On Solving (Non)commutative Weighted Edmonds' Problem

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Abstract

In this paper, we consider computing the degree of the Dieudonné determinant of a polynomial matrix $A = A_\ell + A_{\ell-1}s + \cdots + A_0 s^\ell$, where each A_d is a linear symbolic matrix, i.e., entries of A_d are affine functions in symbols $x_1, \ldots, x_m$ over a field K. This problem is a natural "weighted analog" of Edmonds' problem, which is to compute the rank of a linear symbolic matrix. Regarding $x_1, \ldots, x_m$ as commutative or noncommutative, two different versions of weighted and unweighted Edmonds' problems can be considered. Deterministic polynomial-time algorithms are unknown for commutative Edmonds' problem and have been proposed recently for noncommutative Edmonds' problem.

The main contribution of this paper is to establish a deterministic polynomial-time reduction from (non)commutative weighted Edmonds' problem to unweighed Edmonds' problem. Our reduction makes use of the discrete Legendre conjugacy between the integer sequences of the maximum degree of minors of A and the rank of linear symbolic matrices obtained from the coefficient matrices of A. Combined with algorithms for noncommutative Edmonds' problem, our reduction yields the first deterministic polynomial-time algorithm for noncommutative weighted Edmonds' problem with polynomial bit-length bounds. We also give a reduction of the degree computation of quasideterminants and its application to the degree computation of noncommutative rational functions.

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1 Introduction

The background of this paper goes back to Edmonds [10]. In 1967, Edmonds posed a question whether there exists a polynomial-time algorithm to compute the rank of a *linear* (*symbolic*) *matrix* B over a field K, which is in the form

$$B = B_0 + B_1 x_1 + \cdots + B_m x_m,$$

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where $B_0, B_1 \ldots, B_m \in K^{n\times n}$ and $x_1, \ldots, x_m$ are commutative symbols. Here, B is regarded as a matrix over the polynomial ring $K[x_1, \ldots, x_m]$ or the rational function field $K(x_1, \ldots, x_m)$. In case where B is the Edmonds or Tutte matrix of a bipartite or nonbipartite graph G, the rank computation for B corresponds to solving the maximum matching problem on G. More generally, Lovász [27] showed that Edmonds' problem is equivalent to a linear matroid intersection problem if all B_i are of rank 1, and to a linear matroid parity problem if all B_i are skew-symmetric matrices of rank 2. For general linear matrices, the celebrated Schwartz–Zippel lemma [34] provides a simple randomized algorithm if $|K|$ is large enough [27]. However, no deterministic polynomial-time algorithm still has been known; the existence of such an algorithm would imply nontrivial circuit complexity lower bounds [22, 36].

Recent studies [11, 17, 20] address the noncommutative version of Edmonds' problem (nc-Edmonds' problem). This is a problem of computing the *noncommutative rank* (nc-rank) of B, which is the rank defined by regarding $x_1, \ldots, x_m$ as pairwise noncommutative, i.e., $x_i x_j \neq x_j x_i$ if $i \neq j$. In this way, B is viewed as a matrix over the free ring $K\langle x_1, \ldots, x_m\rangle$ generated by noncommutative symbols $x_1, \ldots, x_m$. The nc-rank of B is precisely the rank of B over a skew (noncommutative) field $K\langle\!(x_1, \ldots, x_m)\!\rangle$, called a *free skew field*, which is the quotient of $K\langle x_1, \ldots, x_m\rangle$ defined by Amitsur [2]. We call a linear matrix over K having noncommutative symbols an *nc-linear matrix* over K. The recent studies [11, 17, 20] revealed that nc-Edmonds' problem is deterministically tractable. For the case where K is the set $\mathbb{Q}$ of rational numbers, Garg et al. [11] proved that Gurvits' *operator scaling algorithm* [16] deterministically computes the nc-rank of B in poly(n, m) arithmetic operations on $\mathbb{Q}$. Algorithms over general field K were later given by Ivanyos et al. [20] and Hamada–Hirai [17] exploiting the min-max theorem established for nc-rank. In [16] and [20] applied to the case of $K = \mathbb{Q}$, bit-lengths of intermediate numbers are proved to be bounded by a polynomial of the input bit-length.

In this paper, we shall consider "weighted" versions of commutative and noncommutative Edmonds' problem introduced by Hirai [18]. First, consider commutative symbols $x_1, \ldots, x_m$ and an extra commutative symbol s. Define a matrix

$$A = A_\ell + A_{\ell-1}s + \cdots + A_0 s^\ell, \tag{1}$$

where $A_d = A_{d,0} + A_{d,1}x_1 + \cdots + A_{d,m}x_m \in K[x_1, \ldots, x_m]^{n\times n}$ is a linear matrix over K for $d = 0, \ldots, \ell$. We call (1) a *linear polynomial matrix* over K. The *weighted Edmonds' problem* (WEP) is the problem to compute the degree (in s) of the determinant of A. Analogously to Edmonds' problem, WEP includes a bunch of weighted combinatorial optimization problems as special cases, such as a maximum weighted perfect matching problem, a weighted linear matroid intersection problem and a weighted linear matroid parity problem; see [18, Section 5].

Defining *noncommutative weighted Edmonds' problem* (nc-WEP) requires some more involved algebraic notions due to noncommutativity. Let $x_1, \ldots, x_m$ be noncommutative symbols and s an extra symbol that commutes with any element in $K\langle x_1, \ldots, x_m\rangle$. An *nc-linear polynomial matrix* A over K is a matrix in the form of (1) with each A_d regarded as an nc-linear matrix. Then A can be viewed as a matrix over the rational function (skew) field $F(s)$ over $F := K\langle\!(x_1, \ldots, x_m)\!\rangle$. Since entries of A are noncommutative, the determinant of A is nontrivial. Here, we employ the *Dieudonné determinant* [9], which is a noncommutative generalization of the usual determinant defined for matrices over skew fields. We denote the Dieudonné determinant of A by $\operatorname{Det} A$. The Dieudonné determinant retains useful properties of the usual determinant such as $\operatorname{Det} AB = \operatorname{Det} A \operatorname{Det} B$. While the value of $\operatorname{Det} A$ is no longer in $F(s)$, its degree (in s) is well-defined [8, 35]. See Section 2.1 for the definition of Dieudonné determinant.

The nc-WEP is the problem to compute $\deg \mathrm{Det}$ of a given nc-linear polynomial matrix. Hirai [18] formulated the dual problem of nc-WEP as the minimization of an *L-convex function* on a *uniform modular lattice*, and gave an algorithm based on the steepest gradient descent. Hirai's algorithm uses $\mathrm{poly}(n, m, \ell)$ arithmetic operations on K while no bit-length bound has been given for $K = \mathbb{Q}$.

A weighted combinatorial optimization problem often reduces to an unweighted problem. This paper explores a reduction from (nc-)WEP (a weighted problem) to (nc-)Edmonds' problem (an unweighted problem). The main result of this paper is the following.

▶ **Theorem 1.** *The (nc-)WEP deterministically reduces to (nc-)Edmonds' problem for an (nc-)linear matrix of size ℓn^2 with m symbols.*

Theorem 1 provides an efficient randomized algorithm for WEP through the Schwartz–Zippel lemma and a deterministic polynomial-time algorithm for nc-WEP via the rank computation algorithms [11, 18, 20] for nc-linear matrices. This algorithm for nc-WEP is much different from Hirai's algorithm [18]; in particular, while Hirai's algorithm calls an oracle of nc-Edmonds' problem polynomially many times, our algorithm calls it only once (the matrix size would be augmented instead). Furthermore, in case of $K = \mathbb{Q}$, our reduction does not exponentially swell the input bit-length because every entry of the nc-linear matrix constructed in this reduction is some coefficient of an entry in the input nc-linear polynomial matrix. Thus, by employing an algorithm [11, 18] for nc-Edmonds' problem with bit-length bounds, we obtain the following.

▶ **Theorem 2.** *We can deterministically solve nc-WEP in $\mathrm{poly}(n, m, \ell)$ arithmetic operations on K. In addition, if $K = \mathbb{Q}$, the bit-lengths of intermediate numbers are bounded by a polynomial of the input bit-length.*

We also give a reduction from computing the degree of a *quasideterminant* [12, 13], which is another noncommutative analogy of the determinant, to computing the degree of Dieudonné determinant. This can be applied to the degree computation of noncommutative rational functions represented as a noncommutative formula with division. See Section 4 for details.

Techniques

Let A be an $n \times n$ (nc-)linear polynomial matrix over a field K and put $F := K(x_1, \ldots, x_m)$ or $K\langle\!\langle x_1, \ldots, x_m \rangle\!\rangle$. Slightly generalizing (nc-)WEP, we consider the problem to compute

$$d_k(A) := \max\{\deg \mathrm{Det}\, A[I, J] \mid |I| = |J| = k\} \tag{2}$$

for given k, where $A[I, J]$ denotes the submatrix of A indexed by a row set I and a column set J. Clearly $\deg \mathrm{Det}\, A = d_n(A)$. In view of combinatorial optimization, computation of $d_k(A)$ corresponds to solving weighted problems under cardinality constraints.

Our reduction scheme is based on a method, which we call *matrix expansion*, that constructs an (nc-)linear matrix $\Omega_\mu(A) \in F^{\mu n \times \mu n}$ obtained by arranging the coefficient matrices of A. Through a canonical form of A called the *Smith–McMillan form*, it is shown that the integer sequences of $(d_0(A), d_1(A), \ldots, d_r(A))$ with $r := \mathrm{rank}\, A$ and $(\omega_0(A), \omega_1(A), \ldots)$ with $\omega_\mu(A) := \mathrm{rank}\, \Omega_\mu(A)$ are concave and convex, respectively. In addition, they are in the relation of the *discrete Legendre conjugate*, that is, they satisfy

$$d_k(A) = \min_{\mu \geq 0}(\omega_\mu(A) - k\mu) \qquad (0 \leq k \leq r), \tag{3}$$

$$\omega_\mu(A) = \max_{0 \leq k \leq r}(d_k(A) + k\mu) \quad (\mu \geq 0). \tag{4}$$

The Legendre conjugacy is an important duality relation on discrete convex and concave functions treated in *discrete convex analysis* [32]. The formulas (3) and (4) are a generalization of results on matrix pencils over fields given by Murota [33] and on polynomial matrices over algebraically closed fields by Moriyama–Murota [28]. For proving the conjugacy, equalities connecting $d_k(A)$ and $\omega_\mu(A)$ are necessary. We present a new short and simple connection which works even on skew fields through the multiplicativity of Ω_μ, i.e.,

$$\Omega_\mu(A)\Omega_\mu(B) = \Omega_\mu(AB). \tag{5}$$

The conjugacy formula (3) reduces the computation of $d_k(A)$ to a one-dimensional discrete convex optimization problem, which can be efficiently done by binary search. In each iteration, the objective function can be evaluated by solving (nc-)Edmonds' problem. Moreover, we derive direct formulas with respect to r and $d_r(A)$ from (3), which proves Theorem 1.

Related Work

In the field of computer algebra, algorithms were proposed for computing various kinds of canonical forms of a polynomial matrix $A \in F[s]^{n\times n}$ (or of its generalization) such as the *Jacobson normal form* [26], the *Hermite normal form* [14], the *Popov normal form* [23] and their weaker form called a *row-reduced form* [1, 4]. These algorithms iteratively solve systems of linear equations over F whose coefficient matrices are variants of expanded matrices $\Omega_\mu(A)$ under the name of "linearized matrices" [23] or "striped Krylov matrices" [4]. While these algorithms can compute $\deg \operatorname{Det} A$, their running time is bounded in terms of the number of operations on F. Hence if $F = K(x_1, \ldots, x_m)$ or $F = K\langle\!\langle x_1, \ldots, x_m \rangle\!\rangle$, the expression size of intermediate numbers might be exponentially large.

Combinatorial relaxation [29, 30] is another framework for deg-det computation based on combinatorial optimization. Hirai's algorithm [18] for nc-WEP can also be viewed as a variant of combinatorial relaxation. Unlike the matrix expansion, it is difficult to give bit complexity bounds for combinatorial relaxation algorithms because they iteratively perform the Gaussian elimination on the same matrix and thus the magnitude of its entries might swell.

Organization

The rest of this paper is organized as follows. Section 2 provides preliminaries on matrices and polynomials over skew fields. Section 3 describes our proposed reductions after introducing the matrix expansion and the Legendre conjugacy. Section 4 describes the computation of the degree of quasideterminants and its application to the degree computation of noncommutative rational functions.

2 Preliminaries

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of nonnegative integers. For $n \in \mathbb{N}$, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and $\{0, 1, 2, \ldots, n\}$ by $[0, n]$.

2.1 Matrices over Skew Fields

A *skew field*, or a *division ring* is a ring F such that every nonzero element has a multiplicative inverse in F. A right (left) F-module is especially called a *right* (*left*) *F-vector space*. The *dimension* of a right (left) F-vector space V is defined as the rank of V as a module, that is, the cardinality of any basis of V. The usual facts from linear algebra on independent sets and generating sets in vector spaces are valid even on skew fields [25].

A square matrix $A \in F^{n\times n}$ is said to be *nonsingular* if there exists a unique $n \times n$ matrix over F, denoted by A^{-1}, such that AA^{-1} and $A^{-1}A$ are the identity matrix I_n of size n. A square matrix is *singular* if it is not nonsingular. The *rank* $\operatorname{rank} A$ of a matrix $A \in F^{n\times n'}$ is the dimension of the right F-vector space spanned by the column vectors of A, and is equal to the dimension of the left F-vector space spanned by the row vectors of A. The rank is invariant under (right and left) multiplications of nonsingular matrices. It is observed that a square matrix $A \in F^{n\times n}$ is nonsingular if and only if $\operatorname{rank} A = n$.

The *Bruhat decomposition* uniquely factors a nonsingular matrix $A \in F^{n\times n}$ into the product of four $n \times n$ matrices over F as $A = LDPU$, where L is lower unitriangular, D is diagonal, P is a permutation matrix and U is upper unitriangular [7, Theorem 2.2 in Section 11.2]. Here, a lower (upper) unitriangular matrix is a lower (upper) triangular matrix whose diagonal entries are 1. Let $F^\times_{\mathrm{ab}} := F^\times / [F^\times, F^\times]$ denote the abelianization of the multiplicative subgroup $F^\times = F \setminus \{0\}$ of F, where $[F^\times, F^\times] := \langle \{aba^{-1}b^{-1} \mid a, b \in F^\times\} \rangle$ is the commutator subgroup of $F^\times$. The *Dieudonné determinant* $\operatorname{Det} A$ of a nonsingular matrix $A \in F^{n\times n}$, which is decomposed as $A = LDPU$, is an element of $F^\times_{\mathrm{ab}}$ defined by

$$\operatorname{Det} A := \operatorname{sgn}(P) e_1 e_2 \cdots e_n \bmod [F^\times, F^\times],$$

where $\operatorname{sgn}(P) \in \{-1, +1\}$ is the sign of the permutation P and $e_1, \dots, e_n \in F^\times$ are the diagonal entries of D [9]. For a singular matrix $A \in F^{n\times n}$, define $\operatorname{Det} A$ as 0 for convenience. In case where F is commutative, the Dieudonné determinant coincides with the usual determinant. As the usual determinant, the Dieudonné determinant satisfies the following properties [3, Chapter 4.1]:

(D1) $\operatorname{Det} AB = \operatorname{Det} A \operatorname{Det} B$ for $A, B \in F^{n\times n}$.

(D2) $\operatorname{Det} \begin{pmatrix} A & * \\ O & B \end{pmatrix} = \operatorname{Det} \begin{pmatrix} A & O \\ * & B \end{pmatrix} = \operatorname{Det} A \operatorname{Det} B$ for $A \in F^{n\times n}$ and $B \in F^{n'\times n'}$, where blocks in O and $*$ represent zero and any matrices of appropriate size, respectively.

2.2 Polynomials over Skew Fields

Let us consider the polynomial ring $F[s]$ over a skew field F, where s is an indeterminate that commutes with any element of F. A nonzero polynomial $p \in F[s]$ is uniquely written as $p = \sum_{d=0}^{\ell} a_{\ell-d} s^d$, where $a_0, \dots, a_\ell \in F$ with $a_0 \neq 0$. The addition and the multiplication in $F[s]$ are naturally defined. The *degree* $\deg p$ of p is defined by $\deg p := \ell$ and we set $\deg 0 := -\infty$. Then the minus of the degree enjoys the *discrete valuation* property, that is, deg satisfies $\deg(p + q) \le \max\{\deg p, \deg q\}$ and $\deg pq = \deg p + \deg q$ for $p, q \in F[s]$.

The polynomial ring $F[s]$ is a (right and left) *Ore domain*, i.e., for each $p, q \in F[s] \setminus \{0\}$, there exists $u, u', v, v' \in F[s] \setminus \{0\}$ such that $pu = pv$ and $u'p = v'q$. This property enables $F[s]$ to have the (right and left) *Ore quotient ring*, which is a skew field of fractions each of whose elements is expressed as $f = pq^{-1} = q'^{-1}p'$ for some $p, p', q, q' \in F[s]$ with $q, q' \neq 0$. Elements of $F(s)$ are called *rational functions* over F and $F(s)$ is called the *rational function field* over F. See [7, Section 9.1] and [15, Chapter 6] for the construction of $F(s)$. The degree on $F[s]$ is uniquely extended to a valuation on $F(s)$ by $\deg f := \deg p - \deg q$ for $f = pq^{-1} \in F(s)$; see [8, Proposition 9.1.1]. A rational function $f \in F(s)$ is said to be *proper* if $\deg f \le 0$.

The *Laurent series field* $F((s^{-1}))$ over F in s^{-1} is the set of formal power series over F in the form of

$$f = \sum_{d=-\ell}^{\infty} a_d s^{-d} \tag{6}$$

for some $\ell \in \mathbb{Z}$ and $a_{-\ell}, a_{-\ell+1}, \ldots \in F$. This skew field has the natural addition and multiplication. The rational function field $F(s)$ can be embedded in $F((s^{-1}))$ [6, Proposition 7.1]. Namely, any rational function $f \in F(s)$ can be uniquely expanded in form of (6). In particular, ℓ coincides with $\deg f$.

Let $A \in F(s)^{n\times n}$ be a square matrix over $F(s)$, called a *rational function matrix* over F. The degree of the Dieudonné determinant of A is well-defined since all commutators of $F(s)^\times$ have degree zero. Note that $\deg \mathrm{Det}$ of singular matrices are $-\infty$. The following properties on $\deg \mathrm{Det}$ are easily seen from (D1) and (D2).

(MV1) $\deg \mathrm{Det}\, AB = \deg \mathrm{Det}\, A + \deg \mathrm{Det}\, B$ for $A, B \in F(s)^{n\times n}$.

(MV2) $\deg \mathrm{Det} \begin{pmatrix} A & * \\ O & B \end{pmatrix} = \deg \mathrm{Det} \begin{pmatrix} A & O \\ * & B \end{pmatrix} = \deg \mathrm{Det}(A) + \deg \mathrm{Det}(B)$ for $A \in F(s)^{n\times n}$ and $B \in F(s)^{n'\times n'}$.

Recall the notation $d_k(A)$ in (2) for $A \in F(s)^{n\times n'}$. Note that $d_1(A)$ is the maximum degree of an entry in A, and we call $d_1(A)$ the *degree* of A. Similarly to (6), A can be uniquely expanded as

$$A = \sum_{d=-\ell}^{\infty} A_d s^{-d} \tag{7}$$

with $\ell = d_1(A)$ and some $A_{-\ell}, A_{-\ell+1}, \ldots \in F(s)^{n\times n'}$. The following proposition gives lower and upper bounds on $d_k(A)$.

▶ **Proposition 3.** *Let $A \in F(s)^{n\times n'}$ be a rational function matrix over a skew field F. For $k \in [0, n^*]$ with $n^* := \min\{n, n'\}$, the following hold:*

(1) *$d_k(As^\ell) = d_k(A) + \ell k$ for $\ell \in \mathbb{Z}$.*

(2) *$d_k(A) \le \ell k$, where ℓ is the degree of A.*

(3) *$d_k(A) > -\infty$ if and only if $k \le \operatorname{rank} A$. In addition, if A is a polynomial matrix, then $d_k(A) \ge 0$ for $k \le \operatorname{rank} A$.*

Proof. (1) follows from the fact that for any $k \times k$ submatrix $A[I, J]$ of A, it holds

$$\begin{aligned}\deg \mathrm{Det}\, A[I,J]s^\ell &= \deg \mathrm{Det}(A[I,J] \cdot s^\ell I_k) \\ &= \deg \mathrm{Det}\, A[I,J] + \deg \det s^\ell I_k \\ &= \deg \mathrm{Det}\, A[I,J] + \ell k.\end{aligned}$$

(2) Let $\alpha_1, \ldots, \alpha_k$ be the exponents of the Smith–McMillan form of a nonsingular $k \times k$ submatrix $A[I, J]$ of A. Then the claim follows from $\deg \mathrm{Det}\, A[I, J] = \alpha_1 + \cdots + \alpha_k$ and $\ell \ge \alpha_1 \ge \cdots \ge \alpha_k$.

(3) The former part is obtained from the fact that $\operatorname{rank} A$ is equal to the maximum size of a nonsingular submatrix of A. The latter part can be proved using the *Smith normal form*, see e.g. [18, Lemma 2.11]. ◀

A rational function matrix is said to be *proper* if its degree is nonpositive. A square rational function matrix is said to be *biproper* if it is proper and nonsingular, and its inverse is also proper. We abbreviate proper and biproper rational function matrices as proper and biproper matrices, respectively. It is easy to see that the product of proper matrices are proper, which implies that the product of biproper matrices are biproper again. Equivalent conditions for proper matrices to be biproper are established as follows.

▶ **Lemma 4** ([18, Lemma 2.10]). *Let $A \in F(s)^{n\times n}$ be a square proper matrix over a skew field F. Then the following are equivalent:*

(1) *A is biproper.*
(2) $\deg \operatorname{Det} A = 0$.
(3) *The coefficient matrix A_0 of s^0 in the expansion* (7) *of A is nonsingular.*

A *biproper transformation* is a transformation of a rational function matrix $A \in F(s)^{n\times n'}$ in the form $A \mapsto SAT$, where $S \in F(s)^{n\times n}$ and $T \in F(s)^{n'\times n'}$ are biproper matrices. Under biproper transformations, we can establish a canonical form of rational function matrices, called the *Smith–McMillan form*. This is well-known for complex rational function matrices as the *Smith–McMillan form at infinity* [31, 37] in the context of control theory.

▶ **Proposition 5** (Smith–McMillan form). *Let $A \in F(s)^{n\times n'}$ be a rational function matrix of rank r over a skew field F. There exist biproper matrices $S \in F(s)^{n\times n}$, $T \in F(s)^{n'\times n'}$ and integers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$ such that*

$$SAT = \begin{pmatrix} \operatorname{diag}(s^{\alpha_1}, \ldots, s^{\alpha_r}) & O \\ O & O \end{pmatrix}.$$

The integer α_i is uniquely determined by

$$\alpha_i = d_i(A) - d_{i-1}(A) \tag{8}$$

for $i \in [r]$. In particular, $d_k(A)$ is invariant under biproper transformations for $k \in [0, r]$.

Proof. The proof is the same as that for nonsingular $A \in F(s)^{n\times n}$ in [18, Proposition 2.9], which iteratively determines α_i from $i = 1$ to n, except that the iterations stops when $i = r$. ◀

Solving (8) for $d_k(A)$, we obtain

$$d_k(A) = \sum_{i=1}^{k} \alpha_i \tag{9}$$

for $k \in [0, r]$. This is a key identity that connects $d_k(A)$ and the Smith–McMillan form of A. It is worth mentioning that all α_i are nonpositive for a proper matrix A since α_1 is equal to the degree $d_1(A)$ of A by (8).

3 Computing the Degree of Dieudonné Determinant

Let $A = \sum_{d=0}^{\ell} A_{\ell-d}s^d \in F[s]^{n\times n'}$ be a polynomial matrix over a skew field F; we typically consider an (nc-)linear polynomial matrix with $F \coloneqq K(x_1, \ldots, x_m)$ or $K\langle\!\langle x_1, \ldots, x_m \rangle\!\rangle$. In this section, we give reductions of computing $d_k(A)$ and rank A to rank computations over F. Instead of A, we deal with a proper matrix obtained from A by

$$As^{-\ell} = \sum_{d=0}^{\ell} A_d s^{-d} \in F(s)^{n\times n'}. \tag{10}$$

The value of $d_k(A)$ can be recovered from that of (10) through Proposition 3 (1).

Section 3.1 introduces *matrix expansion* which is our key tool. Section 3.2 connects the sequence of d_k to the rank of expanded matrices via the *Legendre conjugacy*. Making use of them, we give reductions and algorithms in Section 3.3, which proves Theorem 1.

3.1 Matrix Expansion

For a proper matrix $A \in F(s)^{n\times n'}$ and $\mu \in \mathbb{N}$, we define the *μth-order expanded matrix* $\Omega_\mu(A)$ of A as the following $\mu n \times \mu n'$ block matrix

$$\Omega_\mu(A) := \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & \cdots & A_{\mu-1} \\ O & A_0 & A_1 & A_2 & & \vdots \\ \vdots & O & A_0 & A_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & A_2 \\ \vdots & & & \ddots & A_0 & A_1 \\ O & \cdots & \cdots & \cdots & O & A_0 \end{pmatrix} \in F^{\mu n\times \mu n'},$$

where $A_0, \dots, A_{\mu-1} \in F^{n\times n'}$ are matrices in the expansion (7) of A. Note that $\Omega_\mu(A)$ is an (nc-)linear matrix over K. Expanded matrices satisfy the multiplicativity (5).

▶ **Lemma 6.** *Let $A \in F(s)^{n\times n'}$ and $B \in F(s)^{n'\times n''}$ be proper matrices over a skew field F. Then it holds* (5) *for any $\mu \in \mathbb{N}$.*

Proof. Expand A and B by (7) as $A = \sum_{d=0}^{\infty} A_d s^{-d}$ and $B = \sum_{d=0}^{\infty} B_d s^{-d}$. By

$$AB = \left(\sum_{d=0}^{\infty} A_d s^{-d}\right)\left(\sum_{d=0}^{\infty} B_d s^{-d}\right) = \sum_{d=0}^{\infty} A_d \left(\sum_{j=0}^{d} B_{d-j} s^{-j}\right) = \sum_{j=0}^{\infty}\left(\sum_{d=0}^{j} A_d B_{d-j}\right) s^{-j},$$

the (i,j)th block in $\Omega_\mu(AB)$ is $\sum_{d=0}^{j-i} A_d B_{d-j}$ if $i \le j$ and O otherwise. This coincides with the (i,j)th block in $\Omega_\mu(A)\Omega_\mu(B)$. ◀

Let $\omega_\mu(A)$ denote the rank of $\Omega_\mu(A)$. The following lemma claims that $\omega_\mu(A)$ coincides with that of the Smith–McMillan form of A.

▶ **Lemma 7.** *Let $A \in F(s)^{n\times n'}$ be a proper matrix over a skew field F. Then it holds $\omega_\mu(A) = \omega_\mu(D)$ for $\mu \in \mathbb{N}$, where D is the Smith–McMillan form of A.*

Proof. Let $S \in F(s)^{n\times n}$ and $T \in F(s)^{n'\times n'}$ be biproper matrices such that $SAT = D$. From Lemma 6, we have $\omega_\mu(D) = \operatorname{rank}\Omega_\mu(SAT) = \operatorname{rank}\Omega_\mu(S)\Omega_\mu(A)\Omega_\mu(T)$. Let S_0 and T_0 be the coefficient matrices of s^0 in the expansion (7) of S and T, respectively. Since S_0 and T_0 are nonsingular by Lemma 4,the block matrices $\Omega_\mu(S)$ and $\Omega_\mu(T)$ are also nonsingular. Therefore we have $\omega_\mu(D) = \omega_\mu(A)$. ◀

Let $0 \ge \alpha_1 \ge \cdots \ge \alpha_r$ be the exponents of the Smith–McMillan form of A with $r := \operatorname{rank} A$. Put

$$N_d := |\{i \in [r] \mid -\alpha_i \le d\}| \tag{11}$$

for $d \in \mathbb{N}$. Lemma 7 leads us to the following lemma; a similar result based on the Kronecker canonical form is also known for matrix pencils over a field [21, Theorem 2.3].

▶ **Lemma 8.** *Let $A \in F(s)^{n\times n'}$ be a proper matrix over a skew field F. For $\mu \in \mathbb{N}$, it holds*

$$\omega_\mu(A) = \sum_{d=0}^{\mu-1} N_d, \tag{12}$$

where N_d is defined in (11).

Proof. Let $D = \sum_{d=0}^{\infty} D_d s^{-d}$ be the Smith–McMillan form of A and $\alpha_1, \ldots, \alpha_r$ the exponents of diagonal entries of D, where $r \coloneqq \operatorname{rank} A$. The ith diagonal entry of D_d is 1 if $i \leq r$ and $\alpha_i = -d$, and 0 otherwise. Thus each row and column in $\Omega_\mu(D)$ has at most one nonzero entry. Hence $\omega_\mu(D)$, which is equal to $\omega_\mu(A)$ by Lemma 7, is equal to the number of nonzero entries in $\Omega_\mu(D)$. It is easily checked that the $(\mu - d)$th block row of $\Omega_\mu(D)$ contains N_d nonzero entries for $d = 0, \ldots, \mu - 1$. ◀

The equality (12) is a key identity that connects $\omega_\mu(A)$ and the Smith–McMillan form of A. We remark that, for $d \in \mathbb{N}$, the equality (12) can be rewritten as

$$N_d = \omega_{d+1}(A) - \omega_d(A). \tag{13}$$

3.2 Legendre Conjugacy of $d_k(A)$ and $\omega_\mu(A)$

Let $A \in F(s)^{n \times n'}$ be a proper matrix of rank r and $\alpha_1 \geq \ldots \geq \alpha_r$ the exponents of the Smith–McMillan form of A. Put $d_k \coloneqq d_k(A)$ for $k = 0, \ldots, r$. From $\alpha_k \geq \alpha_{k+1}$ and (8), the inequality $d_{k-1} + d_{k+1} \leq 2d_k$ holds for all $k \in [r-1]$. In addition, for $\mu \in \mathbb{N}$, put $\omega_\mu \coloneqq \omega_\mu(A)$ and define N_μ by (11). From $N_{\mu-1} \leq N_\mu$ and (13), we have $\omega_{\mu-1} + \omega_{\mu+1} \geq 2\omega_\mu$ for all $\mu \geq 1$. These two inequalities for d_k and ω_μ indicate the *concavity* of d_k and the *convexity* of ω_μ in the following sense.

A (discrete) function $f \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ is said to be *convex* if

$$f(x-1) + f(x+1) \geq 2f(x)$$

for all $x \in \mathbb{Z}$. We call a function $g \colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ *concave* if $-g$ is convex. An integer sequence $(a_k)_{k \in K}$ indexed by $K \subseteq \mathbb{Z}$ can be identified with a function $\check{a} \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ by letting $\check{a}(k)$ be a_k if $k \in K$ and $+\infty$ otherwise. We can also identify a with $\hat{a} \colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ defined by $\hat{a}(k) \coloneqq a_k$ if $k \in K$ and $\hat{a}(k) \coloneqq -\infty$ otherwise. In this way, we identify integer sequences $(d_0, d_1, \ldots, d_r)$ and $(\omega_0, \omega_1, \omega_2, \ldots)$ with discrete functions $\check{d} \colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ and $\hat{\omega} \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$, respectively. From the argument in the previous paragraph, $(d_0, d_1, \ldots, d_r)$ is concave and $(\omega_0, \omega_1, \omega_2, \ldots)$ is convex.

Let $f \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ be a function such that $f(x) \in \mathbb{Z}$ for some $x \in \mathbb{Z}$. The *concave conjugate* of f is a function $f^\circ \colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ defined by

$$f^\circ(y) \coloneqq \inf_{x \in \mathbb{Z}} (f(x) - xy)$$

for $y \in \mathbb{Z}$. Similarly, for a function $g \colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ with $g(y) \in \mathbb{Z}$ for some $y \in \mathbb{Z}$, the *convex conjugate* of g is a function $g^\bullet \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ given by

$$g^\bullet(x) \coloneqq \sup_{y \in \mathbb{Z}} (g(y) + xy)$$

for $x \in \mathbb{Z}$. The maps $f \mapsto f^\circ$ and $g \mapsto g^\bullet$ are referred to as the *concave* and *convex discrete Legendre transform*, respectively. In general, f° is concave and $g^\bullet$ is convex. In addition, if f is convex and g is concave,

$$(f^\circ)^\bullet = f, \quad (g^\bullet)^\circ = g \tag{14}$$

hold. Hence the Legendre transformation establishes a one-to-one correspondence between discrete convex and concave functions. See [32] for details of discrete convex/concave functions and their Legendre transform.

Indeed, as explained in Section 1, the sequences of d_k and ω_μ are in the relation of Legendre conjugate. This can be shown from the key identities (9) and (12) that connect $d_k(A)$ and $\omega_\mu(A)$ through the Smith–McMillan form of A.

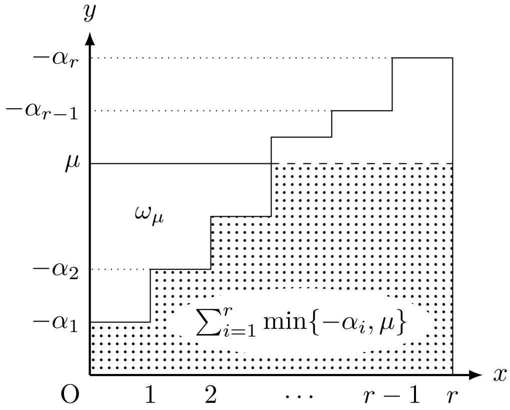

Figure 1 Graphic explanation of (15).

▶ **Theorem 9.** *Let $A \in F(s)^{n \times n'}$ be a proper matrix of rank r over a skew field F. Then* (3) *and* (4) *hold.*

Proof. Put $d_k \coloneqq d_k(A)$ for $k = 0, \ldots, r$ and $\omega_\mu \coloneqq \omega_\mu(A)$ for $\mu \in \mathbb{N}$. Since $(d_0, d_1, \ldots, d_r)$ is concave and $(\omega_0, \omega_1, \omega_2, \ldots)$ is convex, (3) and (4) are equivalent by (14). We show (4).

First we give an equality

$$\omega_\mu = r\mu - \sum_{i=1}^{r} \min\{-\alpha_i, \mu\} \tag{15}$$

for $\mu \in \mathbb{N}$, where $\alpha_1 \geq \ldots \geq \alpha_r$ are the exponents of the Smith–McMillan form of A. Figure 1 graphically shows this equality. Let x and y be the coordinates along the horizontal and vertical axes in Figure 1, respectively. For $i = 1, \ldots, r$, the height of the dotted rectangle with $i - 1 \leq x < i$ is $\min\{-\alpha_i, \mu\}$. Hence the area of the dotted region is equal to $\sum_{i=1}^{r} \min\{-\alpha_i, \mu\}$. In addition, the width of the white rectangle with $d \leq y < d + 1$ is equal to N_d for $d = 0, \ldots, \mu - 1$, where N_d is defined by (11). Hence the area of the white stepped region is equal to $N_0 + \cdots + N_{\mu-1} = \omega_\mu$ by (12). Now we have (15) since the sum of the areas of these two regions is $r\mu$.

Substituting (9) into the right hand side of (4), we have

$$\max_{0 \leq k \leq r} (d_k + k\mu) = \max_{0 \leq k \leq r} \sum_{i=1}^{k} (\alpha_i + \mu) = \sum_{i=1}^{k^*} \alpha_i + k^*\mu, \tag{16}$$

where k^* is the maximum $0 \leq k \leq r$ such that $\alpha_k + \mu \geq 0$. Since $\min\{-\alpha_i, \mu\}$ is $-\alpha_i$ if $i \leq k^*$ and is μ if $i > k^*$, it holds

$$\sum_{i=1}^{r} \min\{-\alpha_i, \mu\} = -\sum_{i=1}^{k^*} \alpha_i + (r - k^*)\mu. \tag{17}$$

From (16) and (17), we have

$$\max_{0 \leq k \leq r} (d_k + k\mu) = r\mu - \sum_{i=1}^{r} \min\{-\alpha_i, \mu\},$$

in which the right hand side is equal to ω_μ by (15). ◀

3.3 Reductions and Algorithms

Let $A = A_0 + A_1 s^{-1} + \cdots + A_\ell s^{-\ell} \in F(s)^{n \times n'}$ be the proper matrix (10) of rank r. The expression (15) of $d_k(A)$ indicates that $d_k(A)$ is equal to the optimal value of an minimization problem with objective function

$$f_k(\mu) \coloneqq \omega_\mu(A) - k\mu. \tag{18}$$

Since f_k is convex, it is minimized by the minimum μ such that $f_k(\mu+1) - f_k(\mu) \geq 0$. This can be found by the binary search in $\mathrm{O}(\log M)$ evaluations of f_k, where M is an upper bound on a minimizer of f_k. The following lemma claims that we can adopt ℓr as the upper bound.

▶ **Lemma 10.** *Let $A = \sum_{d=0}^{\ell} A_d s^{-d} \in F(s)^{n \times n'}$ be the proper matrix (10) of rank r. Then the following hold:*
(1) *The exponents $\alpha_1, \ldots, \alpha_r$ of the Smith–McMillan form of A are at least $-\ell r$.*
(2) *For $k \in [0, r]$, the function f_k in (18) has a minimizer μ^* satisfying $0 \leq \mu^* \leq \ell r$.*

Proof. The claims are trivial if $r = 0$. Suppose $r \geq 1$.

(1) It suffices to show $\alpha_r \geq -\ell r$. Since A is proper, $d_{r-1}(A)$ is nonpositive. In addition, since As^ℓ is a polynomial matrix of rank r, we have $0 \leq d_r(As^\ell) = d_r(A) + \ell r$ by Proposition 3 (1) and (3). Thus $\alpha_r = d_r(A) - d_{r-1}(A) \geq -\ell r$ holds.

(2) From Lemma 8, the objective function f_k can be written as

$$f_k(\mu) = \sum_{d=0}^{\mu-1} (N_d - k)$$

for $\mu \in \mathbb{N}$. Hence f_k is minimized by the maximum $\mu \in \mathbb{N}$ such that $N_\mu + k < 0$. Note that such μ exists since f_k has the minimum value. From the definition (11) of N_d, it holds $N_d = N_{-\alpha_r}$ for all $d \geq -\alpha_r$. Hence f_k has a minimizer less than or equal to $-\alpha_r$, which is at most ℓr by (1). ◀

Finally, we show direct formulas of rank A and $d_r(A)$ for a proper matrix A in (10). These formulas naturally yield efficient algorithms to compute them, which proves Theorem 1.

▶ **Lemma 11.** *Let $A = \sum_{d=0}^{\ell} A_d s^{-d} \in F(s)^{n \times n'}$ be the proper matrix (10) of rank r. Then it holds $r = \omega_{ln^*+1}(A) - \omega_{ln^*}(A)$ and $d_r(A) = \omega_{lr}(A) - lr^2$, where $n^* \coloneqq \min\{n, n'\}$.*

Proof. We first show the formula on r. We have $\omega_{\ell n^*+1}(A) - \omega_{\ell n^*}(A) = N_{\ell n^*}$ by (13). Since $-\alpha_i$ is at most $\ell r \leq \ell n^*$ for all $i \in [r]$ by Lemma 10 (1), we have $r = N_{\ell n^*}$.

Next we show the formula on $d_r(A)$. From (3) and (12), it holds

$$d_r(A) = \min_{\mu \geq 0} \sum_{d=0}^{\mu-1} (N_d - r). \tag{19}$$

Since $N_0 \leq N_1 \leq \cdots \leq N_{\ell r} = N_{\ell r+1} = \cdots = r$ by Lemma 10 (1), the minimum value of the right hand side of (19) is attained by $\mu = \ell r$. Thus we are done. ◀

▶ Remark 12. In view of combinatorial optimization, our algorithms are regarded as pseudo-polynomial time algorithms since the running time depends on a polynomial of the maximum exponent ℓ of s instead of $\mathrm{poly}(\log \ell)$. Thus it is natural to try to solve the following problem:

Sparse Degree of Determinant (SDD)
Input : $A = A_1 s^{w_1} + \cdots + A_m s^{w_m} \in K[s]^{n \times n}$, where $0 \leq w_1 \leq \ldots \leq w_m$ are integers.
Output: $\deg \det A$.

However, setting $w_k := (n+1)^k$ for $k \in [m]$ would make the rank of A the same as that of a linear matrix $A_1 x_1 + \cdots + A_m x_m \in K[x_1, \ldots, x_m]^{n \times n}$ (known as the *Kronecker substitution* [24]). Since giving a deterministic polynomial-time algorithm for Edmonds' problem has still been open for more than half a century, SDD is also a quite challenging problem.

4 Computing the Degree of Quasideterminants

The *quasideterminant* [12, 13] is another noncommutative analogy of the determinant than the Dieudonné determinant. Let $A \in F^{n \times n}$ be a square matrix over a skew field F. Fix $i, j \in [n]$ and put $I := [n] \setminus \{i\}$, $J := [n] \setminus \{j\}$. The (i,j)th *quasideterminant* $|A|_{i,j}$ of A is defined if $A[I,J]$ nonsingular as

$$|A|_{i,j} := A[\{i\},\{j\}] - A[I,\{j\}]A[I,J]^{-1}A[\{i\},J] \in F.$$

Analogous to the usual determinant, A is nonsingular if and only if at least one quasideterminant of A is defined and nonzero [12, Proposition 1.4.6]. When A is nonsingular, $|A|_{i,j}$ is defined if and only if the (i,j)th entry a of A^{-1} is nonzero, and if this is the case, $|A|_{i,j} = a^{-1}$ holds.

Through the Dieudonné determinant, we can compute the degree of a quasideterminant of rational function matrices without computing the quasideterminant.

▶ **Proposition 13.** *Let $A \in F(s)^{n \times n}$ be a square rational function matrix over a skew field F. Then for $i, j \in [n]$ with $S := A[[n] \setminus \{i\}, [n] \setminus \{j\}]$ being nonsingular, it holds*

$$\deg |A|_{i,j} = \deg \operatorname{Det} A - \deg \operatorname{Det} S.$$

Proof. We assume $i = j = 1$ without loss of generality. Express A as

$$A = \begin{pmatrix} a & r \\ c & S \end{pmatrix},$$

where $a \in F(s), r \in F(s)^{1 \times (n-1)}$ and $c \in F(s)^{(n-1) \times 1}$. By elementary row and column operations, it holds

$$A = \begin{pmatrix} a & r \\ c & S \end{pmatrix} = \begin{pmatrix} 1 & rS^{-1} \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} |A|_{i,j} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S^{-1}c & I_{n-1} \end{pmatrix},$$

where we used $|A|_{i,j} = a - rS^{-1}c$. Hence $\deg \operatorname{Det} A = \deg |A|_{i,j} + \deg \operatorname{Det} S$, as required. ◀

Proposition 13 can be applied to the problem of computing the degree of a *noncommutative rational function* (nc-rational function) expressed by a *noncommutative formula* (nc-formula). Let K be a field and consider pairwise noncommutative symbols $x_1, \ldots, x_m$. An *nc-rational function* is an element of the free skew field $K \langle\!(x_1, \ldots, x_m)\!\rangle$. An *nc-formula* (with division) Φ is a binary tree, whose every leaf is labeled with an element of $\{x_1, \ldots, x_m\} \cup K$ and every non-leaf node is labeled with "+", "×" or "÷". Each node computes an nc-rational function in the obvious way, and the output of Φ is the rational function computed by the root. The *size* of Φ is the number of nodes.

Cohn [5] showed that for an nc-formula of size r computing f, we can construct a nonsingular $n \times n$ nc-linear matrix B with $n = \operatorname{poly}(r)$ such that the top-left entry of B^{-1} is f. In addition, the top-left entry of B^{-1} is nonzero if and only if the submatrix of B

without the first row and column is nonsingular as mentioned above. Therefore, as indicated by Hrubeš–Wigderson [19], the problem of checking if an nc-formula represents zero can be reduced to nc-Edmonds' problem.

We can consider a weighted analog of this reduction. Unlike the commutative case, an nc-rational function $f \in K\langle\!\langle x_1, \ldots, x_m \rangle\!\rangle$ cannot always be expressed as the ratio of two noncommutative polynomials. Nevertheless, we can define the (total) degree of f as the degree (in s) of the rational function $g \in K\langle\!\langle x_1, \ldots, x_m \rangle\!\rangle(s)$ obtained by replacing each x_i with $x_i s$. Then given an nc-formula computing f, we construct an nc-linear polynomial matrix A such that $|A|_{1,1} = f^{-1}$ and reduce the degree computation of f to nc-WEP using Proposition 13. By Theorem 2, we have:

▶ **Theorem 14.** *We can deterministically compute the degree of the nc-rational function represented by an nc-formula of size r over a field K in* $\mathrm{poly}(r)$ *arithmetic operations on K. If $K = \mathbb{Q}$, the bit-lengths of intermediate numbers are polynomially bounded.*

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