=1}^{k} r_i = n$, there is a partition $T_1, \dots, T_k$ of P with $|T_1| = r_1, |T_2| = r_2, \dots, |T_k| = r_k$ such that the centroid of the set of lifted points $T := \{T_1 \otimes q_1 \cup \dots \cup T_k \otimes q_k\}$ (this set is also a traversal of $P_1, \dots, P_n$) has distance less than $\delta = \sqrt{\frac{\Delta(\mathcal{G})}{2(n-1)}}\mathbb{D}(P)$ from the origin $\mathbf{0}$.

The bound is better for the case $n = rk$ and $r_1 = \dots = r_k = \frac{n}{k}$. There exists a partition $T_1, \dots, T_k$ of P with $|T_1| = |T_2| = \dots = |T_k| = r$ such that the centroid of $T := \{T_1 \otimes q_1 \cup \dots \cup T_k \otimes q_k\}$ has distance less than $\gamma = \sqrt{\frac{\|\mathcal{G}\|}{k(n-1)}}\mathbb{D}(P)$ from the origin $\mathbf{0}$.

Using Lemma 2.2, we show that there is a ball of bounded radius that intersects the convex hull of each T_i. Let $\alpha_1 = \frac{r_1}{n}, \dots, \alpha_k = \frac{r_k}{n}$ be positive real numbers. The centroid of T, $c(T)$, can be written as

$$c(T) = \frac{1}{n}\sum_{i=1}^{k}\sum_{x \in T_i} x \otimes q_i = \sum_{i=1}^{k}\frac{1}{n}\left(\sum_{x \in T_i} x\right) \otimes q_i = \sum_{i=1}^{k}\frac{r_i}{n}\left(\frac{1}{r_i}\sum_{x \in T_i} x\right) \otimes q_i = \sum_{i=1}^{k}\alpha_i c_i \otimes q_i,$$

where $c_i = c(T_i)$ denotes the centroid of T_i, for $i \in [k]$. Using Equation (1),

$$\|c(T)\|^2 = \left\|\sum_{i=1}^{k}\alpha_i c_i \otimes q_i\right\|^2 = \sum_{v_i v_j \in E}\|\alpha_i c_i - \alpha_j c_j\|^2. \tag{2}$$

Let $x_1 = \alpha_1 c_1, x_2 = \alpha_2 c_2, \dots, x_k = \alpha_k c_k$. Then

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k}\alpha_i c_i = \sum_{i=1}^{k}\frac{r_i}{n}\left(\frac{1}{r_i}\sum_{p \in T_i} p\right) = \frac{1}{n}\sum_{j=1}^{n} p_j = \mathbf{0},$$

so the centroid of $\{x_1, \dots, x_k\}$ coincides with the origin. Using $\|c(T)\| < \delta$ and Equation (2),

$$\sum_{v_i v_j \in E}\|x_i - x_j\|^2 = \sum_{v_i v_j \in E}\|\alpha_i c_i - \alpha_j c_j\|^2 < \delta^2.$$

We bound the distance from x_1 to every other x_j. For each $i \in [k]$, we associate to x_i the node v_i in $\mathcal{G}$. Let the shortest path from v_1 to v_j in $\mathcal{G}$ be denoted by $(v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_z}, v_j)$. This path has length at most $\mathrm{diam}(\mathcal{G})$. Using the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|x_1 - x_j\| &\le \|x_1 - x_{i_1}\| + \|x_{i_1} - x_{i_2}\| + \cdots + \|x_{i_z} - x_j\| \\ &\le \sqrt{\mathrm{diam}(\mathcal{G})}\sqrt{\|x_1 - x_{i_1}\|^2 + \|x_{i_1} - x_{i_2}\|^2 + \cdots + \|x_{i_z} - x_j\|^2} \\ &\le \sqrt{\mathrm{diam}(\mathcal{G})}\sqrt{\sum_{v_i v_j \in E} \|x_i - x_j\|^2} < \sqrt{\mathrm{diam}(\mathcal{G})}\delta. \end{aligned} \tag{3}$$

Therefore, the ball of radius $\beta := \sqrt{\mathrm{diam}(\mathcal{G})}\delta$ centered at x_1 covers the set $\{x_1, \ldots, x_k\}$. That means, the ball covers the convex hull of $\{x_1, \ldots, x_k\}$ and in particular contains the origin. Using triangle inequality, the ball of radius 2β centered at the origin contains $\{x_1, \ldots, x_k\}$. Then the norm of each x_i is at most 2β which implies that the norm of each c_i is at most $2\beta/\alpha_i$. Therefore, the ball of radius $\frac{2\beta}{\min_i \alpha_i} = \frac{2n\sqrt{\mathrm{diam}(\mathcal{G})}\delta}{\min_i r_i}$ centered at $\mathbf{0}$ contains the set $\{c_1, \ldots, c_k\}$. Substituting the value of δ from Lemma 2.2, the ball of radius

$$\frac{2n\sqrt{\mathrm{diam}(\mathcal{G})}}{\min_i r_i}\sqrt{\frac{\Delta(\mathcal{G})}{2(n-1)}}\mathbb{D}(P) = \frac{n\mathbb{D}(P)}{\min_i r_i}\sqrt{\frac{2\mathrm{diam}(\mathcal{G})\Delta(\mathcal{G})}{n-1}}$$

centered at $\mathbf{0}$ covers the set $\{c_1, \ldots, c_k\}$.

Optimizing the choice of $\mathcal{G}$. The radius of the ball has a term $\sqrt{\mathrm{diam}(\mathcal{G})\Delta(\mathcal{G})}$ that depends on the choice of $\mathcal{G}$. For a path graph this term has value $\sqrt{(k-1)2}$. For a star graph, that is, a tree with one root and $k-1$ children, this is $\sqrt{k-1}$. If $\mathcal{G}$ is a balanced s-ary tree, then the Cauchy-Schwarz inequality in Equation (3) can be modified to replace $\mathrm{diam}(\mathcal{G})$ by the height of the tree. Then the term is $\sqrt{\lceil \log_s k\rceil(s+1)}$ which is minimized for $s = 4$. The radius bound for this choice of $\mathcal{G}$ is $\frac{n\mathbb{D}(P)}{\min_i r_i}\sqrt{\frac{10\lceil \log_4 k\rceil}{n-1}}$ as claimed in Theorem 1.1.

Balanced partition. For the case $n = rk$ and $r_1 = \cdots = r_k = r$, we give a better bound for the radius of the ball containing the centroids $c_1, \ldots, c_k$. In this case we have $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \frac{r}{n} = \frac{1}{k}$. Then Equation (2) is

$$\|c(T)\|^2 = \sum_{v_i v_j \in E} \|\alpha_i c_i - \alpha_j c_j\|^2 = \frac{1}{k^2}\sum_{v_i v_j \in E} \|c_i - c_j\|^2.$$

Since $\|c(T)\| < \gamma$, we get

$$\sum_{v_i v_j \in E} \|c_i - c_j\|^2 < k^2\gamma^2. \tag{4}$$

Similar to the general case, we bound the distance from c_1 to any other centroid c_j. For each i, we associate to c_i the node v_i in $\mathcal{G}$. There is a path of length at most $\mathrm{diam}(\mathcal{G})$ from v_1 to any other node. Using the Cauchy-Schwarz inequality and substituting the value of γ, we get

$$\begin{aligned}\|c_1 - c_j\| &\le \sqrt{\mathrm{diam}(\mathcal{G})}\sqrt{\sum_{v_i v_j \in E} \|c_i - c_j\|^2} < \sqrt{\mathrm{diam}(\mathcal{G})}k\gamma = \sqrt{\frac{\mathrm{diam}(\mathcal{G})\|\mathcal{G}\|}{k(n-1)}}k\mathbb{D}(P) \\ &= \sqrt{\frac{k}{n-1}}\sqrt{\mathrm{diam}(\mathcal{G})\|\mathcal{G}\|}\mathbb{D}(P). \end{aligned} \tag{5}$$

Therefore, a ball of radius $\sqrt{\frac{k}{n-1}}\sqrt{\mathrm{diam}(\mathcal{G})\|\mathcal{G}\|}\mathbb{D}(P)$ centered at c_1 contains the set $c_1, \ldots, c_k$. The factor $\sqrt{\mathrm{diam}(\mathcal{G})\|\mathcal{G}\|}$ is minimized when $\mathcal{G}$ is a star graph, which is a tree. We can replace the term $\mathrm{diam}(\mathcal{G})$ by the height of the tree. Then the ball containing $c_1, \ldots, c_k$ has radius $\sqrt{\frac{k(k-1)}{n-1}}\mathbb{D}(P)$, as claimed in Theorem 1.1.

As balanced as possible. When k does not divide n, but we still want a balanced partition, we take any subset of $n_0 = k\lfloor \frac{n}{k} \rfloor$ points of P and get a balanced Tverberg partition on the subset. Then we add the removed points one by one to the sets of the partition, adding at most one point to each set.

As shown above, there is a ball of radius less than $\sqrt{\frac{k(k-1)}{n_0-1}}\mathbb{D}(P)$ that intersects the convex hull of each set in the partition. Noting that $\frac{1}{\sqrt{n_0-1}} \leq \sqrt{\frac{k+2}{k}}\frac{1}{\sqrt{n-1}}$, a ball of radius less than $\sqrt{\frac{(k+2)(k-1)}{(n-1)}}\mathbb{D}(P)$ intersects the convex hull of each set of the partition.

3 Existence of a desired partition

This section is dedicated to the proof of Lemma 2.2. Like Adiprasito et al. [1], we use an averaging argument. More precisely, we bound the average norm δ of the centroid of the lifted points $\{T_1 \otimes q_1 \cup \cdots \cup T_k \otimes q_k\}$ over all partitions of P of the form $T_1, \ldots, T_k$, for which the sets in the partition have sizes $r_1, \ldots, r_k$ respectively, with $\sum_{i=1}^k r_i = n$.

Each such partition can be interpreted as a traversal of the lifted point sets $P_1, \ldots, P_n$ that contains r_i points lifted with q_i for $i \in [k]$. Thus, consider any traversal of this type $X = \{x_1, \ldots, x_n\}$ of $P_1, \ldots, P_n$, where $x_a \in P_a$, for $a \in [n]$. The centroid of X is $c(X) = \frac{\sum_{a=1}^n x_a}{n}$. We bound the expectation $n^2\mathbb{E}\left(\|c(X)\|^2\right) = \mathbb{E}\left(\|\sum_{a=1}^n x_a\|^2\right)$, over all possible traversals X. By the linearity of expectation, $\mathbb{E}\left(\|\sum_{a=1}^n x_a\|^2\right)$ can be written as

$$\mathbb{E}\left(\sum_{a=1}^n \|x_a\|^2 + \sum_{\substack{a,b\in[n]\\a<b}} 2\langle x_a, x_b\rangle\right) = \mathbb{E}\left(\sum_{a=1}^n \|x_a\|^2\right) + 2\mathbb{E}\left(\sum_{\substack{a,b\in[n]\\a<b}} \langle x_a, x_b\rangle\right).$$

We next find the coefficient of each term of the form $\|x_a\|^2$ and $\langle x_a, x_b\rangle$ in the expectation. Using the multinomial coefficient, the total number of traversals X is $\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!r_2!\cdot\cdots\cdot r_k!}$. Furthermore, for any lifted point $x_a = p_{a,j}$, the number of traversals X with $p_{a,j} \in X$ is $\binom{n-1}{r_1, \ldots, r_j - 1, \ldots, r_k} = \frac{(n-1)!}{r_1!\cdot\cdots\cdot(r_j-1)!\cdot\cdots\cdot r_k!}$. So the coefficient of $\|p_{a,j}\|^2$ is $\frac{\frac{(n-1)!}{r_1!\cdot\cdots\cdot(r_j-1)!\cdot\cdots\cdot r_k}}{\frac{n!}{r_1!\cdot\cdots\cdot r_k!}} = \frac{r_j}{n}$. Similarly, for any pair of points $(x_a, x_b) = (p_{a,i}, p_{b,j})$, there are two cases in which they appear in the same traversal: first, if $i = j$, the number of traversals is $\frac{(n-2)!}{r_1!\cdot\cdots\cdot(r_i-2)!\cdot\cdots\cdot r_k!}$. The coefficient of $\langle p_{a,i}, p_{b,j}\rangle$ in the expectation is hence $\frac{r_i(r_i-1)}{n(n-1)}$. Second, if $i \neq j$, the number of traversals is calculated to be $\frac{(n-2)!}{r_1!\cdot\cdots\cdot(r_i-1)!\cdot\cdots\cdot(r_j-1)!\cdot\cdots\cdot r_k!}$. The coefficient of $\langle p_{a,i}, p_{b,j}\rangle$ is $\frac{r_i r_j}{n(n-1)}$. Substituting the coefficients, we bound the expectation as

$$\begin{aligned}
&\mathbb{E}\left(\sum_{a=1}^{n}\|x_a\|^2\right)+2\mathbb{E}\left(\sum_{\substack{a,b\in[n]\\a<b}}\langle x_a,x_b\rangle\right)=\sum_{a=1}^{n}\sum_{j=1}^{k}\|p_{a,j}\|^2\frac{r_j}{n}\\
&+2\sum_{\substack{a,b\in[n]\\a<b}}\left(\sum_{j=1}^{k}\langle p_{a,j},p_{b,j}\rangle\frac{r_j(r_j-1)}{n(n-1)}+\sum_{\substack{i,j\in[k]\\i\neq j}}\langle p_{a,i},p_{b,j}\rangle\frac{r_ir_j}{n(n-1)}\right)\\
&=\sum_{j=1}^{k}\frac{r_j}{n}\sum_{a=1}^{n}\|p_{a,j}\|^2+\frac{2}{n(n-1)}\sum_{\substack{a,b\in[n]\\a<b}}\left(\sum_{i,j\in[k]}\langle p_{a,i},p_{b,j}\rangle r_ir_j-\sum_{j=1}^{k}\langle p_{a,j},p_{b,j}\rangle r_j\right)\\
&=\sum_{j=1}^{k}r_j\left(\frac{1}{n}\sum_{a=1}^{n}\|p_{a,j}\|^2\right)+\sum_{\substack{a,b\in[n]\\a<b}}\sum_{i,j\in[k]}\frac{2\langle p_{a,i},p_{b,j}\rangle r_ir_j}{n(n-1)}-\sum_{\substack{a,b\in[n]\\a<b}}\sum_{j=1}^{k}\frac{2\langle p_{a,j},p_{b,j}\rangle r_j}{n(n-1)}.
\end{aligned}$$

We bound the value of each of the three terms individually to get an upper bound on the value of the expression. The first term can be bounded as

$$\begin{aligned}
&\sum_{j=1}^{k}r_j\left(\frac{1}{n}\sum_{a=1}^{n}\|p_{a,j}\|^2\right)=\frac{1}{n}\sum_{j=1}^{k}r_j\left(\sum_{a=1}^{n}\|p_a\|^2\|q_j\|^2\right)=\frac{1}{n}\left(\sum_{j=1}^{k}r_j\|q_j\|^2\right)\sum_{a=1}^{n}\|p_a\|^2\\
&\leq\frac{1}{n}\left(\Delta(\mathcal{G})\sum_{j=1}^{k}r_j\right)\sum_{a=1}^{n}\|p_a\|^2=\frac{1}{n}\left(\Delta(\mathcal{G})n\right)\sum_{a=1}^{n}\|p_a\|^2<\Delta(\mathcal{G})\left(\frac{n\mathbb{D}(P)^2}{2}\right),
\end{aligned}$$

where we have made use of Claim 2.1 and the fact that $\sum_{a=1}^{n}\|p_a\|^2<\frac{n\mathbb{D}(P)^2}{2}$ (see [1, Lemma 4.1]). The second term can be re-written as

$$\begin{aligned}
&\sum_{\substack{a,b\in[n]\\a<b}}\sum_{i,j\in[k]}\frac{2\langle p_{a,i},p_{b,j}\rangle r_ir_j}{n(n-1)}=\sum_{i,j\in[k]}\frac{2r_ir_j}{n(n-1)}\left(\sum_{\substack{a,b\in[n]\\a<b}}\langle p_{a,i},p_{b,j}\rangle\right)\\
&=\sum_{i,j\in[k]}\frac{2r_ir_j}{n(n-1)}\left(\sum_{\substack{a,b\in[n]\\a<b}}\langle p_a\otimes q_i,p_b\otimes q_j\rangle\right)=\sum_{i,j\in[k]}\frac{2r_ir_j}{n(n-1)}\left(\sum_{\substack{a,b\in[n]\\a<b}}\langle p_a,p_b\rangle\langle q_i,q_j\rangle\right)\\
&=\left(\sum_{i,j\in[k]}\frac{2\langle q_i,q_j\rangle r_ir_j}{n(n-1)}\right)\cdot\sum_{\substack{a,b\in[n]\\a<b}}\langle p_a,p_b\rangle=\left(\sum_{(v_i,v_j)\in E}\frac{2(r_i-r_j)^2}{n(n-1)}\right)\cdot\sum_{\substack{a,b\in[n]\\a<b}}\langle p_a,p_b\rangle\leq 0,
\end{aligned}$$

where we have again made use of Claim 2.1. We also used $c(P)=\mathbf{0}$ to bound the term $\sum_{a,b,\in[n],a<b}\langle p_a,p_b\rangle=-\frac{1}{2}\sum_{a=1}^{n}\|p_a\|^2<0$. The second term is non-positive and therefore can be removed since the total expectation is always non-negative. The third term is

$$\sum_{\substack{a,b\in[n]\\a<b}}\sum_{j=1}^{k}\frac{-2\langle p_{a,j},p_{b,j}\rangle r_j}{n(n-1)}=\sum_{\substack{a,b\in[n]\\a<b}}\sum_{j=1}^{k}\frac{-2\left\langle p_a\otimes q_j,p_b\otimes q_j\right\rangle r_j}{n(n-1)}$$
$$=\sum_{\substack{a,b\in[n]\\a<b}}\sum_{j=1}^{k}\frac{-2\langle p_a,p_b\rangle\|q_j\|^2 r_j}{n(n-1)}=\left(\sum_{j=1}^{k}\|q_j\|^2 r_j\right)\left(\sum_{\substack{a,b\in[n]\\a<b}}\frac{-2\langle p_a,p_b\rangle}{n(n-1)}\right)$$
$$<\left(\sum_{j=1}^{k}\|q_j\|^2 r_j\right)\left(\frac{n\mathbb{D}(P)^2}{2n(n-1)}\right)=\left(\sum_{j=1}^{k}\|q_j\|^2 r_j\right)\frac{\mathbb{D}(P)^2}{2(n-1)}<\frac{n\Delta(\mathcal{G})\mathbb{D}(P)^2}{2(n-1)}.$$

Collecting the three terms, the expression is upper bounded by

$$\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})n}{2}+\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})n}{2(n-1)}=\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})n}{2}\left(1+\frac{1}{n-1}\right)=\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})n^2}{2(n-1)},$$

which bounds the expectation by $\frac{1}{n^2}\left(\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})n^2}{2(n-1)}\right)=\frac{\mathbb{D}(P)^2\Delta(\mathcal{G})}{2(n-1)}$. This shows that there is a traversal such that its centroid has norm less than $\mathbb{D}(P)\sqrt{\frac{\Delta(\mathcal{G})}{2(n-1)}}$, as claimed in Lemma 2.2.

Balanced case. For the case that n is a multiple of k, and $r_1=\cdots=r_k=\frac{n}{k}=r$, the upper bound can be improved: the first term in the expectation is

$$\sum_{j=1}^{k}r_j\left(\frac{1}{n}\sum_{a=1}^{n}\|p_{a,j}\|^2\right)=\frac{r}{n}\sum_{j=1}^{k}\sum_{a=1}^{n}\|p_{a,j}\|^2=\frac{r}{n}\sum_{j=1}^{k}\sum_{a=1}^{n}\|p_a\|^2\|q_j\|^2$$
$$=\frac{r}{n}\left(\sum_{j=1}^{k}\|q_j\|^2\right)\sum_{a=1}^{n}\|p_a\|^2=\frac{r}{n}2\|\mathcal{G}\|\sum_{a=1}^{n}\|p_a\|^2<\frac{r}{n}2\|\mathcal{G}\|\left(\frac{n\mathbb{D}(P)^2}{2}\right)\le r\|\mathcal{G}\|\mathbb{D}(P)^2,$$

The second term is zero, and the third term is less than

$$\left(\sum_{j=1}^{k}\|q_j\|^2 r_j\right)\frac{\mathbb{D}(P)^2}{2(n-1)}=r\left(\sum_{j=1}^{k}\|q_j\|^2\right)\frac{\mathbb{D}(P)^2}{2(n-1)}=2r\|\mathcal{G}\|\frac{\mathbb{D}(P)^2}{2(n-1)}=\frac{r\|\mathcal{G}\|\mathbb{D}(P)^2}{(n-1)}.$$

The expectation is upper bounded as

$$n^2\mathbb{E}\left(\|c(X)\|^2\right)<r\|\mathcal{G}\|\mathbb{D}(P)^2+\frac{r\|\mathcal{G}\|\mathbb{D}(P)^2}{(n-1)}$$
$$\implies\mathbb{E}\left(\|c(X)\|^2\right)<\frac{r\|\mathcal{G}\|\mathbb{D}(P)^2}{n^2}\left(1+\frac{1}{n-1}\right)=\frac{r\|\mathcal{G}\|\mathbb{D}(P)^2}{n(n-1)}=\frac{\|\mathcal{G}\|\mathbb{D}(P)^2}{k(n-1)},$$

which shows that there is at least one balanced traversal X whose centroid has norm less than $\sqrt{\frac{\|\mathcal{G}\|}{k(n-1)}}\mathbb{D}(P)$, as claimed in Lemma 2.2.

4 Computing the Tverberg partition

We now give a deterministic algorithm to compute no-dimensional Tverberg partitions. The algorithm is based on the method of conditional expectations. First, in Section 4.1 we give an algorithm for the general case when the sets in the partitions are constrained to have given sizes $r_1,\ldots,r_k$. The choice of $\mathcal{G}$ is crucial for the algorithm.

The balanced case of $r_1 = \cdots = r_k$ has a better radius bound and uses a different graph $\mathcal{G}$. The algorithm for the general case also extends to the balanced case with a small modification, that we discuss in Section 4.2. We get the same runtime in either case:

▶ **Theorem 4.1.** *Given a set of n points $P \subset \mathbb{R}^d$, and any choice of k positive integers $r_1, \ldots, r_k$ that satisfy $\sum_{i=1}^{k} r_i = n$, a no-dimensional Tverberg k-partition of P with the sets of the partition having sizes $r_1, \ldots, r_k$ can be computed in time $O(nd\lceil \log k \rceil)$.*

4.1 Algorithm for the general case

The input is a set of n points $P \subset \mathbb{R}^d$ and k positive integers $r_1, \ldots, r_k$ satisfying $\sum_{i=1}^{k} r_i = n$. We use the tensor product construction from Section 2 that are derived from a graph $\mathcal{G}$. Each point of P is lifted implicitly using the vectors $\{q_1, \ldots, q_k\}$ to get the set $\{P_1, \ldots, P_n\}$. We then compute a traversal of $\{P_1, \ldots, P_n\}$ using the method of conditional expectations [2], the details of which can be found below. Grouping the points of the traversal according to the lifting vectors used gives us the required partition. We remark that in our algorithm we do not explicitly lift any vector using the tensor product, thereby avoiding costs associated with working on vectors in $d\|\mathcal{G}\|$ dimensions.

We now describe a procedure to find a traversal that corresponds to a desired partition of P. We go over the points in $\{P_1, \ldots, P_n\}$ iteratively in reverse order and find the traversal $Y = (y_1 \in P_1, \ldots, y_n \in P_n)$ point by point. More precisely, we determine y_n in the first step, then y_{n-1} in the second step, and so on. In the first step, we go over all points of P_n and select any point $y_n \in P_n$ that satisfies $\mathbb{E}\left(c\|(x_1, x_2, \ldots, x_{n-1}, y_n)\|^2\right) \leq \mathbb{E}\left(c\|(x_1, x_2, \ldots, x_{n-1}, x_n)\|^2\right)$. For the general step, suppose we have already selected the points $\{y_{s+1}, y_{s+2}, \ldots, y_n\}$. To determine y_s, we choose any point from P_s that achieves

$$\mathbb{E}\left(\|c(x_1, x_2, \ldots, x_{s-1}, y_s, y_{s+1}, \ldots, y_n)\|^2\right) \leq \mathbb{E}\left(\|c(x_1, x_2, \ldots, x_s, y_{s+1}, \ldots, y_n)\|^2\right). \tag{6}$$

The last step gives the required traversal. We expand $\mathbb{E}(\|c(x_1, x_2, \ldots, x_{s-1}, y_s, \ldots, y_n)\|^2)$ to

$$\begin{aligned}
&\mathbb{E}\left(\left\|\frac{1}{n}\left(\sum_{i=1}^{s-1} x_i + \sum_{i=s}^{n} y_i\right)\right\|^2\right) = \frac{1}{n^2}\mathbb{E}\left(\left\|\left(\sum_{i=1}^{s-1} x_i + \sum_{i=s+1}^{n} y_i\right) + y_s\right\|^2\right) \\
=&\frac{1}{n^2}\left(\mathbb{E}\left(\left\|\sum_{i=1}^{s-1} x_i + \sum_{i=s+1}^{n} y_i\right\|^2\right) + \|y_s\|^2 + 2\left\langle y_s, \mathbb{E}\left(\sum_{i=1}^{s-1} x_i + \sum_{i=s+1}^{n} y_i\right)\right\rangle\right) \\
=&\frac{1}{n^2}\left(\mathbb{E}\left(\left\|\sum_{i=1}^{s-1} x_i + \sum_{i=s+1}^{n} y_i\right\|^2\right) + \|y_s\|^2 + 2\left\langle y_s, \mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right) + \sum_{i=s+1}^{n} y_i\right\rangle\right).
\end{aligned}$$

We pick a y_s for which $\mathbb{E}(\|c(x_1, x_2, \ldots, x_{s-1}, y_s, \ldots, y_n)\|^2)$ is at most the average over all choices of $y_s \in P_s$. As the term $\mathbb{E}\left(\left\|\sum_{i=1}^{s-1} x_i + \sum_{i=s+1}^{n} y_i\right\|^2\right)$ is constant over all choices of y_s, and the factor $\frac{1}{n^2}$ is constant, we can remove them from consideration. We are left with

$$\|y_s\|^2 + 2\left\langle y_s, \mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right) + \sum_{i=s+1}^{n} y_i\right\rangle = \|y_s\|^2 + 2\left\langle y_s, \mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right)\right\rangle + 2\langle y_s, \sum_{i=s+1}^{n} y_i\rangle. \tag{7}$$

Let $y_s = p_s \otimes q_i$. The first term is $\|y_s\|^2 = \|p_s \otimes q_i\|^2 = \|p_s\|^2\|q_i\|^2$. Let $r'_1, \ldots, r'_k$ be the number of elements of $T_1, \ldots, T_k$ that are yet to be determined. In the beginning, $r'_i = r_i$ for each i. Using the coefficients from Section 3, $\mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right)$ can be written as

$$\mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right) = \sum_{i=1}^{s-1}\sum_{j=1}^{k} p_{i,j}\frac{r'_j}{s-1} = \sum_{j=1}^{k}\frac{r'_j}{s-1}\sum_{i=1}^{s-1} p_{i,j} = \sum_{j=1}^{k}\frac{r'_j}{s-1}\sum_{i=1}^{s-1} p_i \otimes q_j$$
$$= \frac{1}{s-1}\sum_{j=1}^{k} r'_j\left(\sum_{i=1}^{s-1} p_i\right)\otimes q_j = \left(\frac{1}{s-1}\sum_{i=1}^{s-1} p_i\right)\otimes\left(\sum_{j=1}^{k} r'_j q_j\right) = c_{s-1}\otimes\left(\sum_{j=1}^{k} r'_j q_j\right),$$

where $c_{s-1} = \frac{\sum_{i=1}^{s-1} p_i}{s-1}$ is the centroid of the first $(s-1)$ points. Using this, the second term can be simplified as

$$2\left\langle y_s, \mathbb{E}\left(\sum_{i=1}^{s-1} x_i\right)\right\rangle = 2\left\langle p_s\otimes q_i, c_{s-1}\otimes\left(\sum_{j=1}^{k} r'_j q_j\right)\right\rangle = 2\left\langle p_s, c_{s-1}\right\rangle\left\langle q_i, \sum_{j=1}^{k} r'_j q_j\right\rangle$$
$$= 2\langle p_s, c_{s-1}\rangle\left(r'_i\|q_i\|^2 - \sum_{v_iv_j\in E} r'_j\right) = \langle p_s, c_{s-1}\rangle R_i,$$

where $R_i = 2\left(r'_i\|q_i\|^2 - \sum_{v_iv_j\in E} r'_j\right)$. The third term is

$$2\left\langle y_s, \sum_{j=s+1}^{n} y_j\right\rangle = 2\sum_{j=s+1}^{n}\langle y_s, y_j\rangle = 2\sum_{j=s+1}^{n}\langle p_s\otimes q_i, p_j\otimes q_{i_j}\rangle$$
$$= 2\sum_{j=s+1}^{n}\langle p_s, p_j\rangle\langle q_i, q_{i_j}\rangle = 2\left\langle p_s, \sum_{p\in T_i} p\|q_i\|^2 - \sum_{j:v_iv_j\in E}\sum_{p\in T_j} p\right\rangle$$
$$= \left\langle p_s, 2\left(\|q_i\|^2\sum_{p\in T_i} p - \sum_{j:v_iv_j\in E}\sum_{p\in T_j} p\right)\right\rangle = \langle p_s, U_i\rangle,$$

where $U_i = 2\left(\|q_i\|^2\sum_{p\in T_i} p - \sum_{j:v_iv_j\in E}\sum_{p\in T_j} p\right)$ and T_j represents the set of points in $p_{s+1},\ldots,p_n$ that was lifted using q_j in the traversal. Collecting the three terms, we get

$$\|p_s\|^2\|q_i\|^2 + \langle p_s, c_{s-1}\rangle R_i + \langle p_s, U_i\rangle = \alpha_s N_i + \beta_s R_i + \langle p_s, U_i\rangle, \tag{8}$$

with $N_i = \|q_i\|^2$, $\alpha_s := \|p_s\|^2$, $\beta_s := \langle p_s, c_{s-1}\rangle$. The terms α_s, β_s, p_s are fixed for iteration s.

Algorithm. For each $s \in [1, n]$, we pre-compute the prefix sums $\sum_{a=1}^{s} p_a$, and α_s and β_s. With this information, it is straightforward to compute a traversal in $O(ndk)$ time by evaluating the expression for each choice of p_s. We describe a more careful method that reduces this time to $O(nd\lceil\log k\rceil)$.

We assume that $\mathcal{G}$ is a balanced μ-ary tree. Recall that each node v_i of $\mathcal{G}$ corresponds to a vector q_i. We augment $\mathcal{G}$ with the following additional information for each node v_i:

- $N_i = \|q_i\|^2$: recall that this is the degree of v_i.
- N_i^{st}: this is the average of the N_j over all elements v_j in the subtree rooted at v_i.
- r'_i: as before, this is the number of elements of the set T_i of the partition that are yet to be determined. We initialize each $r'_i := r_i$.
- $R_i = 2\left(r'_i N_i - \sum_{v_iv_j\in E} r'_j\right)$, that is, $r'_i N_i$ minus the r'_j for each node v_j that is a neighbor of v_i in $\mathcal{G}$, times two. We initialize $R_i := 0$.

- R_i^{st}: this is the average of the R_j values over all nodes v_j in the subtree rooted at v_i. We initialize this to 0.
- T_i, u_i: as before, T_i is the set of vectors of the traversal that was lifted using q_i. u_i is the sum of the vectors of T_i. We initialize $T_i = \varnothing$ and $u_i = \mathbf{0}$.
- $U_i = 2\left(\|q_i\|^2 \sum_{p \in T_i} p - \sum_{j: v_i v_j \in E} \sum_{p \in T_j} p\right) = 2\left(u_i N_i - \sum_{v_i v_j \in E} u_j\right)$, initially $\mathbf{0}$.
- U_i^{st}: this is the average of the vectors U_j for all nodes v_j in the subtree of v_i. U^{st} is initialized as $\mathbf{0}$ for each node.

Additionally, each node contains pointers to its children and parents. N^{st}, R^{st} are initialized in one pass over $\mathcal{G}$.

In step s, we find an $i \in [k]$ for which Equation (8) has a value at most the average

$$
\begin{aligned}
A_s &= \frac{1}{k}\left(\sum_{i=1}^{k} \alpha_s N_i + \beta_s R_i + \langle p_s, U_i\rangle\right) = \frac{\alpha_s}{k}\sum_{i=1}^{k} N_i + \frac{\beta_s}{k}\sum_{i=1}^{k} R_i + \left\langle p_s, \frac{1}{k}\sum_{i=1}^{k} U_i\right\rangle \\
&= \alpha_s N_1^{st} + \beta_s R_1^{st} + \langle p_s, U_1^{st}\rangle,
\end{aligned}
$$

where v_1 is the root of $\mathcal{G}$. Then y_s satisfies Equation (6).

To find such a node v_i, we start at the root $v_1 \in \mathcal{G}$. We compute the average A_s and evaluate Equation (8) at v_1. If the value is at most A_s, we report success, setting $i = 1$. If not, then for at least one child v_m of v_1, the average for the subtree is less than A_s, that is, $\alpha_s N_m^{st} + \beta_s R_m^{st} + \langle p_s, U_m^{st}\rangle < A_s$. We scan the children of v_1 and compute the expression to find such a node v_m. We recursively repeat the procedure on the subtree rooted at v_m, and so on until we find a suitable node. There is at least one node in the subtree at v_m for which Equation (8) evaluates to less than A_s, so the procedure is guaranteed to find such a node.

Let v_i be the chosen node. We update the information stored in the nodes of the tree for the next iteration. We set

- $r_i' := r_i' - 1$ and $R_i := R_i - 2N_i$. Similarly we update the R_i values for neighbors of v_i.
- We set $T_i := T_i \cup \{p_s\}$, $u_i := u_i + p_s$ and $U_i := U_i + 2N_i p_s$. Similarly we update the U_i values for the neighbors.
- For each child of v_i and each ancestor of v_i on the path to v_1, we update R^{st} and U^{st}.

After the last step of the algorithm, we get a partition $T_1, \ldots, T_k$ of P. The set of points $\{T_1 \otimes q_1, \ldots, T_k \otimes q_k\}$ is a traversal of $\{P_1, \ldots, P_n\}$, hence using Lemma 2.2 the sets $T_1, \ldots, T_k$ form the required partition of P. This completes the description of the algorithm.

Proof of Theorem 4.1 for the general case. Computing the prefix sums and α_s, β_s takes $O(nd)$ time in total. Creating and initializing the tree takes $O(k)$ time. In step s, computing the average A_s and evaluating Equation 8 takes $O(d)$ time per node. Therefore, computing Equation 8 for the children of a node takes $O(d\mu)$ time, as $\mathcal{G}$ is a μ-ary tree. In the worst case, the search for v_i starts at the root and goes to a leaf, exploring $O(\mu\lceil\log_\mu k\rceil)$ nodes in the process and hence takes $O(d\mu\lceil\log_\mu k\rceil)$ time. For updating the tree, the information local to v_i and its neighbors can be updated in $O(d\mu)$ time. To update R^{st} and U^{st} we travel on the path to the root, which can be of length $O(\lceil\log_\mu k\rceil)$ in the worst case, and hence takes $O(d\mu\lceil\log_\mu k\rceil)$ time. There are n steps in the algorithm, each taking $O(d\mu\lceil\log_\mu k\rceil)$ time. Overall, the running time is $O(nd\mu\lceil\log_\mu k\rceil)$ which is minimized for a 3-ary tree. ◀

4.2 Algorithm for the balanced case

In the case of balanced traversals, $\mathcal{G}$ is chosen to be a star graph as was done in Section 2. Let q_1 correspond to the root of the graph and $q_2, \ldots, q_k$ correspond to the leaves. In this case the objective function $\alpha_s N_i + \beta_s R_i + \langle p_s, U_i\rangle$ from the general case can be simplified:

- for $i = 2, \ldots, k$, we have that $R_i = 2\left(r'_i\|q_i\|^2 - \sum_{v_i v_j \in E} r'_j\right) = 2\,(r'_i - r'_1)$. Also, we have $U_i = 2\left(\sum_{p \in T_i} p\|q_i\|^2 - \sum_{p \in T_j \wedge v_i v_j \in E} p\right) = 2\left(\sum_{p \in T_i} p - \sum_{p \in T_1} p\right)$.
- for the root v_1, $R_i = 2\left(r'_i\|q_i\|^2 - \sum_{v_i v_j \in E} r'_j\right) = 2\left((k-1)r'_1 - \sum_{j=2}^{k} r'_j\right)$. Also, we can write $U_i = 2\left(\|q_i\|^2 \sum_{p \in T_i} p - \sum_{p \in T_j \wedge v_i v_j \in E} p\right) = 2\left((k-1)\sum_{p \in T_i} p - \sum_{p \in T_2 \cup \cdots \cup T_k} p\right)$.

We can augment $\mathcal{G}$ with information at the nodes just as in the general case, and use the algorithm to compute the traversal. However, this would need time $O(nd\mu\lceil\log_\mu k\rceil) = O(ndk)$ since $\mu = (k-1)$ and the height of the tree is 1. Instead, we use an auxiliary balanced ternary rooted tree $\mathcal{T}$ for the algorithm. $\mathcal{T}$ contains k nodes, each associated to one of the vectors $q_1, \ldots, q_k$ in an arbitrary fashion. We augment the tree with the same information as in the general case, but with one difference: for each node v_i, the values of R_i and U_i are updated according to the adjacency in $\mathcal{G}$ and not using the edges of $\mathcal{T}$. Then we can simply use the algorithm for the general case to get a balanced partition. The modification does not affect the complexity of the algorithm.

5 Conclusion and future work

We gave efficient algorithms for a no-dimensional version of Tverberg theorem and for a colorful counterpart. To achieve this end, we presented a refinement of Sarkaria's tensor product construction by defining vectors using a graph. The choice of the graph was different for the general- and the balanced-partition cases and also influenced the time complexity of the algorithms. It would be a worthwhile exercise to look at more applications of this refined tensor product method. Another option could be to look at non-geometric generalizations based on similar ideas.

The radius bound that we obtain for the Tverberg partition is $\sqrt{k}$ off the optimal bound in [1]. This seems to be a limitation in handling Equation (4). It is not clear if this is an artifact of using tensor product constructions. It would be interesting to explore if this factor can be brought down without compromising on the algorithmic complexity. In the general partition case, setting $r_1 = \cdots = r_k$ gives a bound that is $\sqrt{\lceil\log k\rceil}$ worse than the balanced case, so there is some scope for optimization. In the colorful case, the radius bound is again $\sqrt{k}$ off the optimal [1], but with a silver lining. The bound is proportional to $\max_i \mathbb{D}(P_i)$ in contrast to $\mathbb{D}(P_1 \cup \cdots \cup P_n)$ in [1], which is better when the colors are well-separated.

The algorithm for colorful Tverberg has a worse runtime than the regular case. The challenge in improving the runtime lies a bit with selecting an optimal graph as well as the nature of the problem itself. Each iteration in the algorithm looks at each of the permutations $\pi_1, \ldots, \pi_k$ and computes the respective expectations. The two non-zero terms in the expectation are both computed using the chosen permutation. The permutation that minimizes the first term can be determined quickly if $\mathcal{G}$ is chosen as a path graph. This worsens the radius bound by $\sqrt{k-1}$. Further, computing the other (third) term of the expectation still requires $O(k)$ updates per permutation and therefore $O(k^2)$ updates per iteration, thereby eliminating the utility of using an auxiliary tree to determine the best permutation quickly. The optimal approach for this problem is unclear at the moment.

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