Minimum 0-Extension Problems on Directed Metrics

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Abstract

For a metric μ on a finite set T, the minimum 0-extension problem $\mathbf{0\text{-}Ext}[\mu]$ is defined as follows: Given $V \supseteq T$ and $c : \binom{V}{2} \to \mathbb{Q}_+$, minimize $\sum c(xy)\mu(\gamma(x), \gamma(y))$ subject to $\gamma : V \to T$, $\gamma(t) = t$ $(\forall t \in T)$, where the sum is taken over all unordered pairs in V. This problem generalizes several classical combinatorial optimization problems such as the minimum cut problem or the multiterminal cut problem. The complexity dichotomy of $\mathbf{0\text{-}Ext}[\mu]$ was established by Karzanov and Hirai, which is viewed as a manifestation of the dichotomy theorem for finite-valued CSPs due to Thapper and Živný.

In this paper, we consider a directed version $\overrightarrow{\mathbf{0}}\mathbf{\text{-}Ext}[\mu]$ of the minimum 0-extension problem, where μ and c are not assumed to be symmetric. We extend the NP-hardness condition of $\mathbf{0\text{-}Ext}[\mu]$ to $\overrightarrow{\mathbf{0}}\mathbf{\text{-}Ext}[\mu]$: If μ cannot be represented as the shortest path metric of an orientable modular graph with an orbit-invariant "directed" edge-length, then $\overrightarrow{\mathbf{0}}\mathbf{\text{-}Ext}[\mu]$ is NP-hard. We also show a partial converse: If μ is a directed metric of a modular lattice with an orbit-invariant directed edge-length, then $\overrightarrow{\mathbf{0}}\mathbf{\text{-}Ext}[\mu]$ is tractable. We further provide a new NP-hardness condition characteristic of $\overrightarrow{\mathbf{0}}\mathbf{\text{-}Ext}[\mu]$, and establish a dichotomy for the case where μ is a directed metric of a star.

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1 Introduction

A *metric* on a finite set T is a function $\mu : T \times T \to \mathbb{R}_+$ that satisfies $\mu(x, x) = 0$, $\mu(x, y) = \mu(y, x)$, and $\mu(x, y) + \mu(y, z) \geq \mu(x, z)$ for every $x, y, z \in T$, and $\mu(x, y) > 0$ for every $x \neq y \in T$. For a rational-valued metric μ on T, the *minimum* 0*-extension problem* $\mathbf{0\text{-}Ext}[\mu]$ on μ is defined as follows:

$$\begin{aligned} \mathbf{0\text{-}Ext}[\mu]: \quad & \text{Instance}: \ V \supseteq T,\ c : \binom{V}{2} \to \mathbb{Q}_+ \\ & \text{Min.} \sum_{xy \in \binom{V}{2}} c(xy)\mu(\gamma(x), \gamma(y)) \\ & \text{s.t.} \quad \gamma : V \to T \text{ with } \gamma(t) = t \text{ for all } t \in T, \end{aligned} \tag{1}$$

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where $\binom{V}{2}$ denotes the set of all unordered pairs of V, and xy denotes the unordered pair consisting of $x, y \in V$. The minimum 0-extension problem was introduced by Karzanov [11], and also known as the *multifacility location problem* in facility location theory [15]. Note that the formulation (1) of **0-Ext**$[\mu]$ is different from but equivalent to that of [11].

The minimum 0-extension problem generalizes several classical combinatorial optimization problems: If $T = \{s, t\}$, then **0-Ext**$[\mu]$ is nothing but the minimum s-t cut problem in an undirected network. If $T = \{x, y, z\}$ and $\mu(x, y) = \mu(y, z) = \mu(z, x) = 1$, then **0-Ext**$[\mu]$ is the 3-terminal cut problem. Similarly, **0-Ext**$[\mu]$ can formulate the k-terminal cut problem. Moreover, **0-Ext**$[\mu]$ appears as a discretized LP-dual problem for a class of maximum multiflow problems [10, 11] (also see [7, 8]).

The computational complexity of **0-Ext**$[\mu]$ depends on metric μ. In the above examples, the minimum s-t cut problem is in P and the 3-terminal cut problem is NP-hard. In [11], Karzanov addressed the classification problem of the computational complexity of **0-Ext**$[\mu]$ with respect to μ. After [5, 13], the complexity dichotomy of **0-Ext**$[\mu]$ was fully established by Karzanov [12] and Hirai [9], which we explain below.

A metric μ on T is called *modular* if for every $s_0, s_1, s_2 \in T$, there exists an element $m \in T$, called a *median*, such that $\mu(s_i, s_j) = \mu(s_i, m) + \mu(m, s_j)$ holds for every $0 \le i < j \le 2$. The *underlying graph* of μ is defined as the undirected graph $H_\mu = (T, U)$, where $U = \{xy \in \binom{T}{2} \mid \forall z \in T \setminus \{x, y\},\ \mu(x, y) < \mu(x, z) + \mu(z, y)\}$. We say that an undirected graph is *orientable* if it has an edge-orientation such that for every 4-cycle (u, v, w, z, u), uv is oriented from u to v if and only if wz is oriented from z to w.

The dichotomy theorem of the minimum 0-extension problem is the following:

▶ **Theorem 1** ([12]). *Let μ be a rational-valued metric. **0-Ext**$[\mu]$ is strongly NP-hard if*
(i) *μ is not modular, or*
(ii) *H_μ is not orientable.*

▶ **Theorem 2** ([9]). *Let μ be a rational-valued metric. If μ is modular and H_μ is orientable, then **0-Ext**$[\mu]$ is solvable in polynomial time.*

The minimum 0-extension problem constitutes a fundamental class of *valued CSPs* (*valued constraint satisfaction problem*) [9] – a minimization problem of a sum of functions having a small number of variables. More concretely, **0-Ext**$[\mu]$ is precisely the *finite-valued CSP* generated by a single binary function $\mu : T \times T \to \mathbb{Q}_+$ that is a metric. From this viewpoint, the above complexity dichotomy result (Theorem 1 and 2) is a manifestation of the dichotomy theorem for finite-valued CSPs obtained by Thapper and Živný [16]. They gave a complete characterization of tractable finite-valued CSPs in terms of the existence of a certain *fractional polymorphism*. Actually Theorem 2 was proved by utilizing a related tractability condition obtained by Kolmogorov, Thapper, and Živný [14]. However, it is a strong characterization specialized for the minimum 0-extension problem, which yields an efficient and combinatorial polynomial time testing algorithm for the tractability of **0-Ext**$[\mu]$. Indeed, we can verify modularity of μ by checking whether m is a median of triple t_1, t_2, t_3 for every $m, t_1, t_2, t_3 \in T$. We can also verify the orientability of H_μ by orienting each edge in depth first order with respect to an adjacency relation such that edges uv and zw in each 4-cycle (u, v, w, z, u) are said to be adjacent. Moreover, a known polynomial time testing algorithm [17] (that is applicable to **0-Ext**$[\mu]$) is based on the ellipsoid method, and of much worse complexity. So it is a natural direction to seek such an efficient combinatorial characterization for a more general binary function $\mu : T \times T \to \mathbb{Q}_+$ for which the corresponding valued CSP is tractable.

Motivated by these facts, in this paper, we consider a *directed* version of the minimum 0-extension problem, aiming to extend the above results. Here, by "directed" we mean that symmetry of μ and c is not assumed. A *directed metric* on a finite set T is a function

$\mu : T \times T \to \mathbb{R}_+$ that satisfies $\mu(x,x) = 0$ and $\mu(x,y) + \mu(y,z) \geq \mu(x,z)$ for every $x,y,z \in T$, and $\mu(x,y) + \mu(y,x) > 0$ for every $x \neq y \in T$. For a rational-valued directed metric μ on T, the *directed minimum 0-extension problem* $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ on μ is defined as follows:

$$\overrightarrow{\mathbf{0}}\textbf{-Ext}[\mu]: \quad \text{Instance}: \; V \supseteq T, \; c : V \times V \to \mathbb{Q}_+$$
$$\text{Min.} \sum_{(x,y) \in V \times V} c(x,y)\mu(\gamma(x), \gamma(y))$$
$$\text{s.t.} \;\; \gamma : V \to T \text{ with } \gamma(t) = t \text{ for all } t \in T.$$

The minimum s-t cut problem on a directed network is a typical example of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ in the case of $T = \{s,t\}$, $\mu(s,t) = 1$, and $\mu(t,s) = 0$. Also, the directed minimum 0-extension problem contains the undirected version. Hence, the complexity classification of the directed version is an extension of that of the undirected version.

In this paper, we explore sufficient conditions for which $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is tractable, and for which $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is NP-hard. Our first contribution is an extension of Theorem 1 to the directed version:

▶ **Theorem 3.** *Let μ be a rational-valued directed metric. $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is strongly NP-hard if one of the following holds:*
- **(i)** *μ is not modular.*
- **(ii)** *H_μ is not orientable.*
- **(iii)** *μ is not directed orbit-invariant.*

The modularity and the underlying graph H_μ of a directed metric μ are natural extensions of those of a metric. In **0-Ext**$[\mu]$, the condition (i) in Theorem 1 contains the condition (iii) in Theorem 3. See Section 3 for the precise definitions of the terminologies.

We next consider the converse of Theorem 3. It is known [1] that a canonical example of a modular metric is the graph metric of the covering graph of a modular lattice with respect to an orbit-invariant edge-length. Moreover, a tractable metric μ in Theorem 2 is obtained by *gluing* such metrics of modular lattices [9]. It turns out in Section 3 that a directed metric excluded by (i), (ii), and (iii) in Theorem 3 also admits an amalgamated structure of modular lattices. Our second contribution is the tractability for the building block of such a directed metric.

▶ **Theorem 4.** *Let μ be a rational-valued directed metric. Suppose that H_μ is the covering graph of a modular lattice and μ is directed orbit-invariant. Then $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is solvable in polynomial time.*

See Sections 2 and 3 for the undefined terminologies.

The converse of Theorem 3 is not true: Even if H_μ is a tree (that is excluded by (i), (ii), and (iii) in Theorem 3), $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ can be NP-hard. On the other hand, **0-Ext**$[\mu]$ for which H_μ is a tree is always tractable (see [15]). This is a notable difference between **0-Ext**$[\mu]$ and $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$. Our third contribution is a new hardness condition capturing this difference. For $x,y \in T$, let $I_\mu(x,y) := \{z \in T \mid \mu(x,y) = \mu(x,z) + \mu(z,y)\}$, which is called the *interval* from x to y. We denote $I := I_\mu$ if μ is clear in the context. For $x,y \in T$, the *ratio* $R_\mu(x,y)$ from x to y is defined as $R_\mu(x,y) := \mu(x,y)/\mu(y,x)$ (if $\mu(y,x) = 0$, then $R_\mu(x,y) := \infty$). A pair $(x,y) \in \binom{T}{2}$ is called a *biased pair* if $R_\mu(x,z) > R_\mu(z,y)$ holds for every $z \in I(x,y) \cap I(y,x) \setminus \{x,y\}$, or $R_\mu(x,z) < R_\mu(z,y)$ holds for every $z \in I(x,y) \cap I(y,x) \setminus \{x,y\}$. A triple (s_0,s_1,s_2) is called a *non-collinear triple* if $s_i \notin I(s_{i-1},s_{i+1}) \cap I(s_{i+1},s_{i-1})$ holds for every $i \in \{0,1,2\}$ (the indices of s_i are taken modulo 3). A non-collinear triple (s_0, s_1, s_2) is also called a *biased non-collinear triple* if (s_i, s_j) is a biased pair for every $i \neq j$. We now state an additional NP-hardness condition of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$:

▶ **Theorem 5.** *Let μ be a rational-valued directed metric on T. If there exists a biased non-collinear triple for μ, then $\overrightarrow{\mathbf{0}}$-**Ext**$[\mu]$ is strongly NP-hard.*

Our forth contribution says that the non-existence of a biased non-collinear triple implies tractability, provided the underlying graph is a star.

▶ **Theorem 6.** *Let μ be a rational-valued directed metric on T. If H_μ is a star and there exists no biased non-collinear triple for μ, then $\overrightarrow{\mathbf{0}}$-**Ext**$[\mu]$ is solvable in polynomial time.*

The organization of this paper is as follows. Section 2 provides preliminary arguments which are necessary for the proofs. Section 3 introduces some notions and shows several properties in directed metric spaces. Section 4 provides a proof of Theorem 4 (see the full version for a proof of Theorem 6). To show Theorem 4 and 6, we utilize the tractablity condition of valued CSPs by Kolmogorov, Thapper, and Živný [14], as in the spirit of [9] to prove Theorem 2. Section 5 shows one of the hardness results of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ (see the full version for proofs of the other hardness results). We prove Theorem 3 and 5 by showing polynomial-time reductions from the maximum cut problem, which are originated from the hardness proof of the multiterminal cut problem [6], and were also used by [11, 12] to prove Theorem 1. Also see the full version for proofs omitted in Section 2, 3, 4, 5.

Notation

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of reals, rationals, integers, and positive integers, respectively. Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, where ∞ is an infinity element. We also denote the sets of nonnegative reals and rationals by $\mathbb{R}_+$ and $\mathbb{Q}_+$.

2 Preliminaries

2.1 Modular graphs

Let $G = (V, E)$ be a connected graph. The *graph metric* $d_G : V \times V \to \mathbb{Z}$ is defined as follows:

$$d_G(x, y) := \text{the number of edges in a shortest path from } x \text{ to } y \text{ in } G \ (x, y \in V). \tag{2}$$

We denote d_G simply by d if G is clear in the context. We say that G is *modular* if its graph metric d_G is modular.

▶ **Lemma 7** ([2]). *A connected graph $G = (V, E)$ is modular if and only if the following two conditions hold:*

(i) *G is a bipartite graph.*

(ii) *For vertices $p, q \in V$ and neighbors p_1, p_2 of p with $d(p, q) = 1 + d(p_1, q) = 1 + d(p_2, q)$, there exists a common neighbor p' of p_1, p_2 with $d(p, q) = 2 + d(p', q)$.*

Let (T, μ) be a metric space. For $x, y \in T$, we denote the interval of x, y by $I_\mu(x, y) := \{z \in T \mid \mu(x, y) = \mu(x, z) + \mu(z, y)\}$. We denote $I := I_\mu$ if μ is clear in the context. A subset $X \subseteq T$ is called a *convex set* if $I(p, q) \subseteq X$ for every $p, q \in X$. A subset $X \subseteq T$ is called a *gated set* if for every $p \in T$, there exists $p' \in X$, called the *gate* of p at X, such that $\mu(p, q) = \mu(p, p') + \mu(p', q)$ for every $q \in X$. The gate of p at X is unique. Chepoi [4] showed the following relation between convex sets and gated sets:

▶ **Lemma 8** ([4]). *Let $G = (V, E)$ be a modular graph. For the metric space (V, d) and a subset $X \subseteq V$, the following conditions are equivalent:*
(i) *X is convex.*
(ii) *X is gated.*

2.2 Modular lattices

Let $\mathcal{L}$ be a partially ordered finite set with a partial order $\preceq$. By $a \prec b$ we mean $a \preceq b$ and $a \neq b$. For $a, b \in \mathcal{L}$, we denote by $a \vee b$ the minimum element of the set $\{c \in \mathcal{L} \mid c \succeq a$ and $c \succeq b\}$, and denote by $a \wedge b$ the maximum element of the set $\{c \in \mathcal{L} \mid c \preceq a$ and $c \preceq b\}$. If for every $a, b \in \mathcal{L}$ there exist $a \vee b$ and $a \wedge b$, then $\mathcal{L}$ is called a *lattice*. A lattice $\mathcal{L}$ is called *modular* if for every $a, b, c \in \mathcal{L}$ with $a \preceq c$ it holds that $a \vee (b \wedge c) = (a \vee b) \wedge c$. For $a \preceq b \in \mathcal{L}$, we let $[a, b]$ denote the interval $\{c \in \mathcal{L} \mid a \preceq c \preceq b\}$. For $a \prec b \in \mathcal{L}$, a sequence $(a = u_0, u_1, \ldots, u_n = b)$ is called a *chain* from a to b if $u_{i-1} \prec u_i$ holds for all $i \in \{1, 2, \ldots, n\}$. Here the length of a chain $(u_0, u_1, \ldots, u_n)$ is n. We denote by $r[a, b]$ the length of the longest chain from a to b. For a lattice $\mathcal{L}$, let $\mathbf{0}$ denote the minimum element of $\mathcal{L}$, and let $\mathbf{1}$ denote the maximum element of $\mathcal{L}$. The rank $r(a)$ of an element a is defined by $r(a) := r[\mathbf{0}, a]$.

▶ **Lemma 9** (see [3, Chapter II]). *Let $\mathcal{L}$ be a modular lattice. For $a \preceq b \in \mathcal{L}$, the following condition (called Jordan-Dedekind chain condition) holds:*

All maximal chains from a to b have the same length. (3)

By Lemma 9, we can see that for a modular lattice $\mathcal{L}$ and $a \in \mathcal{L}$, $r(a)$ is equal to the length of a maximal chain from $\mathbf{0}$ to a. A modular lattice is also characterised by rank as follows:

▶ **Lemma 10** (see [3, Chapter II]). *A lattice $\mathcal{L}$ is modular if and only if for every $a, b \in \mathcal{L}$, $r(a) + r(b) = r(a \wedge b) + r(a \vee b)$ holds.*

For a poset $\mathcal{L}$ and $a, b \in \mathcal{L}$, we say that b *covers* a if $a \prec b$ holds and there is no $c \in \mathcal{L}$ with $a \prec c \prec b$. The *covering graph* of $\mathcal{L}$ is the undirected graph obtained by linking all pairs a, b of $\mathcal{L}$ such that a covers b, or b covers a. Here we have the following relation between modular lattices and modular graphs:

▶ **Lemma 11** ([18]). *A lattice $\mathcal{L}$ is modular if and only if the covering graph of $\mathcal{L}$ is modular.*

Let $\mathcal{L}$ be a lattice. A function $f : \mathcal{L} \to \mathbb{R}$ is called *submodular* if $f(p) + f(q) \geq f(p \vee q) + f(p \wedge q)$ holds for every $p, q \in \mathcal{L}$. If $a, b \in \mathcal{L}$ are covered by $a \vee b$, then the pair (a, b) is called a 2-*covered pair*. We have the following characterization of submodular functions on modular lattices:

▶ **Lemma 12.** *Let $\mathcal{L}$ be a modular lattice. A function $f : \mathcal{L} \to \mathbb{R}$ is submodular if and only if $f(a) + f(b) \geq f(a \vee b) + f(a \wedge b)$ holds for every 2-covered pair (a, b).*

2.3 Valued CSP

Let D be a finite set. For a positive integer k, a function $f : D^k \to \overline{\mathbb{Q}}$ is called a *k-ary cost function* on D. For a cost function f, let $\mathrm{dom} f := \{x \in D^k \mid f(x) < \infty\}$. Let $k_f := k$ denote the arity of f. Let $\mathcal{C}_n(D)$ be the set of all pairs of a cost function f on D of arity at most n and an assignment $\sigma : \{1, 2, \ldots, k_f\} \to \{1, 2, \ldots, n\}$. The *valued constraint satisfaction* *problem* (*VCSP*) on D is defined as follows [19]:

$$\textbf{VCSP:}\quad \text{Instance}: \ n \in \mathbb{N},\ \mathcal{C} \subseteq \mathcal{C}_n(D)$$
$$\text{Min.} \sum_{(f,\sigma)\in\mathcal{C}} f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k_f)})$$
$$\text{s.t.}\ \ x = (x_1, x_2, \dots, x_n) \in D^n.$$

Without loss of generality, we may assume that for any $i \in \{1, 2, \dots, n\}$, there exist j and $(f, \sigma) \in \mathcal{C}$ that satisfy $\sigma(j) = i$ (otherwise, we can erase unused variables).

Let Γ be a set of cost functions, which is called a *language*. The instance of VCSP is called a Γ-*instance* if all cost functions in the instance belong to Γ. Let VCSP[Γ] denote the class of the optimization problems whose instances are restricted to Γ-instances.

Let μ be a directed metric on T. The directed minimum 0-extension problem $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is viewed as a language-restricted VCSP. Indeed, let

$$\begin{aligned} D &:= T, \\ f(x, y) &:= \mu(x, y), \\ g_t(x) &:= \mu(x, t), \\ h_t(x) &:= \mu(t, x), \end{aligned}$$

and let

$$\Gamma := \{Cf \mid C \in \mathbb{Q}_+\} \cup \{Cg_t \mid t \in T,\ C \in \mathbb{Q}_+\} \cup \{Ch_t \mid t \in T,\ C \in \mathbb{Q}_+\}. \tag{4}$$

Then we can conclude that $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is an instance-restricted VCSP[Γ] on D.

Kolmogorov, Thapper, and Živný [14] discovered a powerful criterion for a language Γ such that VCSP[Γ] is tractable. A special case of this criterion is the following:

▶ **Theorem 13** ([14]). *Let D be a lattice, and Γ be a set of cost functions on D. Suppose that for every $f \in \Gamma$ and $x, y \in \operatorname{dom} f$, we have*

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y). \tag{5}$$

Then VCSP[Γ] can be solved in polynomial time.

3 Directed metric spaces

3.1 Modular directed metrics

We first extend the notions of modularity, medians, and underlying graphs to directed metric spaces. Let μ be a directed metric on T. We say that μ is *modular* if and only if for every $s_0, s_1, s_2 \in T$, there exists an element $m \in T$, called a *median*, such that $\mu(s_i, s_j) = \mu(s_i, m) + \mu(m, s_j)$ for every $0 \leq i, j \leq 2$ $(i \neq j)$. See Figure 1 (a) for an example of modular directed metrics. We define the *underlying graph* of μ as the undirected graph $H_\mu = (T, U)$, where

$$\begin{aligned} U := \{xy \in \binom{T}{2} \mid\ & \forall z \in T \setminus \{x, y\},\ \mu(x, y) < \mu(x, z) + \mu(z, y) \\ & \text{or } \forall z \in T \setminus \{x, y\},\ \mu(y, x) < \mu(y, z) + \mu(z, x)\}. \end{aligned} \tag{6}$$

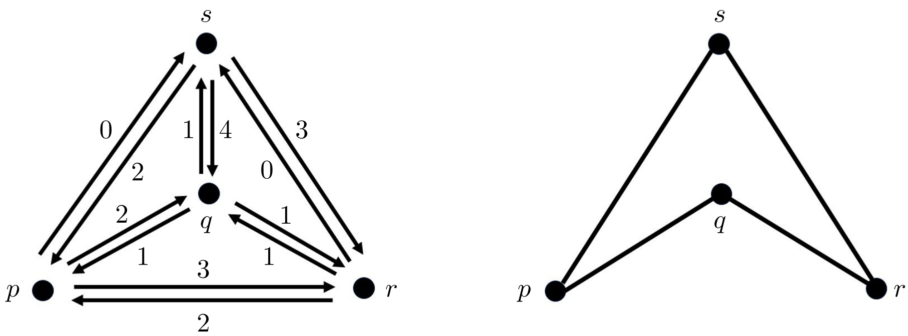

Figure 1 (a) a modular directed metric. (b) the underlying graph of (a).

For a directed metric μ on T and $v_0, v_1, \ldots, v_n \in T$, we say that a sequence $(v_0, v_1, \ldots, v_n)$ is *μ-shortest* if $\mu(v_0, v_n) = \sum_{i=0}^{n-1} \mu(v_i, v_{i+1})$. Bandelt [1] showed that for a modular (undirected) metric μ, a μ-shortest sequence is also d_{H_μ}-shortest. We have the following directed version of this property:

▶ **Lemma 14.** *Let μ be a modular directed metric on T, and let $v_0, v_1, \ldots, v_n \in T$.*

(1) *If a sequence $(v_0, v_1, \ldots, v_n)$ is μ-shortest, then the inverted sequence $(v_n, v_{n-1}, \ldots, v_0)$ is also μ-shortest.*

(2) *If a sequence $(v_0, v_1, \ldots, v_n)$ is μ-shortest, then this sequence is also d_{H_μ}-shortest.*

For a modular directed metric μ on T, let m be a median of $x, y, z \in T$ in μ. Then, by Lemma 14 m is also a median of x, y, z in H_μ. Hence, we have the following lemma:

▶ **Lemma 15.** *If a directed metric μ is modular, then H_μ is also modular.*

3.2 Directed orbits and directed orbit invariance

Let $G = (V, E)$ be an undirected graph. Let $\overleftrightarrow{E} := \{(u, v) \mid uv \in E\} \subseteq V \times V$, and $\overleftrightarrow{G} := (V, \overleftrightarrow{E})$. An element of $\overleftrightarrow{E}$ is called an *oriented edge* of E. For a path P from s to t in G, we orient each edge of P along the direction of P, and we denote by $\overrightarrow{P}$ the corresponding path in $\overleftrightarrow{G}$. Let $\overrightarrow{P}$ and $\overrightarrow{W}$ be paths in $\overleftrightarrow{G}$ such that the end point of $\overrightarrow{P}$ and the start point of $\overrightarrow{W}$ are identified. Then we denote by $\overrightarrow{P} \cup \overrightarrow{W}$ the path obtained by concatenating $\overrightarrow{P}$ and $\overrightarrow{W}$ in this order. In particular, if $\overrightarrow{W}$ consists of one oriented edge (p, q), then we simply denote $\overrightarrow{W} := (p, q)$ and $\overrightarrow{P} \cup \overrightarrow{W} := \overrightarrow{P} \cup (p, q)$. For $\overrightarrow{e}, \overrightarrow{e'} \in \overleftrightarrow{E}$, we say that $\overrightarrow{e}$ and $\overrightarrow{e'}$ are *projective* if there exists a sequence $(\overrightarrow{e} = \overrightarrow{e_0}, \overrightarrow{e_1}, \ldots, \overrightarrow{e_m} = \overrightarrow{e'})$ $(\overrightarrow{e_i} = (p_i, q_i) \in \overleftrightarrow{E}$ for each $i)$ such that $(p_i, q_i, q_{i+1}, p_{i+1}, p_i)$ is a 4-cycle in G for each i. An equivalence class of the projectivity relation is called a *directed orbit*. Then we have the following lemma about the number of oriented edges of each directed orbit included in a shortest path. This is a sharpening of the result for undirected graphs due to Bandelt [1], and is similarly shown by the proof of the undirected version.

▶ **Lemma 16** ([1])**.** *Let $G = (V, E)$ be a modular graph, and let $\overrightarrow{Q}$ be a directed orbit. For $x, y \in V$, let P be a path from x to y, and let P^* be a shortest path from x to y. Then we have*

$$|\overrightarrow{P^*} \cap \overrightarrow{Q}| \leq |\overrightarrow{P} \cap \overrightarrow{Q}|. \tag{7}$$

Let $G = (V, E)$ be an undirected graph. If a function $h : \overleftrightarrow{E} \to \mathbb{R}_+$ satisfies $h(\overrightarrow{e}) = h(\overrightarrow{e'})$ for every $\overrightarrow{e}, \overrightarrow{e'} \in \overleftrightarrow{E}$ belonging to the same directed orbit, then we say that h is *directed orbit-invariant*. Let μ be a directed metric on T with the underlying graph $H_\mu = (T, U)$. We say that μ is directed orbit-invariant if $\mu(u_1, u_2) = \mu(u'_1, u'_2)$ holds for every $\overrightarrow{u} = (u_1, u_2), \overrightarrow{u'} = (u'_1, u'_2) \in \overleftrightarrow{U}$ belonging to the same directed orbit in H_μ. A 4-cycle (p, q, r, s, p) in H_μ is called a *directed orbit-varying modular cycle* if $\mu(p, q) - \mu(s, r) = \mu(r, s) - \mu(q, p) = \mu(p, s) - \mu(q, r) = \mu(r, q) - \mu(s, p) \neq 0$. The cycle (p, q, r, s, p) in Figure 1 (b) is an example of a directed orbit-varying modular cycle.

Bandelt [1] showed that a metric μ is orbit-invariant if μ is modular. A directed metric μ is not necessarily directed orbit-invariant even if μ is modular. For example, if H_μ is a directed orbit-varying modular cycle, then μ is modular but not directed orbit-invariant. The name "directed orbit-varying modular cycle" is motivated by this fact. We now have the following sufficient condition of a directed metric to be directed orbit-invariant.

▶ **Lemma 17.** *Let μ be a modular directed metric. Suppose that H_μ has no directed orbit-varying modular cycle. Then, μ is directed orbit-invariant.*

This lemma is used to prove Theorem 3 (iii) in the full version.

We now consider a sufficient condition for which the converse of Lemma 14 (2) holds. For an undirected metric μ, Bandelt [1] showed that if μ is orbit-invariant and H_μ is modular, then a d_{H_μ}-shortest sequence is also μ-shortest. The similar property also holds for a directed metric as follows:

▶ **Lemma 18.** *Let μ be a directed metric on T, and let $v_0, v_1, \dots, v_n \in T$. If μ is directed orbit-invariant and H_μ is modular, then the following condition holds:*

$$\text{If } (v_0, v_1, \dots, v_n) \text{ is } d_{H_\mu}\text{-shortest, then it is also } \mu\text{-shortest.} \tag{8}$$

4 Proof of tractablity

In this section, we give a proof of Theorem 4. Let μ be a directed metric on T, and Γ be the language defined in (4). Then we see that $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is a subclass of VCSP$[\Gamma]$. Hence, by Theorem 13 we can prove the tractability of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ by showing submodularity of μ. To show submodularity, we imitate the proof of submodularity of metric functions on modular semilattices in the undirected version [9].

4.1 Proof of Theorem 4

Note that the underlying graph H_μ of μ is the covering graph of a modular lattice $\mathcal{L}$ with a partial order $\preceq$. We define a partial order $\preceq$ on $\mathcal{L} \times \mathcal{L}$ by $(a, b) \preceq (c, d) \iff a \preceq c$ and $b \preceq d$ $(a, b, c, d \in \mathcal{L})$. Then $\mathcal{L} \times \mathcal{L}$ is also a modular lattice. If μ is a submodular function on $\mathcal{L} \times \mathcal{L}$, then by Theorem 13 we can conclude that $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is solvable in polynomial time. Hence, the following property completes the proof:

▶ **Theorem 19.** *Let μ be a directed metric. Suppose that H_μ is the covering graph of a modular lattice $\mathcal{L}$ and μ is directed orbit-invariant. Then the function $\mu : \mathcal{L} \times \mathcal{L} \to \mathbb{R}_+$ is submodular.*

Proof. Note that $\mathcal{L} \times \mathcal{L}$ is a modular lattice. By Lemma 12, μ is a submodular function on $\mathcal{L} \times \mathcal{L}$ if and only if $\mu(a) + \mu(b) \geq \mu(a \vee b) + \mu(a \wedge b)$ holds for every 2-covered pair (a, b) $(a, b \in \mathcal{L} \times \mathcal{L})$. Thus, it suffices to show that $\mu(a) + \mu(b) \geq \mu(a \vee b) + \mu(a \wedge b)$ holds for any 2-covered pair (a, b). Let $a = (a_1, a_2), b = (b_1, b_2)$ $(a_1, a_2, b_1, b_2 \in \mathcal{L})$. Then, it suffices to consider the following two cases:

(i) $a_1 = b_1$, and $a_2 \vee b_2$ covers a_2, b_2.

(ii) a_1 covers b_1, and b_2 covers a_2.

We first consider the case (i). It suffices to show that $\mu(a_1, a_2) + \mu(a_1, b_2) \geq \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$. Let $Y := [a_2 \wedge b_2, a_2 \vee b_2]$. Then, for every $y \in Y \setminus \{a_2 \wedge b_2, a_2 \vee b_2\}$, it holds that $a_2 \vee b_2$ covers y and y covers $a_2 \wedge b_2$, because of Lemma 9 and Lemma 10. Hence, Y is a convex set in the metric space (T, d) (in this proof, we denote $d := d_{H_\mu}$ for simplicity). Since H_μ is modular, by Lemma 8 Y is a gated set. Hence, there exists $y^* \in Y$ such that $d(a_1, y) = d(a_1, y^*) + d(y^*, y)$ holds for every $y \in Y$. Therefore, by Lemma 18, (a_1, y^*, y) is μ-shortest for every $y \in Y$. If $y^* = a_2$, then we have

$$\begin{aligned} \mu(a_1, a_2 \vee b_2) &= \mu(a_1, a_2) + \mu(a_2, a_2 \vee b_2), \\ \mu(a_1, a_2 \wedge b_2) &= \mu(a_1, a_2) + \mu(a_2, a_2 \wedge b_2), \\ \mu(a_1, b_2) &= \mu(a_1, a_2) + \mu(a_2, b_2). \end{aligned} \tag{9}$$

Furthermore, since μ is directed orbit-invariant, we have $\mu(a_2 \vee b_2, b_2) = \mu(a_2, a_2 \wedge b_2)$. In addition, by Lemma 18 we have $\mu(a_2, b_2) = \mu(a_2, a_2 \vee b_2) + \mu(a_2 \vee b_2, b_2)$. Hence, we have $\mu(a_1, b_2) = \mu(a_1, a_2) + \mu(a_2, a_2 \vee b_2) + \mu(a_2, a_2 \wedge b_2)$. Therefore, we obtain $\mu(a_1, a_2) + \mu(a_1, b_2) = \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$. Similarly, if $y^* = b_2$, we obtain $\mu(a_1, a_2) + \mu(a_1, b_2) = \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$. If $y^* = a_2 \vee b_2$, then we have

$$\begin{aligned} \mu(a_1, a_2) &= \mu(a_1, a_2 \vee b_2) + \mu(a_2 \vee b_2, a_2), \\ \mu(a_1, b_2) &= \mu(a_1, a_2 \vee b_2) + \mu(a_2 \vee b_2, b_2), \\ \mu(a_1, a_2 \wedge b_2) &= \mu(a_1, a_2 \vee b_2) + \mu(a_2 \vee b_2, a_2) + \mu(a_2, a_2 \wedge b_2). \end{aligned} \tag{10}$$

Since μ is directed orbit-invariant, we have $\mu(a_2 \vee b_2, b_2) = \mu(a_2, a_2 \wedge b_2)$. Hence, by (10) we obtain $\mu(a_1, a_2) + \mu(a_1, b_2) = \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$. Similarly, if $y^* = a_2 \wedge b_2$, then we obtain $\mu(a_1, a_2) + \mu(a_1, b_2) = \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$. Thus, it suffices to consider the case when $y^* \neq a_2, b_2, a_2 \vee b_2, a_2 \wedge b_2$. In this case, we have

$$\begin{aligned} \mu(a_1, a_2) &= \mu(a_1, y^*) + \mu(y^*, a_2 \vee b_2) + \mu(a_2 \vee b_2, a_2) \geq \mu(a_1, a_2 \vee b_2), \\ \mu(a_1, b_2) &= \mu(a_1, y^*) + \mu(y^*, a_2 \wedge b_2) + \mu(a_2 \wedge b_2, b_2) \geq \mu(a_1, a_2 \wedge b_2). \end{aligned} \tag{11}$$

Hence, we have $\mu(a_1, a_2) + \mu(a_1, b_2) \geq \mu(a_1, a_2 \vee b_2) + \mu(a_1, a_2 \wedge b_2)$.

For the next, we consider the case (ii). The submodularity is $\mu(a_1, a_2) + \mu(b_1, b_2) \geq \mu(a_1, b_2) + \mu(b_1, a_2)$. Since H_μ is bipartite, $d(a_1, b_2)$ is equal to either $d(b_1, a_2)$ or $d(b_1, a_2) + 2$ or $d(b_1, a_2) - 2$. If $d(a_1, b_2)$ is equal to $d(b_1, a_2) + 2$ or $d(b_1, a_2) - 2$, then by Lemma 18 we have $\mu(a_1, a_2) + \mu(b_1, b_2) = \mu(a_1, b_2) + \mu(b_1, a_2)$. Thus, it suffices to consider the case when $d(a_1, b_2) = d(b_1, a_2)$. In this case, $d(a_1, a_2)$ is equal to either $d(a_1, b_2) - 1$ or $d(a_1, b_2) + 1$. Suppose that $d(a_1, a_2) = d(a_1, b_2) + 1$. Then, by Lemma 18 we have $\mu(a_1, a_2) = \mu(a_1, b_2) + \mu(b_2, a_2)$. Hence, we obtain $\mu(a_1, a_2) + \mu(b_1, b_2) = \mu(a_1, b_2) + \mu(b_2, a_2) + \mu(b_1, b_2) \geq \mu(a_1, b_2) + \mu(b_1, a_2)$. Consider the case when $d(a_1, a_2) = d(a_1, b_2) - 1$. Similarly, $d(b_1, b_2)$ is equal to either $d(a_1, b_2) - 1$ or $d(a_1, b_2) + 1$, and by the similar argument, we may assume that $d(b_1, b_2) = d(a_1, b_2) - 1$. Let P be a shortest path in H_μ from a_1 to a_2. Let z be the vertex in P that is adjacent to a_1. Then, we have $d(z, b_2) = d(b_1, b_2) = d(a_1, b_2) - 1$. Hence, by Lemma 7, there exists a common neighbor w of z, b_1 with $d(w, b_2) = d(a_1, b_2) - 2$. Then, we have $d(z, b_2) = d(w, a_2) = d(a_1, b_2) - 1$, $d(z, a_2) = d(z, b_2) - 1$, and $d(w, b_2) = d(z, b_2) - 1$. Furthermore, since a_1 covers b_1, we see that z covers w. Hence, we can apply the same argument to z, w, a_2, b_2 which we apply to a_1, b_1, a_2, b_2 above. By repeating this argument, we can see that a_2 covers b_2, but this is a contradiction. ◀

5 Proof of hardness

In this section, we give a proof of Theorem 3 for the case (i) (see the full version for proofs of the other cases and Theorem 5). We prove them by reductions from the maximum cut problem (MAX CUT). In each reduction, we construct a "gadget" which is a counterexample to submodularity of the objective function of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ (in a certain sense). This type of reduction is originated from the proof of hardness of the 3-terminal cut problem [6]. Also, Karzanov [11, 12] showed the hardness of the minimum 0-extension problems on undirected metrics by using similar reductions (Theorem 1). We extend these reductions to directed cases. We first describe the main idea of a reduction from MAX CUT to $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ in Section 5.1. For the next, we prove Theorem 3 for the case (i) by using this reduction in Section 5.2.

5.1 Approach

Let μ be a rational-valued directed metric on T. Suppose that we are given $V \supseteq T$ and $c : V \times V \to \mathbb{Q}$ as an instance of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$. For $s_0, s_1, \ldots, s_k \in T$ and $x_0, x_1, \ldots, x_k \in V \setminus T$, we denote by $\tau_c(s_0, x_0|s_1, x_1|\cdots|s_k, x_k)$ the optimal value of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ subject to $\gamma(x_0) = s_0, \gamma(x_1) = s_1, \ldots, \gamma(x_k) = s_k$. We simply denote $\tau(s_0, x_0|s_1, x_1|\cdots|s_k, x_k) := \tau_c(s_0, x_0|s_1, x_1|\cdots|s_k, x_k)$ if c is clear in the context. Let τ^* be the optimal value of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ subject to no constraint. Imitating the constructions in [6, 11, 12], we call a pair (V, c) a *gadget* if it satisfies the following properties (in other words, "violates submodularity," cf. [6]) for specified elements $s, t \in T$ and $x, y \in V \setminus T$.

$$\begin{aligned} &\text{(i)} \quad \tau(s, x|t, y) = \tau(t, x|s, y) = \tau^*, \\ &\text{(ii)} \quad \tau(s, x|s, y) = \tau(t, x|t, y) = \tau^* + \delta \quad \text{for some } \delta > 0, \\ &\text{(iii)} \quad \tau(s', x|t', y) \geq \tau^* + \delta \quad \text{for all other pairs } (s', t') \in T \times T. \end{aligned} \tag{12}$$

We now show that there exists a polynomial-time reduction from MAX CUT to $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ if there exists a gadget (V, c) that satisfies (12) with respect to some $s, t \in T$ and $x, y \in V \setminus T$. Suppose that we are given a graph $G = (U, E)$ and a positive integer k as an instance of MAX CUT. In MAX CUT, we are asked whether there exists a partition $(S, U \setminus S)$ such that the number of edges between S and $U \setminus S$ is at least k. Let (V, c) be a gadget which satisfies (12) with respect to $s, t \in T$, $x, y \in V \setminus T$. For each edge $e = uv \in E$, we replace e by a copy of (V, c), identifying x with u, and y with v. We also identify copies of each element in T which belong to different copies of (V, c). The other copied elements are distinct. Then, for the gadget (V', c') constructed above, the optimal value of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ with respect to (V', c') is at most $|E|\tau^* + (|E| - k)\delta$ if and only if there exists a cut in G whose size is at least k (if a partition $(S, U \setminus S)$ cuts the maximum number of edges, then the optimal value of $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is achieved when $\gamma(u) = s$ for every $u \in S$, and $\gamma(u) = t$ for every $u \in U \setminus S$). This is a polynomial-time reduction from MAX CUT to $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$.

5.2 Proof of Theorem 3 for the case (i)

We first introduce the following lemma, which is originated from the proof of Theorem 1 (i) in [12].

▶ **Lemma 20.** *Let μ be a rational-valued directed metric on T. If there exists a gadget (V, c) which satisfies the following properties for a non-collinear triple (s_0, s_1, s_2) in T and distinct elements $z_i \in V \setminus T$ $(i = 0, \ldots, 5)$, then $\overrightarrow{\mathbf{0}}$**-Ext**$[\mu]$ is strongly NP-hard.*

(i) $\tau(s_{i_0+1}, z_0|s_{i_1-1}, z_1|s_{i_2}, z_2|s_{i_3+1}, z_3|s_{i_4-1}, z_4|s_{i_5}, z_5) = \tau^*$ $\quad$ ($i_j \in \{0,1\}$ *for each* j),

(ii) $\tau(s'_0, z_0|s'_1, z_1|s'_2, z_2|s'_3, z_3|s'_4, z_4|s'_5, z_5) \geq \tau^* + \delta$
for all other sextuplets $s'_0, s'_1, s'_2, s'_3, s'_4, s'_5, s'_6$ *and some* $\delta > 0$, (13)

where the indices of s_i *are taken modulo 3.*

We now show Theorem 3 for the case (i) by use of Lemma 20. The proof we describe below is a directed version of that of Theorem 1 (i) in [12]. Let μ be a nonmodular rational-valued directed metric on T. For $x, y, z \in T$, we denote $\Delta(x, y, z) := \mu(x, y) + \mu(y, x) + \mu(y, z) + \mu(z, y) + \mu(z, x) + \mu(x, z)$. Let (s_0, s_1, s_2) be a medianless triple such that $\Delta(s_0, s_1, s_2)$ is minimum. Let $\bar{\Delta} := \Delta(s_0, s_1, s_2)$. Take six elements $z_0, z_1, \ldots, z_5$, and let $V := T \cup \{z_0, z_1, \ldots, z_5\}$. Let $\mu_i := \mu(s_{i-1}, s_{i+1}) + \mu(s_{i+1}, s_{i-1})$ and $a_i := (\mu_{i-1} + \mu_{i+1} - \mu_i)/\mu_{i-1}\mu_{i+1}$ for $i = 0, 1, 2$, where the indices of s_i and μ_i are taken modulo 3. Then we define a function $c : V \times V \to \mathbb{Q}_+$ as follows:

$$
\begin{aligned}
&c(s_i, z_{i+1}) = c(z_{i+1}, s_i) = 1 \quad (0 \leq i \leq 5),\\
&c(s_i, z_{i+2}) = c(z_{i+2}, s_i) = 1 \quad (0 \leq i \leq 5),
\end{aligned} \tag{14}
$$

where the indices of z_i are taken modulo 6. Also we define a function $c' : V \times V \to \mathbb{Q}_+$ as follows:

$$
c'(s_i, z_j) = c'(z_j, s_i) = a_i \quad (0 \leq i \leq 2,\ 0 \leq j \leq 5). \tag{15}
$$

Let N be a sufficiently large positive rational. We define a function $\tilde{c}$ by $\tilde{c} := Nc + c'$. We now show that a gadget $(V, \tilde{c})$ satisfies (13). We first observe that $\tau_{\tilde{c}}(\gamma)$ is not the optimal or nearly optimal value if $\gamma(z_i) \notin I(s_{i-1}, s_{i+1}) \cap I(s_{i+1}, s_{i-1})$ for some i. Consider the case when $\gamma(z_i) \in I(s_{i-1}, s_{i+1}) \cap I(s_{i+1}, s_{i-1})$ holds for each i. We show the following claim:

▷ Claim 21. *Let* $x \in I(s_{i-1}, s_{i+1}) \cap I(s_{i+1}, s_{i-1})$. *Then at least one of the following conditions holds:*

(i) *Both of sequences* (s_i, s_{i-1}, x) *and* (x, s_{i-1}, s_i) *are* μ*-shortest.*

(ii) *Both of sequences* (s_i, s_{i+1}, x) *and* (x, s_{i+1}, s_i) *are* μ*-shortest.*

Proof. Suppose that (ii) does not hold. Let $i = 1$. By the assumption, we have $\mu(s_1, x) < \mu(s_1, s_2) + \mu(s_2, x)$ or $\mu(x, s_1) < \mu(x, s_2) + \mu(s_2, s_1)$. Then we have $\Delta(s_0, s_1, x) < \Delta(s_0, s_1, s_2)$. Hence, there exists a median m of s_0, s_1, x. If $m = s_0$, then (i) holds. If $m \neq s_0$, then we have

$$
\begin{aligned}
\Delta(s_1, m, s_2) &= \mu(s_1, s_2) + \mu(s_2, s_1) + \mu(s_1, m) + \mu(m, s_1) + \mu(s_2, m) + \mu(m, s_2)\\
&< \mu(s_1, s_2) + \mu(s_2, s_1) + \mu(s_1, s_0) + \mu(s_0, s_1) + \mu(s_2, x) + \mu(x, m)\\
&\quad + \mu(m, x) + \mu(x, s_2)\\
&< \Delta(s_0, s_1, s_2).
\end{aligned} \tag{16}
$$

Hence, there exists a median w of s_1, m, s_2. However, w is also a median of s_0, s_1, s_2, and this is a contradiction. ◀

For each $i \in \{0, 1, \ldots, 5\}$, let g_i be the contribution to the value $\tau_{\tilde{c}}(\gamma)$ from $c'(z_i, s_0)$, $c'(z_i, s_1)$, $c'(z_i, s_2)$, $c'(s_0, z_i)$, $c'(s_1, z_i)$, $c'(s_2, z_i)$. If $\gamma(z_i) = s_0$, then we have $g_i = a_1\mu_2 + a_2\mu_1 = (\mu_0 + \mu_2 - \mu_1)/\mu_0 + (\mu_1 + \mu_0 - \mu_2)/\mu_0 = 2$. Similarly, we have $g_i = 2$ when $\gamma(z_i) = s_1$ or s_2. We next consider the case when $\gamma(z_i) \in I(s_{i-1}, s_{i+1}) \cap I(s_{i+1}, s_{i-1}) \setminus \{s_{i-1}, s_{i+1}\}$. Let $i = 0$ and

$\epsilon := \mu(s_1, \gamma(z_0)) + \mu(\gamma(z_0), s_1)$. By Claim 21, we may assume that $\mu(s_0, \gamma(z_o)) + \mu(\gamma(z_0), s_0) = \mu_2 + \epsilon$ holds. Hence, we have

$$\begin{aligned} g_0 &= a_0(\mu_2 + \epsilon) + a_1\epsilon + a_2(\mu_0 - \epsilon) \\ &= a_0\mu_2 + a_2\mu_0 + \epsilon(a_0 + a_1 - a_2) \\ &= 2 + \epsilon(a_0 + a_1 - a_2). \end{aligned} \tag{17}$$

Note that we have

$$\begin{aligned} \mu_0\mu_1\mu_2(a_0 + a_1 - a_2) &= \mu_0(\mu_1 + \mu_2 - \mu_0) + \mu_1(\mu_0 + \mu_2 - \mu_1) - \mu_2(\mu_0 + \mu_1 - \mu_2) \\ &= 2\mu_0\mu_1 - \mu_1^2 - \mu_0^2 + \mu_2^2 \\ &= \mu_2^2 - (\mu_0 - \mu_1)^2 > 0. \end{aligned} \tag{18}$$

Hence, we have $g_0 > 2$. Similarly, for each $i \in \{0, 1, \dots, 5\}$, we have $g_i > 2$ if $\gamma(z_i) \in I(s_{i-1}, s_{i+1}) \cap I(s_{i+1}, s_{i-1}) \setminus \{s_{i-1}, s_{i+1}\}$. Hence, the gadget $(V, \tilde{c})$ satisfies (13).

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