Bounding Radon Number via Betti Numbers

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Abstract

We prove general topological Radon-type theorems for sets in $\mathbb{R}^d$, smooth real manifolds or finite dimensional simplicial complexes. Combined with a recent result of Holmsen and Lee, it gives fractional Helly theorem, and consequently the existence of weak ε-nets as well as a (p, q)-theorem.

More precisely: Let X be either $\mathbb{R}^d$, smooth real d-manifold, or a finite d-dimensional simplicial complex. Then if $\mathcal{F}$ is a finite, intersection-closed family of sets in X such that the ith reduced Betti number (with $\mathbb{Z}_2$ coefficients) of any set in $\mathcal{F}$ is at most b for every non-negative integer i less or equal to k, then the Radon number of $\mathcal{F}$ is bounded in terms of b and X. Here k is the smallest integer larger or equal to $d/2 - 1$ if $X = \mathbb{R}^d$; $k = d - 1$ if X is a smooth real d-manifold and not a surface, $k = 0$ if X is a surface and $k = d$ if X is a d-dimensional simplicial complex.

Using the recent result of the author and Kalai, we manage to prove the following optimal bound on fractional Helly number for families of open sets in a surface: Let $\mathcal{F}$ be a finite family of open sets in a surface S such that the intersection of any subfamily of $\mathcal{F}$ is either empty, or path-connected. Then the fractional Helly number of $\mathcal{F}$ is at most three. This also settles a conjecture of Holmsen, Kim, and Lee about an existence of a (p, q)-theorem for open subsets of a surface.

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1 Introduction

The classical Radon's theorem [15] states that it is possible to split any $d + 2$ points in $\mathbb{R}^d$ into two disjoint parts whose convex hulls intersect. It is natural to ask what happens to the statement, if one starts varying the notion of convexity.

Perhaps the most versatile generalization of the convex hull is the following. Let X be an underlying set and let $\mathcal{F}$ be a finite family of subsets of X. Let $S \subseteq X$ be a set. The *convex hull* $\operatorname{conv}_{\mathcal{F}}(S)$ *of* S *relative to* $\mathcal{F}$ is defined as the intersection of all sets from $\mathcal{F}$ that contain S. If there is no such set, the convex hull is, by definition, X. If $\operatorname{conv}_{\mathcal{F}} S = S$, the set S is called $\mathcal{F}$*-convex*.

This definition is closely related to so called *convexity spaces*[1], as defined for example in [18, 2, 17]. The only difference is that most authors require that in a convexity space $\operatorname{conv}\emptyset = \emptyset$, which is not needed in any of our considerations. Moreover, it can be easily forced by including $\emptyset$ to $\mathcal{F}$.

In our examples we are also going to use the definition of $\operatorname{conv}_{\mathcal{F}}$ for the family $\mathcal{F}$ of all (standard) convex sets in $\mathbb{R}^d$. We note that in this case $\operatorname{conv}_{\mathcal{F}}$ coincides with the standard convex hull.

We say that $\mathcal{F}$ has *Radon number* $r(\mathcal{F})$ if $r(\mathcal{F})$ is the smallest integer r such that any set $S \subseteq X$ of cardinality r can be split into two parts $S = P_1 \sqcup P_2$ satisfying $\operatorname{conv}_{\mathcal{F}}(P_1) \cap \operatorname{conv}_{\mathcal{F}}(P_2) \neq \emptyset$. If no such r exists, we put $r(\mathcal{F}) = \infty$. We note that Radon number is anti-monotone in the sense that $r(\mathcal{F}) \leq r(\mathcal{G})$ for $\mathcal{G} \subseteq \mathcal{F}$.

In this paper we show that very mild topological conditions are enough to force a bound on Radon number for sets in Euclidean space (Theorem 1). A simple trick allows us to give a version of the result for smooth manifolds or simplicial complexes, see Section 2.1. Furthermore, the proof technique also works for surfaces (Theorem 2). In Section 2.2 we list some important consequences, most notably a fractional Helly theorem (Theorem 3), which allows us to solve a conjecture of Holmsen, Kim, and Lee (a special case of Theorem 6).

2 New results

One can observe that bounded Radon number is not a property of a standard convexity since it is preserved by topological deformations of $\mathbb{R}^d$. In fact, we can even show that if the family $\mathcal{F}$ is *"not too topologically complicated"*, its Radon number is bounded. Let us first explain what "not too topologically complicated" means.

Topological complexity. Let $k \geq 1$ be an integer or ∞ and $\mathcal{F}$ a family of sets in a topological space X. We define the *k-level topological complexity* of $\mathcal{F}$ as:

$$\sup\left\{\widetilde{\beta}_i\left(\bigcap \mathcal{G}; \mathbb{Z}_2\right) : \mathcal{G} \subseteq \mathcal{F}, 0 \leq i < k\right\}$$

and denote it by $TC_k(\mathcal{F})$. We call the number $TC_\infty(\mathcal{F})$ the *(full) topological complexity.*

Examples. Let us name few examples of families with bounded topological complexity: family of convex sets in $\mathbb{R}^d$, good covers[2], families of spheres and pseudospheres in $\mathbb{R}^d$, finite families of *semialgebraic sets* in $\mathbb{R}^d$ defined by a constant number of polynomial inequalities, where all polynomials have a constant degree, etc.

We can now state our main theorem.

▶ **Theorem 1** (Bounded mid-level topological complexity implies Radon). *For every non-negative integers b and d there is a number $r(b, d)$ such that the following holds: If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ with $TC_{\lceil d/2\rceil}(\mathcal{F}) \leq b$, then $r(\mathcal{F}) \leq r(b, d)$.*

Qualitatively, Theorem 1 is sharp in the sense that all (reduced) Betti numbers $\widetilde{\beta}_i$, $0 \leq i < \lceil d/2\rceil$, need to be bounded in order to obtain a bounded Radon number, see [7, Example 3].

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[1] A pair $(X, \mathcal{C})$ is called a *convexity space* on X if $\mathcal{C} \subset 2^X$ is a family of subsets of X such that $\emptyset, X \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking intersections; and unions of chains. The sets in $\mathcal{C}$ are called *convex*. Note that the last condition is trivially satisfied whenever $\mathcal{C}$ is finite.

[2] A family of sets in $\mathbb{R}^d$ where intersection of each subfamily is either empty or contractible.

2.1 Embeddability

We have seen that for a finite family of sets in $\mathbb{R}^d$, in order to have a bounded Radon number, it suffices to restrict the reduced Betti numbers up to $\lceil d/2 \rceil - 1$. Which Betti numbers do we need to restrict, if we replace $\mathbb{R}^d$ by some other topological space X? The following paragraphs provide some simple bounds if X is a simplicial complex or a smooth real manifold. The base for the statements is the following simple observation: Given a topological space X embeddable into $\mathbb{R}^d$, we may view any subset of X as a subset of $\mathbb{R}^d$ and use Theorem 1.

Since any (finite) k-dimensional simplicial complex embeds into $\mathbb{R}^{2k+1}$, we have:

If K is a (finite) k-dimensional simplicial complex and $\mathcal{F}$ is a finite family of sets in K with $TC_{k+1}(\mathcal{F}) \leq b$, then $r(\mathcal{F}) \leq r(b, 2k+1)$.

Again, this bound is qualitatively sharp in the sense that all $\widetilde{\beta}_i$, $0 \leq i \leq k$, need to be bounded in order to have a bounded Radon number, see [7, Example 3].

Using the strong Whitney's embedding theorem [19], stating that any smooth real k-dimensional manifold embeds into $\mathbb{R}^{2k}$, we obtain the following:

If M is a smooth k-dimensional real manifold and $\mathcal{F}$ is a finite family of sets in M with $TC_k(\mathcal{F}) \leq b$, then $r(\mathcal{F}) \leq r(b, 2k)$.

Unlike in the previous statements we do not know whether bounding all reduced Betti numbers $\widetilde{\beta}_i$, $0 \leq i \leq k-1$, is necessary. The following result about surfaces indicates that it possibly suffices to bound less. Let $\mathcal{F}$ be a finite family $\mathcal{F}$ of sets in a surface[3] S. In order to have a finite Radon number $r(\mathcal{F})$, it is enough to require that $TC_1(\mathcal{F})$ is bounded, that is, it only suffices to have a universal bound on the number of connected components.

▶ **Theorem 2.** *For each surface S and each integer $b \geq 0$ there is a number $r_S(b)$ such that each finite family $\mathcal{F}$ of sets in S satisfying $TC_1(\mathcal{F}) \leq b$ has $r(\mathcal{F}) \leq r_S(b)$.*

See Section 3.2 for the proof.

However, at the present time the author does not know how to generalize this result to higher dimensional manifolds. Given a d-dimensional manifold M, it is an open question whether $r(\mathcal{F})$ is bounded for all families $\mathcal{F} \subseteq M$ with bounded $TC_{\lceil d/2 \rceil}(\mathcal{F})$.

2.2 Consequences and related results

We say that $\mathcal{F}$ has *Helly number* $h(\mathcal{F})$, if $h(\mathcal{F})$ is the smallest integer h with the following property: If in a finite subfamily $\mathcal{S} \subseteq \mathcal{F}$ each h members of $\mathcal{S}$ have a point in common, then all the sets of $\mathcal{S}$ have a point in common. If no such h exists, we put $h(\mathcal{F}) = \infty$. By older results, bounded Radon number implies bounded Helly number [12] as well as bounded Tverberg numbers[4] [10, (6)]. From these consequences only the fact that for sets in $\mathbb{R}^d$ bounded $TC_{\lceil d/2 \rceil}$ implies bounded Helly number has been shown earlier [7].

Due to recent results by Holmsen and Lee, bounded Radon number implies colorful Helly theorem [9, Lemma 2.3] and bounded fractional Helly number [9, Theorem 1.1]. Thus, in combination with Theorem 1 and the results from the previous section, we have obtained the following fractional Helly theorem.

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[3] By a *surface* we mean a compact two-dimensional real manifold.

[4] Given an integer $k \geq 3$, we say that $\mathcal{F}$ has k^{th} *Tverberg number* $r_k(\mathcal{F})$, if $r_k(\mathcal{F})$ is the smallest integer r such that any set $S \subseteq X$ of size r_k can be split into k parts $S = P_1 \sqcup P_2 \sqcup \ldots \sqcup P_k$ satisfying $\bigcap_{i=1}^{k} \operatorname{conv}_{\mathcal{F}} P_i \neq \emptyset$. We set $r_k(\mathcal{F}) = \infty$ if there is no such r_k.

▶ **Theorem 3.** *Let X be either $\mathbb{R}^d$, in which case we set $k = \lceil d/2 \rceil$, or a smooth real d-dimensional manifold, $d \geq 3$, in which case we set $k = d$, or a surface, in which case we set $k = 1$, or a (finite) d-dimensional simplicial complex, in which case we set $k = d + 1$. Then for every integer $b \geq 0$ there is a number $h_f = h_f(b, X)$ such that the following holds. For every $\alpha \in (0, 1]$ there exists $\beta = \beta(\alpha, b, X) > 0$ with the following property. Let $\mathcal{F}$ be a family of sets in X with $TC_k(\mathcal{F}) \leq b$ and $\mathcal{G}$ be a finite family of $\mathcal{F}$-convex sets, having at least an α fraction of the h_f-tuples with non-empty intersection, then there is a point contained in at least $\beta|\mathcal{G}|$ sets of $\mathcal{G}$.*

We note that Theorem 3 can be applied to many spaces X that are often encountered in geometry. Let us mention $\mathbb{R}^d$, Grassmanians, or flag manifolds.

We refer to the number h_f from the theorem as the *fractional Helly number*. Bounded fractional Helly number in turn provides a weak ε-net theorem [1] and a (p, q)-theorem [1]. The existence of a fractional Helly theorem for sets with bounded topological complexity might be seen as the most important application of Theorem 1, not only because it implies an existence of weak ε-nets and a (p, q)-theorem, but also in its own right. Its existence answers positively a question by Matoušek (personal communication), also mentioned in [3, Open Problem 3.6].

The bound on h_f we obtain from the proof is not optimal. So what is the optimal bound? The case of $(d-1)$-flats in $\mathbb{R}^d$ in general position shows that we cannot hope for anything better than $d + 1$. In Section 4 we establish a reasonably small bound for a large class of families $\mathcal{F}$ of open subsets of surfaces using a bootstrapping method based on the result of the author and Kalai [11]. In particular, for families $\mathcal{F}$ of open sets with $TC_1(\mathcal{F}) = 0$, we obtain the optimal bound.

▶ **Theorem 4** (Fractional Helly for surfaces)**.** *Let $b \geq 0$ be an integer. We set $k = 3$ for $b = 0$ and $k = 2b + 4$ for $b \geq 1$, respectively. Then for any surface S and $\alpha \in (0, 1)$ there exists $\beta = \beta(\alpha, b, S) > 0$ with the following property. Let $\mathcal{A}$ be a family of n open subsets of a surface S with $TC_1(\mathcal{A}) \leq b$. If at least $\alpha\binom{n}{k}$ of the k-tuples of $\mathcal{A}$ are intersecting, then there is intersecting subfamily of $\mathcal{A}$ of size at least βn.*

We note that the statement holds also for a family of open sets in $\mathbb{R}^2$, since the plane can be seen as an open subset of a 2-dimensional sphere.

The author conjectures that k in Theorem 4 is independent of b, more precisely, the conjectured value is three. The author also conjectures that the fractional Helly number for families in $\mathbb{R}^d$ is $d + 1$.

▶ **Conjecture 5.** *For any integers $b \geq 1, d \geq 2$ and $\alpha \in (0, 1)$ there exists $\beta = \beta(\alpha, b, d) > 0$ with the following property. Let $\mathcal{A}$ be a family of $n \geq d + 1$ sets in $\mathbb{R}^d$ with $TC_{\lceil d/2 \rceil}(\mathcal{A}) \leq b$. If at least $\alpha\binom{n}{d+1}$ of the $(d + 1)$-tuples of $\mathcal{A}$ intersect, then there is an intersecting subfamily of $\mathcal{A}$ of size at least βn.*

The proof of Theorem 4 is given in Section 4. By the results in [1], the fractional Helly theorem is the only ingredient needed to prove a (p, q)-theorem, hence combining Theorem 4 with results in [1] immediately gives Theorem 6. Let us recall that a family $\mathcal{F}$ of sets has the *(p, q)-property* if among every p sets of $\mathcal{F}$, some q have a point in common.

▶ **Theorem 6.** *Let $b \geq 0$ be an integer. Set $k = 3$ for $b = 0$ and $k = 2b + 4$ for $b \geq 1$, respectively. For any integers $p \geq q \geq k$ and a surface S, there exists an integer $C = C(p, q, S)$ such that the following holds. Let $\mathcal{F}$ be a finite family of open subsets of S with $TC_1(\mathcal{F}) \leq b$. If $\mathcal{F}$ has the (p, q)-property, then there is a set X that intersects all sets from $\mathcal{F}$ and has at most C elements.*

The case $b = 0$ in Theorem 6 settles a conjecture by Holmsen, Kim, and Lee [8, Conj. 5.3].

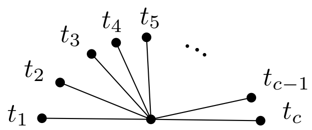
We have seen that bounded topological complexity has many interesting consequences. However, there is one parameter of $\mathcal{F}$ that cannot be bounded by the topological complexity alone. We say that $\mathcal{F}$ has *Carathéodory number* $c(\mathcal{F})$, if c is the smallest integer c with the following property: For any set $S \subseteq X$ and any point $x \in \operatorname{conv}_{\mathcal{F}}(S)$, there is a subset $S' \subseteq S$ of size at most c such that $x \in \operatorname{conv}_{\mathcal{F}}(S')$. If no such c exists, we put $c(\mathcal{F}) = \infty$.

It is easy to construct an example of a finite $\mathcal{F}$ of bounded full-level topological complexity with arbitrarily high Carathéodory's number.

▶ **Theorem 7** (Bounded topological complexity does not imply Carathéodory). *For every positive integers $c \geq 2$ and $d \geq 2$ there is a finite family $\mathcal{F}$ of sets in $\mathbb{R}^d$ of full-level topological complexity zero, satisfying $c(\mathcal{F}) = c$.*

Proof. Indeed, consider a star with c spines $T_1, T_2, \ldots, T_c$ each containing a point t_i. Let $A_i := \bigcup_{j \neq i} T_j$ and $\mathcal{F} = \{A_1, A_2, \ldots, A_c\}$.

Then any intersection of the sets A_i is contractible, and hence topologically trivial. Let $S = \{t_1, \ldots, t_c\}$. Observe that $\operatorname{conv}_{\mathcal{F}} S = \mathbb{R}^d$. Let x be any point in $(\operatorname{conv}_{\mathcal{F}} S) \setminus \bigcup_{i=1}^{c} A_i$. Then $x \in \operatorname{conv}_F S$, and $x \notin \operatorname{conv}_F S'$ for any $S' \subsetneq S$. Thus $c(\mathcal{A}) = c$.



◀

3 Technique

The introduction of relative convex hulls allows us to strengthen and polish the techniques developed in [7]. Independently of these changes we also manage to separate the combinatorial and topological part of the proof, which improves the overall exposition. We start with the topological tools (Sections 3.1 and 3.2) including the proof of Theorem 1 modulo Proposition 13. We divide the proof of the main ingredient (Proposition 13) into two parts: Ramsey-type result (Section 3.3) and induction (Section 3.4).

Notation & convention. For an integer $n \geq 1$, let $[n] = \{1, \ldots, n\}$. If P is a set, we use the symbol 2^P to denote the set of all its subsets and $\binom{P}{n}$ to denote the family of all n-element subsets of P. We denote by Δ_n the standard n-dimensional simplex. If K is a simplicial complex, $V(K)$ stands for its set of vertices and $K^{(k)}$ stands for its k-dimensional skeleton, i.e. the subcomplex formed by all its faces of dimension up to k. Unless stated otherwise, we only work with abstract simplicial complexes.[5] All chain groups and chain complexes are considered with $\mathbb{Z}_2$-coefficients.

3.1 Homological almost embeddings

Homological almost embeddings are the first ingredient we need. Before defining them, let us first recall (standard) almost embeddings. Let $\mathbf{R}$ be a topological space.

[5] The definition of singular homology forces us to use the geometric standard simplex Δ_n on some places.

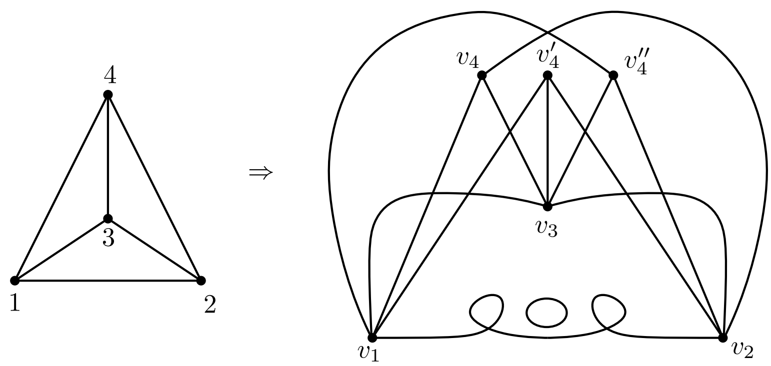


Figure 1 An example of a homological almost-embedding of K_4 into the plane.

▶ **Definition 8.** *Let K be an (abstract) simplicial complex with geometric realization $|K|$ and $\mathbf{R}$ a topological space. A continuous map $f\colon |K| \to \mathbf{R}$ is an* almost-embedding *of K into $\mathbf{R}$, if the images of disjoint simplices are disjoint.*

▶ **Definition 9.** *Let K be a simplicial complex, and consider a chain map $\gamma\colon C_*(K;\mathbb{Z}_2) \to C_*(\mathbf{R};\mathbb{Z}_2)$ from the simplicial chains in K to singular chains in $\mathbf{R}$.*

(i) *The chain map γ is called* nontrivial[6] *if the image of every vertex of K is a finite set of points in $\mathbf{R}$ (a 0-chain) of* odd *cardinality.*

(ii) *The chain map γ is called a* homological almost-embedding *of K in $\mathbf{R}$ if it is nontrivial and if, additionally, the following holds: whenever σ and τ are disjoint simplices of K, their image chains $\gamma(\sigma)$ and $\gamma(\tau)$ have disjoint supports, where the support of a chain is the union of (the images of) the singular simplices with nonzero coefficient in that chain.*

In analogy to almost-embeddings, there is no homological almost-embedding of the k-skeleton of $(2k+2)$-dimensional simplex into $\mathbb{R}^{2k}$:

▶ **Theorem 10** (Corollary 13 in [7]). *For any $k \geq 0$, the k-skeleton $\Delta^{(k)}_{2k+2}$ of the $(2k+2)$-dimensional simplex has no homological almost-embedding in $\mathbb{R}^{2k}$.*

Let us say a few words about the proof. It is based on the standard cohomological proof of the fact that $\Delta^{(k)}_{2k+2}$ does not "almost-embed" into $\mathbb{R}^{2k}$ and combined with the fact that cohomology "does not distinguish" between maps and non-trivial chain maps. For details see [7].

3.2 Constrained chain maps

We continue developing the machinery from [7] in order to capture our more general setting. To prove Theorem 1, we need one more definition (Definition 11). A curious reader may compare our definition of constrained chain map with the definition from [7]. Let us just remark that the definition presented here is more versatile. (Although it might not be obvious on the first sight.) Unlike the previous definition, the current form allows us to prove the bound on the Radon number. Nevertheless, both definitions are equivalent under some special circumstances.

[6] If we consider augmented chain complexes with chain groups also in dimension -1, then being nontrivial is equivalent to requiring that the generator of $C_{-1}(K) \cong \mathbb{Z}_2$ (this generator corresponds to the empty simplex in K) is mapped to the generator of $C_{-1}(\mathbf{R}) \cong \mathbb{Z}_2$.

Let $\mathbf{R}$ be a topological space, let K be a simplicial complex and let $\gamma : C_*(K) \to C_*(\mathbf{R})$ be a chain map from the simplicial chains of K to the singular chains of $\mathbf{R}$.

▶ **Definition 11** (Constrained chain map). *Let $\mathcal{F}$ be a finite family of sets in $\mathbf{R}$ and P be a (multi-)set*[7] *of points in $\mathbf{R}$. Let $\gamma : C_*(K) \to C_*(\mathbf{R})$ be the aforementioned chain map. We say that γ is* constrained by $(\mathcal{F}, \Phi)$ *if:*

(i) *Φ is a map from K to 2^P such that $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$ for all $\sigma, \tau \in K$ and $\Phi(\emptyset) = \emptyset$.*
(ii) *For any simplex $\sigma \in K$, the support of $\gamma(\sigma)$ is contained in $\operatorname{conv}_{\mathcal{F}} \Phi(\sigma)$.*

If there is some Φ such that a chain map γ from K is constrained by $(\mathcal{F}, \Phi)$, we say that γ is constrained by $(\mathcal{F}, P)$.

We can now prove an analogue of Lemma 26 from [7] and relate constrained maps and homological almost embeddings.

▶ **Lemma 12.** *Let $\gamma : C_*(K) \to C_*(\mathbf{R})$ be a nontrivial chain map constrained by $(\mathcal{F}, P)$. If $\operatorname{conv}_{\mathcal{F}} S \cap \operatorname{conv}_{\mathcal{F}} T = \emptyset$ whenever $S \subseteq P$ and $T \subseteq P$ are disjoint, then γ is a homological almost-embedding of K to $\mathbf{R}$.*

Proof. Let σ and τ be two disjoint simplices of K. The supports of $\gamma(\sigma)$ and $\gamma(\tau)$ are contained, respectively, in $\operatorname{conv}_{\mathcal{F}} \Phi(\sigma)$ and $\operatorname{conv}_{\mathcal{F}} \Phi(\tau)$. By the definition of Φ, $\Phi(\sigma)$ and $\Phi(\tau)$ are disjoint. Thus, by the assumption

$$\operatorname{conv}_{\mathcal{F}} \Phi(\sigma) \cap \operatorname{conv}_{\mathcal{F}} \Phi(\tau) = \emptyset.$$

Therefore, γ is a homological almost-embedding of K. ◀

The most important ingredient for the proof of Theorem 1 is the following proposition:

▶ **Proposition 13.** *For any finite simplicial complex K and a non-negative integer b there exists a constant $r_K(b)$ such that the following holds. For any finite family $\mathcal{F}$ in $\mathbf{R}$ with $TC_{\dim K}(\mathcal{F}) \leq b$ and a set P of at least $r_K(b)$ points in $\mathbf{R}$ there exists a nontrivial chain map $\gamma : C_*(K) \to C_*(\mathbf{R})$ that is constrained by $(\mathcal{F}, P)$.*

Furthermore, if $\dim K \leq 1$, one can even find such γ that is induced by some continuous map $f : |K| \to \mathbf{R}$ from the geometric realization $|K|$ of K to $\mathbf{R}$.

Before proving Theorems 1 and 2, let us relate Proposition 13 to the Radon number.

▶ **Proposition 14.** *Let $\mathbf{R}$ be a topological space and K a simplicial complex that does not homologically embed into $\mathbf{R}$. Then for each integer $b \geq 0$ and each finite family $\mathcal{F}$ of sets in $\mathbf{R}$ satisfying $TC_{\dim K}(\mathcal{F}) \leq b$, one has $r(\mathcal{F}) \leq r_K(b)$, where $r_K(b)$ is as in Proposition 13.*

Moreover, if $\dim K \leq 1$, it suffices to assume that K does not almost embed into $\mathbf{R}$.

Proof. If $r(\mathcal{F}) > r_K(b)$, then there is a set P of $r_K(b)$ points such that for any two disjoint subsets $P_1, P_2 \subseteq P$ we have $\operatorname{conv}_{\mathcal{F}}(P_1) \cap \operatorname{conv}_{\mathcal{F}}(P_2) = \emptyset$. Let $\gamma\colon C_*(K) \to C_*(\mathbf{R})$ be a nontrivial chain map constrained by $(\mathcal{F}, P)$ given by Proposition 13. By Lemma 12, γ is a homological almost-embedding of K, a contradiction.

[7] However, the switch to multisets requires some minor adjustments. If $P = \{p_i \mid i \in I\}$ is a multiset, one needs to replace the multiset P by the index set I in all definitions and proofs; and if $J \subseteq I$ consider $\operatorname{conv}_{\mathcal{F}}(J)$ as a shorthand notation for $\operatorname{conv}_{\mathcal{F}}(\{p_i \mid i \in J\})$. However, we have decided not to clutter the main exposition with such technical details.

If $\dim K \le 1$, one can take γ to be induced by a continuous map $f\colon |K| \to \mathbf{R}$. However, one can easily check that in that case γ is a homological almost embedding if and only if f is an almost embedding. ◀

Theorems 1 and 2 are now immediate consequences of Proposition 14.

Proof of Theorem 1. Let $k = \lceil d/2 \rceil$. By Theorem 10, $\Delta^{(k)}_{2k+2}$ does not homologically almost embeds into $\mathbb{R}^d$, so Proposition 14 applies and yields Theorem 1. ◀

Proof of Theorem 2. By results in [6], for each surface S there is a finite graph G that does not almost embed[8] into S, so Proposition 14 applies. ◀

3.3 Combinatorial part of the proof

The classical Ramsey theorem [16] states that for all positive integers k, n and c there is a number $R_k(n; c)$ such that the following holds. For each set X satisfying $|X| \ge R_k(n; c)$ and each coloring[9] $\rho\colon \binom{X}{k} \to [c]$, there is a *monochromatic* subset $Y \subseteq X$ of size n, where a subset Y is monochromatic, if all k-tuples in Y have the same color. Note that the case $k = 1$ corresponds to the pigeon hole principle and $R_1(n; c) = n(c-1) + 1$.

In order to perform the induction step in the proof of Proposition 13, we need the following Ramsey type theorem.

▶ **Proposition 15.** *For any positive integers k, m, n, c there is a constant $N_k = N_k(n; m; c)$ such that the following holds. Let X be a set and for every $V \subseteq X$ let $\rho_V\colon \binom{V}{k} \to [c]$ be a coloring[10] of the k-element subsets of V. If $|X| \ge N_k$, then there always exists an n-element subset $Y \subseteq X$ and a map $M_{(\cdot)}\colon \binom{Y}{m} \to 2^{X \setminus Y}$ such that all sets M_Z for $Z \in \binom{Y}{m}$ are disjoint, and each $Z \in \binom{Y}{m}$ is monochromatic in $\rho_{Z \cup M_Z}$.*

The fact that each k-tuple is colored by several different colorings ρ_V reflects the fact that we are going to color a cycle z by the singular homology of $\gamma(z)$ inside $\operatorname{conv}_{\mathcal{F}} \Phi(V)$ for various different sets V. There, it may easily happen that z and z' have the same color in V but different in V'.

Proof. Let $r = R_k(m; c)$. We claim that it is enough to take

$$N_k = R_r\left(n + \binom{n}{m} \cdot (r - m); \binom{r}{m}\right).$$

Suppose that $|X| \ge N_k$ and choose an arbitrary order of the elements of X.

By the choice of r, if $V \in \binom{X}{r}$, then there is a subset $A \subseteq V$ of size m such that ρ_V assigns the same color to all k-tuples in A. Let us introduce another coloring, $\eta\colon \binom{X}{r} \to \binom{[r]}{m}$, that colors each $V \in \binom{X}{r}$ by the relative[11] position of the first monochromatic A inside V (with respect to the lexicographic ordering).

By the definition of N_k and the fact that $|X| \ge N_k$, there is a subset U of size $n + \binom{n}{m} \cdot (r - m)$, such that all r-tuples in U have the same color in η, say color Ω.

[8] Compared to [6], recent works by Paták, Tancer [14], and Fulek, Kynčl [5] provide much smaller graphs which are not almost-embeddable into S.
[9] A coloring is just another name for a map. However, it is easier to say "the color of z", instead of "the image of z under ρ".
[10] If $|V| < k$, the coloring c_V is, by definition, the empty map.
[11] For illustration: If $V = \{2, 4, 6, 8, \dots, 36\}$ and $A = \{2, 4, 34, 36\}$ we assign V the "color" $\{1, 2, 17, 18\}$, since the elements of A are on first, second, 17th and 18th position of V.

Consider the set $Y' = \{1, 2, \dots, n\}$. Since the rational numbers are dense, we can find an assignment

$$\begin{aligned} N\colon \binom{Y'}{m} &\to \binom{\mathbb{Q} \setminus Y'}{r-m} \\ Z' &\mapsto N_{Z'} \end{aligned}$$

of mutually disjoint sets $N_{Z'}$ such that Z' is on the position Ω inside $Z' \cup N_{Z'}$.

The unique order-preserving isomorphism from $Y' \cup \bigcup N_{Z'}$ to U then carries Y' to the desired set Y and $N_{Z'}$ to the desired sets M_Z. ◀

3.4 The induction

Proof of Proposition 13. We proceed by induction on $\dim K$, similarly as in [7]. If the reader finds the current exposition too fast, we encourage him/her to consult [7] which goes slower and shows motivation and necessity of some ideas presented here. Note however, that our current setup is much more general.

Induction basis. If K is 0-dimensional with vertices $V(K) = \{v_1, \dots, v_m\}$, we set $r_K(b) = m$. If $P = \{x_1, \dots, x_n\}$ is a point set in $\mathbf{R}$ with $|P| \geq m$, we can take as Φ the map $\Phi(v_i) = \{x_i\}$. It remains to define γ. We want it to "map" v_i to x_i. However, γ should be a chain map from simplicial chains of K to singular chains in $\mathbb{R}^d$. Therefore for each vertex v_i we define $\gamma(v_i)$ as the unique map from[12] Δ_0 to x_i; and extend this definition linearly to the whole $C_0(K)$. By construction, γ is nontrivial and constrained by $(\mathcal{F}, \Phi)$.

Induction step. Let $\dim K = k \geq 1$. The aim is to find a chain map $\gamma\colon C_*(K^{(k-1)}) \to C_*(\mathbf{R})$ and a suitable map Φ such that γ is nontrivial, constrained by $(\mathcal{F}, \Phi)$ and $\gamma(\partial\sigma)$ has trivial homology inside $\operatorname{conv}_{\mathcal{F}} \Phi(\sigma)$ for each k-simplex $\sigma \in K$. Extending such γ to the whole complex K is then straightforward.

Let $s \geq 1$ be some integer depending on K which we determine later. To construct γ we will define three auxiliary chain maps

$$C_*\left(K^{(k-1)}\right) \xrightarrow{\ \alpha\ } C_*\left((\operatorname{sd} K)^{(k-1)}\right) \xrightarrow{\ \beta\ } C_*\left(\Delta_s^{(k-1)}\right) \xrightarrow{\ \gamma'\ } C_*(\mathbf{R}),$$

where $\operatorname{sd} K$ is the barycentric subdivision[13] of K.

Definition of α. We start with the easiest map, α. It maps each l-simplex σ from $K^{(k-1)}$ to the sum of the l-simplices in the barycentric subdivision of σ.

Definition of γ'. The map γ' is obtained from induction. Let the cardinality of P be large enough. Since $\dim \Delta_s^{(k-1)} = k - 1$, by induction hypothesis, there is a nontrivial chain map $\gamma' : C_*(\Delta_s^{(k-1)}) \to C_*(\mathbf{R})$ and a map $\Psi\colon \Delta_s^{(k-1)} \to 2^P$ such that γ' is constrained by $(\mathcal{F}, \Psi)$.

In order to define Φ easily, we need to extend Ψ to Δ_s, hence for $\sigma \in \Delta_s$ we define

$$\Psi(\sigma) = \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq \sigma} \Psi(\tau). \tag{1}$$

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[12] This is the only place where Δ_n is considered to be a geometric simplex.

[13] The *barycentric subdivision* $\operatorname{sd} K$ of an abstract simplicial complex K is the complex formed by all the chains contained in the partially ordered set $(K \setminus \{\emptyset\}, \subseteq)$, so called the *order complex* of $(K \setminus \{\emptyset\}, \subseteq)$.

If $\tau \subseteq \sigma \in \Delta_s^{(k-1)}$, then $\Psi(\tau) \cap \Psi(\sigma)$ was equal to $\Psi(\tau \cap \sigma) = \Psi(\tau)$. Thus the equality (1) does not change the value of $\Psi(\sigma)$ if $\sigma \in \Delta_s^{(k-1)}$ and it is indeed an extension of Ψ. Moreover, easy calculation shows that $\Psi(A) \cap \Psi(B) = \Psi(A \cap B)$ for any $A, B \in \Delta_s$.

Definition of β. With the help of Proposition 15 it is now easy to find the map β. Indeed, for each simplex $\tau \in \Delta_s$, let c_τ be the coloring that assigns to each k-simplex $\sigma \subseteq \tau$ the singular homology class of $\gamma'(\partial\sigma)$ inside $\mathrm{conv}_{\mathcal{F}}(\Psi(\tau))$. Let m be the number of vertices of $\mathrm{sd}\,\Delta_k$, n the number of vertices of $\mathrm{sd}\,K$ and c the maximal number of elements in $\widetilde{H}_k(\bigcap \mathcal{G}; \mathbb{Z}_2)$, where $\mathcal{G} \subseteq \mathcal{F}$. Clearly $c \leq 2^b$.

Thus if $s \geq N_{k+1}(n; m; c)$ from Proposition 15, the following holds.

(a) There is an inclusion j of $(\mathrm{sd}\,K)^{(k-1)}$ to a simplex $Y \subseteq \Delta_s$. We let $\varphi\colon K \to 2^{V(\Delta_s)}$ be the map that to each $\sigma \in K$ assigns the set $j(V(\mathrm{sd}\,\sigma))$.
(b) For each k-simplex μ in K there is a simplex M_μ in Δ_s with the following three properties:

- **(i)** For all k-simplices τ inside $\mathrm{sd}\,\mu$, the singular homology class of $\gamma'(j(\partial\tau))$ inside $\mathrm{conv}_{\mathcal{F}}\,\Psi(M_\mu \cup \varphi(\mu))$ is the same,
- **(ii)** each M_μ is disjoint from Y,
- **(iii)** all the simplices M_μ are mutually disjoint.

We define $M_\mu := \emptyset$ for $\mu \in K$ a simplex of dimension at most $k-1$. We set $\Phi(\mu) := \Psi(M_\mu \cup \varphi(\mu))$. Note that for a simplex $\sigma \in K^{(k-1)}$, $\Phi(\sigma)$ reduces to $\Psi(\varphi(\sigma))$.

Let β be the chain map induced by j. Observe that Φ satisfies $\Phi(\emptyset) = \emptyset$ and $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, $A, B \in K$. Indeed, the first claim is obvious and for the second one let σ, τ be distinct simplices in K:

$$\begin{aligned}\Phi(\mu) \cap \Phi(\tau) &= \Psi\left(M_\mu \cup \varphi(\mu)\right) \cap \Psi\left(M_\tau \cup \varphi(\tau)\right) = \Psi\left([M_\mu \cup \varphi(\mu)] \cap [M_\tau \cup \varphi(\tau)]\right)\\ &= \Psi(\varphi(\mu) \cap \varphi(\tau)),\end{aligned}$$

where the the second equality express the fact that Ψ respects intersections and the last equality uses both (bii) and (biii). Then

$$\Phi(\mu) \cap \Phi(\tau) = \Psi(\varphi(\mu) \cap \varphi(\tau)) = \Psi(\varphi(\mu \cap \tau)) = \Phi(\mu \cap \tau)$$

since φ obviously respects intersections and $\dim(\mu \cap \tau) \leq k-1$.

We define γ on $K^{(k-1)}$ as the composition $\gamma' \circ \beta \circ \alpha$. Then, by the definition, γ is a nontrivial chain map constrained by $(\mathcal{F}, \Phi)$. It remains to extend it to the whole complex K.

If σ is a k-simplex of K, all the k-simplices ζ in $\mathrm{sd}\,\sigma$ have the same value of $\gamma'\beta(\partial\zeta)$ inside $\mathrm{conv}_{\mathcal{F}}\,\Phi(\sigma)$. Since there is an even number of them and we work with $\mathbb{Z}_2$-coefficients, $\gamma(\partial\sigma)$ has trivial homology inside $\mathrm{conv}_{\mathcal{F}}\,\Phi(\sigma)$. So for each such σ we may pick some $\gamma_\sigma \in C_k\left(\mathrm{conv}_{\mathcal{F}}\,\Phi(\sigma); \mathbb{Z}_2\right)$ such that $\partial\gamma_\sigma = \gamma(\partial\sigma)$ and extend γ by setting $\gamma(\sigma) := \gamma_\sigma$. Then, by definition, γ is a non-trivial chain map from $C_*(K; \mathbb{Z}_2)$ to $C_*(\mathbf{R}; \mathbb{Z}_2)$ constrained by $(\mathcal{F}, \Phi)$ and hence by $(\mathcal{F}, P)$.

It remains to show that if $\dim K \leq 1$, we can take γ that is induced by a continuous map $f\colon |K| \to \mathbf{R}$. If $\dim K = 0$, we map each point to a point, so the statement is obviously true.

If $\dim K = 1$, we inspect the composition $\gamma = \gamma' \circ \beta \circ \alpha$. It maps points of K to points in $\mathbf{R}$ in such a way that the homology class of $\gamma(\partial\tau)$ inside $\mathrm{conv}_{\mathcal{F}}(\Psi(\tau))$ is trivial for each edge τ of K. But this means that the endpoints of τ get mapped to points in the same path-component of $\mathrm{conv}_{\mathcal{F}}(\Psi(\tau))$ and can be connected by an actual path. ◀

4 A fractional Helly theorem on surfaces

The aim is to bring the constant h_f from Theorem 3 (applied to a surface S) down to three for $b = 0$ and to $2b + 4$ for $b \geq 1$, respectively. This will give Theorem 4. The presented method is based on the recent result of Kalai and the author [11] and allow us to significantly decrease h_f to a small value as soon as we have a finite upper bound on h_f.

Before we perform the bootstrapping, we need few definitions. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be subsets of a surface S. Set $A_I = \bigcap_{i \in I} A_i$ and let $N(\mathcal{A}) = \{I \in [n] \colon A_I \neq \emptyset\}$ be the nerve of $\mathcal{A}$. We put $f_k(\mathcal{A}) = |\{I \in N(\mathcal{A}) \colon |I| = k+1\}|$. In words, f_k counts the number if intersecting $(k+1)$-tuples from $\mathcal{A}$.

The main tool for the bootstrapping is the following proposition.

▶ **Proposition 16.** *Let $b \geq 0$ and $k \geq 2$ be integers satisfying that for $b = 0$, $k \geq 2$ and for $b \geq 1$, $k \geq 2b + 3$, respectively. Let S be a surface. Then for every $\alpha_1 \in (0, 1)$ there exists $\alpha_2 = \alpha_2(\alpha_1, b, k, S) > 0$ such that for any sufficiently large family $\mathcal{A}$ of n open sets in S with $TC_1(\mathcal{A}) \leq b$ the following holds:*

$$f_k(\mathcal{A}) \geq \alpha_1 \binom{n}{k+1} \quad \Rightarrow \quad f_{k+1}(\mathcal{A}) \geq \alpha_2 \binom{n}{k+2}.$$

Let $b \geq 0$ and let $k_0 = k_0(b)$ be an integer depending on b. Namely, we set $k_0(0) = 3$ and $k_0(b) = 2b + 4$ for $b \geq 1$. Let $k \geq k_0 + 1$. By a successive application of the proposition we get that if at least an α-fraction of all k_0-tuples intersect, then also some α'-fraction of all k-tuples intersect. By the (non-optimal) fractional Helly theorem (Theorem 3), we already know that if some α'-fraction of all h_f-tuples intersect, there is some β-fraction of all sets that have a point in common. Putting $k = h_f$ proves Theorem 4.

As mentioned, the proof of Proposition 16 heavily relies on [11, Theorem 4], which can be reformulated[14], in terms of bounded topological complexity, as follows:

▶ **Theorem 17** ([11]). *Let S be a surface, $b \geq 0$ an integer and let $k = k(b)$ be an integer depending on b, namely $k(0) \geq 2$ and $k(b) \geq 2b + 3$ for $b \geq 1$. Let $\mathcal{A}$ be a finite family of open sets in S with $TC_1(\mathcal{A}) \leq b$. Then*

$$f_{k+1}(\mathcal{A}) = 0 \quad \Rightarrow \quad f_k(\mathcal{A}) \leq c_1 f_{k-1}(\mathcal{A}) + c_2,$$

where $c_1 > 0, c_2 \geq 0$ are constants depending only on k, b and the surface S.

Hypergraphs. A hypergraph is *ℓ-uniform* if all its edges have size ℓ. A hypergraph is *ℓ-partite*, if its vertex set V can be partitioned into ℓ subsets $V_1, \dots, V_\ell$, called *classes*, so that each edge contains at most one point from each V_i. Let $K^\ell(t)$ denote the complete ℓ-partite ℓ-uniform hypergraph with t vertices in each of its ℓ vertex classes.

We need the following theorem of Erdős and Simonovits [4] about super-saturated hypergraphs (see also [13, Chapter 9.2]):

▶ **Theorem 18** ([4]). *For any positive integers ℓ and t and any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: Let H be an ℓ-uniform hypergraph on n vertices and with at least $\varepsilon\binom{n}{\ell}$ edges. Then H contains at least $\lfloor \delta n^{\ell t} \rfloor$ copies (not necessarily induced) of $K^\ell(t)$.*

[14] We note that our reformulation is slightly weaker, however, we prefer a simpler exposition which is moreover adapted to our notion of topological complexity.

Proof of Proposition 16. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a family of sets in S satisfying the assumptions of the proposition. By Theorem 17, there exist constants $c_1 > 0, c_2 \geq 0$ depending on b, k and S such that $f_k(\mathcal{A}) \leq c_1 f_{k-1}(\mathcal{A}) + c_2$ provided $f_{k+1}(\mathcal{A}) = 0$. Since $f_{k-1}(\mathcal{A}) \leq \binom{n}{k}$, we have

$$f_{k+1}(\mathcal{A}) = 0 \quad \Rightarrow \quad f_k(\mathcal{A}) \leq (c_1 + c_2)\binom{n}{k}. \tag{2}$$

Let H be a $(k+1)$-uniform hypergraph whose vertices and edges correspond to the vertices and k-simplices of the nerve N of $\mathcal{A}$. Set

$$t := \left\lceil (c_1 + c_2) \cdot \frac{(k+1)^k}{k!} \right\rceil$$

By Erdős-Simonovits theorem ($\varepsilon = \alpha_1, \ell = k+1$), there is at least $\delta n^{(k+1)t}$ copies of $K^{k+1}(t)$ in H.

Since $K^{k+1}(t)$ has $(k+1)t$ vertices and t^{k+1} edges, it follows by (2) that for every copy of $K^{k+1}(t)$ in H there is an intersecting subfamily of size $k+2$ among the corresponding members of $\mathcal{A}$. Indeed, the implication (2) translates into checking that for $k \geq 2$,

$$t^{k+1} \quad > \quad (c_1 + c_2)\binom{(k+1)t}{k}.$$

On the other hand, each such intersecting $(k+2)$-tuple is contained in at most $n^{(k+1)t-(k+2)}$ distinct copies of $K^{k+1}(t)$ (this is the number of choices for the vertices not belonging to the considered $(k+2)$-tuple), and the result follows (i.e. $f_{k+1}(\mathcal{A}) \geq \delta n^{k+2} \geq \alpha_2\binom{n}{k+2}$). ◀

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