A General Kernelization Technique for Domination and Independence Problems in Sparse Classes

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Abstract

We unify and extend previous kernelization techniques in sparse classes [6, 17] by defining *water lilies* and show how they can be used in bounded expansion classes to construct linear bikernels for (r, c)-Dominating Set, (r, c)-Scattered Set, Total r-Domination, r-Roman Domination, and a problem we call $(r, [\lambda, \mu])$-Domination (implying a bikernel for r-Perfect Code). At the cost of slightly changing the output graph class our bikernels can be turned into kernels. We also demonstrate how these constructions can be combined to create "multikernels", meaning graphs that represent kernels for multiple problems at once.

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1 Introduction

Dominating Set is arguably one of the touchstone for kernelization in sparse graph classes: after a linear kernel in planar graphs [1] and a polynomial kernel in graphs defined by an excluded topological minor [2, 12] results for linear kernels in bounded genus graphs [3] apex-minor-free graphs [9], H-minor-free graphs [10], and finally H-topological-minor-free graphs [11] followed in quick succession. The most general results to date are linear kernels for bounded expansion classes [6] (generalizing all aforementioned classes) and an almost-linear kernels for nowhere dense classes [14] (generalizing bounded expansion classes). These latter two results even hold for the general problem of r-Dominating Set, where a vertex dominates everything in its closed r-neighbourhood. Together with an almost-linear kernel for the related r-Independence problem [17], these results led us to the guiding question: Do the kernelization techniques developed for r-Domination/r-Independence in sparse classes carry over to related problems?

Bounded expansion classes. Nešetřil and Ossona de Mendez introduced bounded expansion classes as a generalization of classes excluding a (topological) minor and various useful notions of sparsity (*e.g.* embeddability in a surface, bounded degree). In short, a class $\mathcal{G}$ has *bounded expansion* (BE) if any minor obtained by contracting disjoint subgraphs of radius at most r in any member $G \in \mathcal{G}$ is $\nabla_r(\mathcal{G})$-degenerate, where $\nabla_r(\mathcal{G})$ is a class constant independent of G. There are various equivalent definitions for BE classes [15, 19, 18, 16], all of which have in common that they define families of graph invariants $\{f_r\}_{r \in \mathbb{N}}$ where r is a parameter governing the "depth" at which the invariant is measured. BE classes then are precisely those graph classes for which f_r is finite for every member of the class. We will not need

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to work with these invariants directly, instead building on higher-level results discussed in Section 2. Consequently, we broadly refer to these invariants as *expansion characteristics*. For an in-depth discussion see [16].

A selection of problems. The commonality of the following problems is that they can be expressed via *universal neighbourhood constraints*, meaning that a solution X needs to intersect every "neighbourhood" (a slightly flexible term as we will see in the following) in at least/at most a certain value. We define an *r-dominating set* of a graph G to be any set D that satisfies $|N^r[u] \cap D| \geq 1$ for all $u \in V(G)$, where $N^r[u]$ contains all vertices at distance $\leq r$ from u. We arrive at a natural extension of the problem by replacing the right hand side of this *domination constraint* by an arbitrary constant. We call a set that satisfies the constraint $|N^r[u] \cap D| \geq c$ an *(r, c)-dominating set* and the corresponding decision problem

(r, c)-DOMINATION parametrised by k
Input: A graph G and an integer k.
Problem: Is there a set $D \subseteq V(G)$ of size at most k such that $|N^r[v] \cap D| \geq c$ for all $v \in G$?

For $r = 1$ this problem has received some attention in the literature under the name "k-DOMINATION" (*e.g.* [4]), for $c = 1$ we recover the above discussed r-DOMINATION. Two other domination problems of interest, TOTAL r-DOMINATION and r-ROMAN DOMINATION, can be found in the full version.

The problem of *independence* turns out to be closely related to that of domination. We define an *r-scattered set* of a graph G to be any set I that satisfies $|N^r[u] \cap I| \leq 1$ for all $u \in V(G)$. Note that an r-scattered set is equivalent to a *$2r$-independent set* (all vertices in I are pairwise at distance $> 2r$) and the domination/independence duality that holds in BE-classes (see below) has usually been described with this terminology. However, the natural extension to *(r, c)-scattered sets* that satisfy the constraints $|N^r[u] \cap I| \leq c$ does not correspond to independent sets. We therefore opt to speak in terms of scattered instead of independent sets, in particular, we consider the following parameterized problem:

(r, c)-SCATTERED SET parametrised by k
Input: A graph G and an integer k.
Problem: Is there a set $I \subseteq V(G)$, $|I| \geq k$ such that $|N^r[v] \cap I| \leq c$ for all $v \in V(G)$?

Finally, we consider the problem that arises when combining the domination- and scatter-constraints into the form $\lambda \leq |N^r[u] \cap D| \leq \mu$, which leads to the following, rather general, parameterized problem:

$(r, [\lambda, \mu])$-DOMINATION parametrised by k
Input: A graph G and an integer k.
Problem: Is there a set $D \subseteq V(G)$, $|D| \leq k$ s.t. every $v \in G$ satisfies $\lambda \leq |N^r[v] \cap D| \leq \mu$?

$(r, [c, \infty])$-DOMINATION is equivalent to (r, c)-DOMINATING SET and $(r, [0, c])$-DOMINATION to (r, c)-SCATTERED SET. For $\lambda = \mu = 1$ it is equivalent to PERFECT CODE (see full version).

Kernelization in sparse classes. The definition of a kernel (see [5]) for a problem restricted to a certain input class demands that the output belongs to this class as well, *e.g.* a planar kernelization needs to output a planar graph. This turns out to be too restrictive for very general notions of sparseness and we are left with the choice of either outputting an annotated instance belonging to a different problem, called a *bikernel*, or to modify the graph to "simulate" the annotation in the original problem, but these modifications take the instance out of the original graph class. Here we settle for the following compromise: a parametrised graph problem $\mathcal{P} \subseteq \mathcal{G} \times \mathbb{N}$ for a BE-class $\mathcal{G}$ admits a *BE kernel* if there is a kernelization that outputs an instance in $\mathcal{G}' \times \mathbb{N}$ with $\nabla_r(\mathcal{G}') \leq g(\nabla_r(\mathcal{G}))$ for some function g and all $r \in \mathbb{N}$. This is justified by the idea that all nice algorithmic properties stemming from $\mathcal{G}$ being BE carry over from $\mathcal{G}$ to $\mathcal{G}'$ with only changes to some constants – if other properties of the class are of primary interest (embedding in a surface, excluded minors, *etc.*) then the BE-view is simply too coarse.

Our results. Inspired by the kernelization for r-Dominating Set [6] and r-Independent Set [17] in sparse classes, we unify and extend these techniques by defining a structure we call *water lilies* and show how their existence can be used to find small *cores*, that is, subset of vertices that either are guaranteed to contain a solution (*solution core*) or that already fully represent the neighbourhood-constraints governing the problem (*constraint core*). We define and prove the existence of water lilies in BE-classes in Section 4, building on our proof of a constant-factor approximation for (r, c)-Dominating Set in BE-classes from Section 3.

In Section 5 we use water lilies to prove linear bikernels for all the above listed problems into appropriate annotated variants and how most of these bikernels can be turned into BE-kernels. Finally we demonstrate how these constructions can be combined to create "multikernels", meaning graphs that represent kernels for multiple problems at once.

As mentioned above, we only present a selection of kernels obtainable by our method and we also omit some proofs (marked with $\star$). Please see the full version of this paper[1] for more kernels and further details.

2 Notation and previous results

For a maximization problem $\mathcal{P}$ defined via universal neighbourhood constraints and a graph G we call a set $L \subseteq V(G)$ a *constraint core* if for every set $D \subseteq V(G)$ it holds that D is a solution to $\mathcal{P}$ in G already if the constraints only hold for vertices in L. Analogous, for a minimization problem $\mathcal{P}$ defined via universal neighbourhood constraints, we call a set $U \subseteq V(G)$ a *solution core* if a minimum solution to P already exists inside U. In both cases, note that $V(G)$ is always a trivial core and that a superset of any core is a core as well.

A set $D \subseteq V(G)$ is an *(r, c)-dominating set* if for every vertex $v \in V(G)$ it holds that $|N^r[v] \cap D| \geq c$. Importantly, this constraint must also hold for vertices contained in D, therefore such a set can only exist if $|N^r[v]| \geq c$ for all $v \in G$. We write $\mathbf{dom}_r^c(G)$ to denote the size of a minimum (r, c)-dominating set in G and let $\mathbf{dom}_r^c(G) = \infty$ if no such set exists. A set $I \subseteq V(G)$ is *$2r$-independent* if every pair of vertices $u, v \in I$ has distance at least $2r + 1$. We write $\mathbf{ind}_{2r}(G)$ to denote the size of a maximum $2r$-independent set in G. Related, a set $I \subseteq V(G)$ is an *(r, c)-scattered set* if for all vertices $v \in G$ it holds that $|N^r[v] \cap I| \leq c$. An $(r, 1)$-scattered set is equivalent to a $2r$-independent set, but this relationship breaks down for $c > 1$. We defined $\mathbf{sct}_r^c(G)$ as the size of a maximum (r, c)-scattered set in G. In all cases, for $c = 1$ we will omit the superscript.

Important BE properties

We adapted the following results to use the notation introduced above for the sake of a unified presentation. In particular, we will be using $\mathbf{sct}_r$ instead of $\mathbf{ind}_{2r}$. The function wcol_r is one of the expansion characteristics mentioned above (see *e.g.* [19] for a definition), here it is enough to know that for every member G of a BE-class, $\mathrm{wcol}_r(G)$ is bounded by a constant for every $r \in \mathbb{N}$.

[1] Available at `https://arxiv.org/abs/2002.09028`.

▶ **Theorem 1** (Dvořák [7]). *For every graph G and integer $r \in \mathbb{N}$ it holds that* $\mathbf{sct}_r(G) \leq \mathbf{dom}_r(G) \leq \mathrm{wcol}_{2r}^2(G)\,\mathbf{sct}_r(G)$.

Dvořák recently showed an improved bound [8], we will use the above simpler expression. In the same work he also proved the following relationship between r-scattered sets and (r,c)-scattered sets (translated into our terminology):

▶ **Theorem 2** (Dvořák [8]). *For every graph G and integers $c, r \in \mathbb{N}$ it holds that* $\frac{1}{2c\,\mathrm{wcol}_{2r}(G)}\,\mathbf{sct}_r^c(G) \leq \mathbf{sct}_r(G) \leq \mathbf{sct}_r^c(G)$.

▶ **Theorem 3** (Dvořák's algorithm [7]). *For every BE class $\mathcal{G}$ and $r \in \mathbb{N}$ there exists a constant c_r^{dvrk} and a polynomial-time algorithm that computes an r-dominating set D of G and an r-scattered set $A \subseteq D$ with $|D| \leq \mathsf{c}_r^{dvrk}|A|$.*

In particular, the r-scattered set A witnesses that D is indeed a $\mathsf{c}_r^{\mathrm{dvrk}}$-approximation of a minimum r-dominating of G. This algorithm can further be modified to compute a dominating set for a specific set $X \subseteq V(G)$ only; in that case it outputs the sets A and D, $A \subseteq D \cap X$, where D dominates all of X in G and A is r-scattered in G. We will call this algorithm the *warm-start* variant since we only need to mark the vertices $V(G) \setminus X$ as already dominated and then run the original algorithm (an alternative is a small gadget construction [6]).

Given a vertex set $X \subseteq V(G)$ we call a path X*-avoiding* if its internal vertices are not contained in X. A *shortest X-avoiding path* between vertices x, y is shortest among all X-avoiding paths between x and y.

▶ **Definition 4** (r-projection). *For a vertex set $X \subseteq V(G)$ and a vertex $u \notin X$ we define the* r-projection *of u onto X as the set*

$$P_X^r(u) := \{v \in X \mid \textit{there exists an } X\textit{-avoiding } u\textit{-}v\textit{-path of length} \leq r\}$$

▶ **Definition 5** (r-shadow). *For a vertex set $X \subseteq V(G)$ and a vertex $u \notin X$ we define the* r-shadow *of u onto X as the set*

$$S_X^r(u) := \{v \in V(G) \mid \textit{every } u\textit{-}v\textit{-path of length} \leq r \textit{ has an internal vertex in } X\}$$

The shadow $S_X^r(u)$ contains precisely those vertices that are "cut off" by the set $P_X^r(u)$. We will frequently need the union of shadow and projection and therefore introduce the shorthand $SP_X^r(u) := S_X^r(u) \cup P_X^r(u)$.

Two vertices that have the same r-projection onto X do not, however, necessarily have the same shadow since the precise distance at which the projection lies might differ. To distinguish such cases, it is useful to consider the *projection profile* of a vertex to its projection:

▶ **Definition 6** (r-projection profile). *For a vertex set $X \subseteq V(G)$ and a vertex $u \notin X$ we define the* r-projection profile *of u wrt X as a function $\pi_{G,X}^r[u]\colon X \to [r] \cup \infty$ where $\pi_{G,X}^r[u](v)$ for $v \in X$ is the length of a shortest X-avoiding path from u to v if such a path of length at most r exists and ∞ otherwise.*

We say that a function $\nu\colon X \to [r] \cup \infty$ is *realized on X* (as a projection profile) if there exists a vertex $u \notin X$ for which $\nu = \pi_{G,X}^r[u]$ and we denote the set of all realized profiles by $\Pi_G^r(X)$. We will usually drop the subscript G if the graph is clear from the context. It will be convenient to define an equivalence relation that groups vertices outside of X by their projection profile. Define $u \sim_X^r v \iff \pi_X^r[u] = \pi_X^r[v]$ for pairs $u, v \in V(G) \setminus X$.

It turns out that in BE classes, the number of possible projection profiles realised on a set X is bounded linearly in the size of X.

▶ **Lemma 7** (Adapted from [6, 14]). *For every BE class $\mathcal{G}$ and $r \in \mathbb{N}$ there exists a constant c_r^{proj} such that for every $G \in \mathcal{G}$ and $X \subseteq V(G)$, the number of r-projection profiles realised on X is at most $\mathsf{c}_r^{proj}|X|$.*

In our notation this can alternatively be written as $|\Pi^r(X)| = |(V(G) \setminus X)/\sim_X^r| \leq \mathsf{c}_r^{\mathrm{proj}}|X|$. We will crucially rely on the following two results for BE classes:

▶ **Lemma 8** (Projection closure [6]). *For every BE class $\mathcal{G}$ and $r \in \mathbb{N}$ there exists a constant c_r^{projcl} and a polynomial-time algorithm that, given $G \in \mathcal{G}$ and $X \subseteq V(G)$, computes a superset $X' \supseteq X$, $|X'| \leq \mathsf{c}_r^{projcl}|X|$, such that $|P_{X'}^r(u)| \leq \mathsf{c}_r^{projcl}$ for all $u \in V(G) \setminus X'$.*

▶ **Lemma 9** (Shortest path closure [6]). *For every BE class $\mathcal{G}$ and $r \in \mathbb{N}$ there exists a constant c_r^{pathcl} and a polynomial-time algorithm that, given $G \in \mathcal{G}$ and $X \subseteq V(G)$, computes a superset $X' \supseteq X$, $|X'| \leq \mathsf{c}_r^{pathcl}|X|$, such that for all $u, v \in X$ with $dist(u, v) \leq r$ it holds that $dist_{G[X']}(u, v) = dist(u, v)$.*

It will be useful to combine the above two lemmas in the following way:

▶ **Definition 10** (Projection kernel). *Given a graph G and a set $X \subseteq V(G)$, an (r, c)-*projection kernel *of (G, X) is an induced subgraph $\hat{G}$ of G with $X \subseteq V(\hat{G})$ and the following properties:*
1. *$N_{\hat{G}}^d(v) \cap X = N_G^d(v) \cap X$ for all $v \in X$ and $d \leq r$; and*
2. *if the signature $\nu\colon X \to [r] \cup \infty$ is realized on X by p distinct vertices in G, then ν is realized by at least $\min\{c, p\}$ distinct vertices in $\hat{G}$.*

▶ **Lemma 11.** *For every BE class $\mathcal{G}$ and $c, r \in \mathbb{N}$ there exists a constant $\mathsf{c}_{r,c}^{total}$ and a polynomial-time algorithm that, given $G \in \mathcal{G}$ and $X \subseteq V(G)$, computes an (r, c)-projection kernel $\hat{G}$ of (G, X) with $|\hat{G}| \leq \mathsf{c}_{r,c}^{total}|X|$.*

Proof. We first apply Lemma 8 to X and obtain a set $X_1 \supset X$, $|X_1| \leq \mathsf{c}_r^{\mathrm{projcl}}|X|$, such that the projections of outside vertices onto X_1 have size at most $\mathsf{c}_r^{\mathrm{projcl}}$.

Next, we apply Lemma 9 to X_1 and receive a set $X_2 \supset X_1$, $|X_2| \leq \mathsf{c}_r^{\mathrm{pathcl}}|X_1|$, such that the graph $G[X_2]$ preserves short distances (less than or equal to r) between vertices in X_1. Finally, let U contain up to c representatives for every equivalence class $[u] \in V(G)/\sim_{X_1}^r$ (if the class is smaller than c we include all of it). By Lemma 7 we have that $|U| \leq c \cdot \mathsf{c}_r^{\mathrm{proj}}|X_1|$.

Construct now X_3 by taking the union $X_2 \cup U$ as well as shortest paths from every member $u \in X_2 \cup U$ to all of $P_{X_1}^r(u)$. By definition, each of these paths has length at most r and therefore contains at most $r - 1$ internal vertices. Since, by construction of X_1, $|P_{X_1}^r(u)| \leq \mathsf{c}_r^{\mathrm{proj}}$; it follows that we add at most $\mathsf{c}_r^{\mathrm{proj}}(r-1)$ vertices per vertex in $X_2 \cup U$. Taking the above bounds together, we have that $|X_3| \leq (r-1)\,\mathsf{c}_r^{\mathrm{proj}}(\mathsf{c}_r^{\mathrm{pathcl}} + c \cdot \mathsf{c}_r^{\mathrm{proj}})|X| =: \mathsf{c}_{r,c}^{\mathrm{total}}|X|$. It remains to be shown that $\hat{G} := G[X_3]$ has the desired properties.

Property 1 follows directly from the fact that already $G[X_2] \subseteq \hat{G}$ preserves short distances among vertices inside $X_1 \supseteq X$. In particular, each vertex in $X_1 \setminus X$ has the same r-projection profile onto X in G and $\hat{G}$.

To see that Property 2 holds, consider any profile ν realized on X by vertices $S \subseteq V(G) \setminus X$ in G. First consider the case $S \setminus X_1 \neq \emptyset$. Then by construction, the set U contains $\min\{c, |S \setminus X_1|\}$ vertices from $S \setminus X_1$ that realize ν in G and whose projection onto X_1 is the same in G and $\hat{G}$. Since $X_1 \supseteq X$, we conclude that their projection on X in $\hat{G}$ must be ν. By the above, the vertices in $S \cap X_1$ must have the profile ν as well. Now assume $S \subseteq X_1$, therefore no vertex outside of X_1 has the profile ν in G. As argued above, S has the profile ν in $\hat{G}$ as well, therefore $\hat{G}$ contains $|S| \geq \min\{c, |S|\}$ vertices with profile ν, as claimed. ◀

Note that the above construction implies that $\Pi_{\hat{G}}^r(X) \supseteq \Pi_G^r(X)$, however, it is not necessarily true that $\Pi_{\hat{G}}^r(X) = \Pi_G^r(X)$.

The following is a slight restatement of Theorem 4 in [13]. We emphasise that the proof by Kreutzer *et al.* is actually constructive and can be implemented to run in polynomial time.

▶ **Lemma 12** (UQW in BE classes [13]). *For every BE class $\mathcal{G}$ and* distance $d \in \mathbb{N}$ *there exists a constant c_d^{UQW} and a polynomial-time algorithm that, given $G \in \mathcal{G}$, a size $t \in \mathbb{N}$ and $X \subseteq V(G)$ with $|X| \geq \mathsf{c}_d^{UQW} \cdot 2^t$, computes a set S of size at most $(\mathsf{c}_d^{UQW})^2$ and $X' \subseteq X \setminus S$ of size at least t such that X' is d-scattered in $G - S$.*

3 Approximating (r, c)-Dominating Set

▶ **Theorem 13.** *Let $\mathcal{G}$ be a BE class and fix $r, c \in \mathbb{N}$. There exists a constant $\mathsf{c}_{r,c}^{cdom}$ and an algorithm that, for every $G \in \mathcal{G}$, computes in polynomial time an (r, c)-dominating set of size at most $\mathsf{c}_{r,c}^{cdom}\,\mathbf{dom}_r^c(G)$ or concludes correctly that G cannot be (r, c)-dominated.*

Proof. We compute a sequence of dominating sets $D_1, D_2, \ldots, D_c$ with the invariants that a) D_i (r, i)-dominates G and b) $|D_{i+1}| \leq 5\mathsf{c}_r^{\mathrm{dvrk}}\mathsf{c}_r^{\mathrm{proj}}|D_i| + \mathsf{c}_r^{\mathrm{dvrk}}\,\mathbf{dom}_r^{i+1}(G)$.

To start the process, let D_1 be an $\mathsf{c}_r^{\mathrm{dvrk}}$-approximate r-dominating set for G, this set clearly satisfies invariant a). We proceed in two steps to construct D_{i+1} from D_i. Build the set U_i as follows: for every projection $\mu \in \Pi^r(D_i)$ realized by an equivalence class $[v] \in (V(G)\setminus D_i)/{\sim_{D_i}^r}$ we pick one (arbitrary) vertex from $S_{D_i}^r(v) \setminus D_i$ and add it to U_i, if such a vertex exists. Then for every vertex $u \in D_i$ that is not $(i+1)$-dominated by $D_i \cup U_i$, we add an arbitrary vertex from $N^r[u] \setminus D_i$ to U_i (note that if no such vertex exists we conclude that G cannot be (r, c)-dominated).

By construction, the size of U_i is bounded by $|U_i| \leq |\Pi^r(D_i)| + |D_i| \leq (\mathsf{c}_r^{\mathrm{proj}} + 1)|D_i|$. Further note that every vertex in $D_i \cup U_i$ is $(r, i+1)$-dominated by $D_i \cup U_i$: due to invariant a), the set D_i (r, i)-dominates $D_i \cup U_i$ and U_i now additionally dominates itself (at least) once and, by construction, those vertices in D_i that are not yet $(r, i+1)$-dominated by D_i.

Define the set R_i to contain all vertices that are *not* $(r, i+1)$-dominated by $D_i \cup U_i$, note that in particular $N^r[R_i] \cap U_i = \emptyset$. Let $G' = G - (D_i \cup U_i)$. Apply Dvořák's warm-start algorithm to find a distance-r dominator D_i' for R_i in G' and a r-scattered set $A_i' \subseteq D_i' \cap R_i$ with $|A_i'| \leq |D_i'| \leq \mathsf{c}_r^{\mathrm{dvrk}}|A_i'|$.

▷ Claim. $|A_i'| \leq (\mathsf{c}_r^{\mathrm{proj}} + 1)|D_i| + \mathbf{dom}_r^{i+1}(G)$.

Proof. Let X be an $(r, i+1)$-dominating set of G of minimum size and assume that $|A_i'| > (\mathsf{c}_r^{\mathrm{proj}}+1)|D_i|+\mathbf{dom}_r^{i+1}(G) \geq |U_i \cup X|$. Then there exists $a \in A_i'$ such that $N_{G'}^r[a] \cap (U_i \cup X) = \emptyset$. Since X $(r, i+1)$-dominates a but $D_i \cup U_i$ does not (because $a \in R_i$) there must be at least one vertex $b \in X \cap (N_G^r[a] \setminus N_{G'}^r[a])$ that is not contained in $D_i \cup U_i$. This means that $b \in S_{D_i \cup U_i}^r(a)$ and since $N_G^r[a] \cap U_i = \emptyset$, we have that $S_{D_i \cup U_i}^r(a) = S_{D_i}^r(a)$ and therefore even $b \in S_{D_i}^r(a)$. But then, since $b \notin D_i \cup U_i$, we could have added b to U_i during the first construction phase in order to dominate the class $[a]$. The existence of a leads us to a contradiction and we conclude that $|A_i'| \leq (\mathsf{c}_r^{\mathrm{proj}} + 1)|D_i| + \mathbf{dom}_r^{i+1}(G)$. ◁

Finally, construct the set $D_{i+1} = D_i' \cup D_i \cup U_i$. Since D_i' r-dominates R_i which, by construction, were the only vertices not yet $(r, i+1)$-dominated by $D_i \cup U_i$, we conclude that D_{i+1} is indeed an $(r, i+1)$-dominating set of G; thus invariant a) is preserved. To see that invariant b) holds, let us bound the size of D_{i+1}:

$$
\begin{aligned}
|D_{i+1}| &\leq |D_i'| + |D_i| + |U_i| \leq \mathsf{c}_r^{\mathrm{dvrk}}|A_i'| + |D_i| + (\mathsf{c}_r^{\mathrm{proj}} + 1)|D_i| \\
&\leq \mathsf{c}_r^{\mathrm{dvrk}}(\mathsf{c}_r^{\mathrm{proj}} + 1)|D_i| + \mathsf{c}_r^{\mathrm{dvrk}}\,\mathbf{dom}_r^{i+1}(G) + (\mathsf{c}_r^{\mathrm{proj}} + 2)|D_i| \\
&\leq 5\mathsf{c}_r^{\mathrm{dvrk}}\mathsf{c}_r^{\mathrm{proj}}|D_i| + \mathsf{c}_r^{\mathrm{dvrk}}\,\mathbf{dom}_r^{i+1}(G).
\end{aligned}
$$

Resolving the recurrence provided by this inequality, we finally obtain the bound $|D_c| \leq (5\mathsf{c}_r^{\mathrm{dvrk}}\mathsf{c}_r^{\mathrm{proj}})^{c+1}\,\mathbf{dom}_r^c(G)$, and the claim follows with $\mathsf{c}_{r,c}^{\mathrm{cdom}} := (5\mathsf{c}_r^{\mathrm{dvrk}}\mathsf{c}_r^{\mathrm{proj}})^{c+1}$. ◀

4 Water lilies

▶ **Definition 14** (Water lily). *A* water lily *of* radius r*, depth* $d \le r$ *and* adhesion c *in a graph* G *is a tuple* (R, C) *of disjoint vertex sets with the following properties:*

- C *is* r*-scattered in* $G - R$*,*
- $N^r_{G-R}[C]$ *is* (d, c)*-dominated by* R *in* G*.*

We call R *the* roots*,* C *the* centres*, and the sets* $\{N^r_{G-R}[x]\}_{x \in C}$ *the* pads *of the water lily. A water lily is* uniform *if all centres have the same* d*-projection onto* R*,* e.g. $\pi^d_R[x]$ *is the same function for all* $x \in C$*. The* ratio *of a water lily is any guaranteed lower bound on* $|C|/|R|$*.*

The following lemma lies at the heart of our unification of previous techniques [6, 14, 17]. It streamlines the construction of BE-kernels considerably, as we will see in the following section.

▶ **Lemma 15.** *For every BE class* $\mathcal{G}$ *and* $c, r, d \in \mathbb{N}$*,* $d \le r$*, there exist constants* $\mathsf{c}^{scale}_{c,r,d}$*,* $\mathsf{c}^{margin}_{c,r,d}$*,* $\mathsf{c}^{base}_{r,d}$ *with the following property: for every* $G \in \mathcal{G}$ *which has an* (r, c)*-dominating set,* $t \in \mathbb{N}$ *and* $A \subseteq V(G)$ *with* $|A| \ge \mathsf{c}^{scale}_{c,r,d} \cdot (\mathsf{c}^{base}_{r,d})^t \cdot \mathbf{dom}^c_d(G)$ *there exists a uniform water lily* (R, C)*,* $C \subseteq A$*, with depth* d*, radius* r*, adhesions* c *and with* $|R| \le \mathsf{c}^{margin}_{c,r,d}$*,* $|C| \ge t$*. Moreover, such a water lily can be computed in polynomial time.*

Proof. Given G, we use Theorem 13 to compute a (d, c)-dominating set D' of size at most $\mathsf{c}^{\mathrm{cdom}}_{r,c} \cdot \mathbf{dom}_d(G)$ in polynomial time or conclude that no such set exits. Afterwards, we compute the $(r + d)$-projection closure D of D', by Lemma 8 we have that $|D| \le \mathsf{c}^{\mathrm{projcl}}_{r+d}|D'|$ and thus $|D| \le \mathsf{c}^{\mathrm{projcl}}_{r+d}\mathsf{c}^{\mathrm{cdom}}_{r,c}\,\mathbf{dom}_d(G)$. Let $A'' := A \setminus D$, we will later choose $\mathsf{c}^{\mathrm{scale}}_{c,r,d}$ so that A'' is still large enough for the following arguments to go through.

Define the equivalence relation $\sim_D$ over A'' via $a \sim_D a' \iff \pi^{r+d}_D[a] = \pi^{r+d}_D[a']$. By Lemma 7, the number of classes in $A''/\sim_D$ is bounded by $\mathsf{c}^{\mathrm{proj}}_{r+d}|D|$; by an averaging argument we have at least one class $[a] \in A''/\sim_D$ of size $\big|[a]\big| \ge |A''|/(\mathsf{c}^{\mathrm{proj}}_{r+d}|D|) \ge (|A| - |D|)/(\mathsf{c}^{\mathrm{proj}}_{r+d}|D|)$.

Let R''' be $P^{r+d}_D(a)$, i.e. the $(r + d)$-projection of $[a]$'s members on D. By our earlier application of Lemma 8 we have that $|R'''| = |P^{r+d}_D(a)| \le \mathsf{c}^{\mathrm{projcl}}_{r+d}$. Again, we will choose $\mathsf{c}^{\mathrm{scale}}_{c,r,d}$ large enough to apply Lemma 12 with distance r and size $\mathsf{c}^{\mathrm{proj}}_d|R'''|t$ to the set $[a]$ and receive a subset $A' \subseteq [a]$ of size at least $\mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d}) \cdot t$ and a set $R'' \subseteq V(G) \setminus A'$, $|R''| \le \mathsf{c}^{\mathrm{UQW}}_r$, such that A' is r-scattered in $G - R''$. Let $R' := R'' \cup R'''$, by the above bounds on R'' and R''' it follows that $|R'| \le \mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d}$. By Lemma 7 and the fact that $|A'| \ge \mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d}) \cdot t \ge |\Pi^d(R')| \cdot t$ there exists a set $C \subseteq A'$ of size at least t such that all members of C have the same d-projection onto R'.

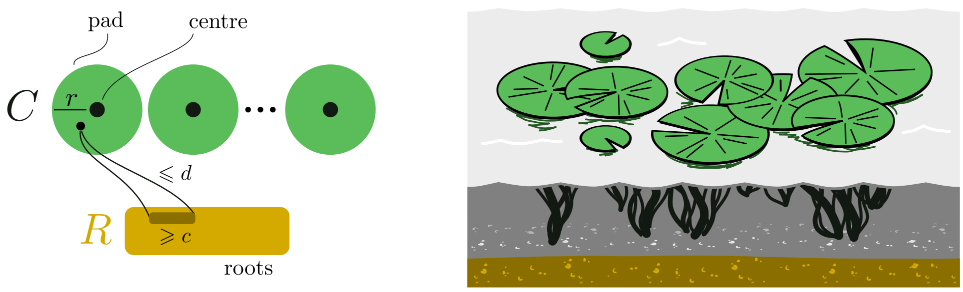

Figure 1 Schematic of a water lily (R, C) with radius r, depth d and adhesion c. Removing the "tangled" roots R creates disjoint r-neighbourhoods around C which we imagine like lily pads floating on a pond.

We construct the set R from R' as follows: for every projection profile $\mu \in \Pi^d(R')$ realized by a class $[u] \in N^r_{G-R'}[C]/{\sim^d_{R'}}$ we add $\max\{0, c - |P^d_{R'}(u)|\}$ vertices from the shadow $S^d_{R'}(u) \cap D'$. Since D' (d, c)-dominates all of G, such vertices must exist. By construction, $|R| \le c|R'|$ and R (c, d)-dominates all of $N^r_{G-R'}[C]$ and thus in particular $N^r_{G-R}[A']$. Note further that all vertices we added lie inside $S^{r+d}_{R'}[C]$, therefore the projection profiles of C are not changed by this operation (all paths of length at most $r + d$ from C to vertices in R/R' pass through R'). We conclude that the uniformity condition holds on (R, C). This construction also provides us with the bound $|R| \le c(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d}) =: \mathsf{c}^{\mathrm{margin}}_{c,r,d}$.

Finally, let us determine a value for $\mathsf{c}^{\mathrm{scale}}_{c,r,d}$ that suffices for the above construction to go through. In order to apply Lemma 12, we need that $|[a]| \ge \mathsf{c}^{\mathrm{UQW}}_r \cdot 2^{\mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d})\cdot t}$, accordingly we need that $(|A| - |D|)/(\mathsf{c}^{\mathrm{proj}}_{r+d}|D|) \ge \mathsf{c}^{\mathrm{UQW}}_r \cdot 2^{\mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d})\cdot t}$, which in particular holds if we ensure that $|A|/(2\mathsf{c}^{\mathrm{proj}}_{r+d}\mathsf{c}^{\mathrm{projcl}}_{r+d}\mathsf{c}^{\mathrm{cdom}}_{r,c}\,\mathbf{dom}^c_d(G)) \ge \mathsf{c}^{\mathrm{UQW}}_r \cdot 2^{\mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d})\cdot t}$. Hence setting $\mathsf{c}^{\mathrm{scale}}_{c,r,d} = 2\mathsf{c}^{\mathrm{proj}}_{r+d}\mathsf{c}^{\mathrm{projcl}}_{r+d}\mathsf{c}^{\mathrm{cdom}}_{r,c}\mathsf{c}^{\mathrm{UQW}}_r$ and $\mathsf{c}^{\mathrm{base}}_{r,d} = 2^{\mathsf{c}^{\mathrm{proj}}_d(\mathsf{c}^{\mathrm{UQW}}_r + \mathsf{c}^{\mathrm{projcl}}_{r+d})}$ suffices. ◀

We can impose even more structure on a water lily in the following sense: let us define a *pad signature* as a function $\sigma\colon C \to \Sigma^*$ (for some alphabet Σ) that can be computed by a polynomial-time algorithm receiving the following inputs:

- The depth d, radius r and adhesion c of the water lily;
- the centre a, its pad $N^r_{G-R}[a]$, the roots R;
- the subgraph $G[R \cup N^r_{G-R}[a]]$ alongside potential vertex/edge labels from the host graph G.

We say that σ is *bounded* if the size of its image can be bounded by a constant.

Every pad signature σ gives rise to an equivalence relation $\sim_\sigma \subseteq C \times C$ for a water lily (R, C) via $a \sim_\sigma a' \iff \sigma(a) = \sigma(a')$. Note that if σ is bounded, then $\sim_\sigma$ has finite index. A water lily is *σ-uniform* if all its centres belong to the same equivalence class under $\sim_\sigma$; or alternatively if all centres have the same image under σ. For a bounded signature σ, we find a $\sim_\sigma$-uniform water lily of ratio τ by first finding a water lily (R', C') with ratio $p \cdot \tau$, where p is an upper bound on the image of σ, and then return R' together with the largest class in $C'/\sim_\sigma$. Accordingly:

▶ **Corollary 16.** *For every BE class $\mathcal{G}$, $c, r, \tau \in \mathbb{N}$ and pad signature σ with finite index there exists a constant $\mathsf{c}^{lily} = \mathsf{c}^{lily}_{c,2r,r,\tau,\sigma}$ with the following property: for every $G \in \mathcal{G}$ which has an (r, c)-dominating set and $A \subseteq V(G)$ with $|A| \ge \mathsf{c}^{lily} \cdot \mathbf{dom}^c_d(G)$ there exists a σ-uniform water lily (R, C), $C \subseteq A$, $|R| \le \mathsf{c}^{lily}$, of depth r, radius $2r$, adhesion c and ratio τ. Moreover, such a water lily can be computed in polynomial time.*

Let us define a particular bounded pad signature that will be useful in the remainder: let $\nu(a) := \big(\{\pi^d_R[x] \mid x \in N^i_{G-R}(a)\} \mid 0 \le i \le r\big)$, where the right-hand side is to be understood as encoded in a string by some suitable scheme. Two centres are equivalent under $\sim_\nu$ if they have the same projection-types at the same distance (though potentially at different multiplicities) inside their respective pads. Since $|R|$ has constant size according to Lemma 15 and there are at most $\mathsf{c}^{\mathrm{proj}}_d|R|$ possible projection profiles according to Lemma 7, the image of ν has size at most $r^{\mathsf{c}^{\mathrm{proj}}_d|R|} \le r^{\mathsf{c}^{\mathrm{proj}}_d\mathsf{c}^{\mathrm{lily}}}$ and therefore ν is a bounded pad signature.

We will sometimes combine ν with a finite number of vertex labels that arise during the construction of bikernels. If vertices are labelled by $f\colon V(G) \to \Sigma$ for some finite alphabet Σ, then we understand ν to be the above equivalence relation further refined by the equivalence relation $u \sim_f v \iff f(u) = f(v)$.

5 Bikernels into annotated problems

We show in the following that a range of problems over hereditary BE-classes admit linear bikernels in the same class (see the full version for r-Roman Domination and Total r-Domination). The target problem in all three cases is a suitable annotated version of the original problem, which we define just ahead of each proof.

Annotated (r, c)-Domination parametrised by k
Input: A graph G, a set $L \subseteq V(G)$ and an integer k.
Problem: Is there a set $D \subseteq V(G)$ of size at most k such that $|N^r[v] \cap D| \geq c$ for all $v \in L$?

▶ **Theorem 17.** *(r, c)-Dominating Set over a hereditary BE-class $\mathcal{G}$ admits a linear bikernel into Annotated (r, c)-Dominating Set over the same class $\mathcal{G}$. Moreover, the resulting graph is an (r, c)-projection kernel of the original graph.*

Proof. Let (G, k) be an input where G is taken from a BE class. As a first step, we deal with the case $\mathbf{dom}_r^\mu(G)$ large by computing an (r, μ)-dominating set using the algorithm from Theorem 13. If it returns a solution larger than $\mathsf{c}_{r,c}^{\mathrm{cdom}} k$, we conclude that $\mathbf{dom}_r^\mu(G) > k$ in which case we return a trivial no-instance. Otherwise, we show that (r, c)-Dominating Set admits a linear constraint core and then show how to construct a BE-kernel from that core.

▷ Claim. (r, c)-Dominating Set has a linear constraint core in BE classes.

Proof. Let $L \subseteq V(G)$ be a constraint core of G with $|L| \geq \mathsf{c}_{c,2r,r,2}^{\mathrm{lily}} \mathbf{dom}_r^c(G)$. By Corollary 16, we can find in polynomial time a uniform water lily (R, C), $C \subseteq L$, $|R| \leq \mathsf{c}^{\mathrm{lily}}$ of depth r, radius $2r$, adhesion c and ratio 2. Let $a \in C$ be an arbitrary centre, we claim that $L \setminus \{a\}$ is still a constraint core, that is, every set that (r, c)-dominates $L \setminus \{a\}$ will also (r, c)-dominate a.

To that end, let D be a minimum (r, c)-dominating set and define $D' := D \setminus N_{G-R}^r[C]$. If D' (r, c)-dominates any part of C, it dominates all of C (and therefore a) as (R, C) is uniform. Thus assume that D' does not (r, c)-dominate C. Consider the case where a set $S \subseteq D \cap N_{G-R}^r[C]$ exists such that every vertex in S dominates more than one vertex in C. If $|S| \geq c$ then S alone already (r, c)-dominates all of C and thus in particular a. In all remaining cases, every set $N_{G-R}^r[a']$, $a' \in C$ must contain at least one vertex from D and we conclude that $|D \setminus D'| \geq |C| \geq 2|R|$. Let $\tilde{D} := D' \cup R$, we claim that $\tilde{D}$ is an (r, c)-dominating set of G. Simply note that the only vertices that are not (r, c)-dominated by D' lie inside $N_{G-R}^{2r}[C]$ – but this is precisely the set that is (r, c)-dominated by R. We arrive at a contradiction since $|D| = |D \setminus D'| + |D'| \geq 2|R| + |D'| > |R| + |D'| \geq |\tilde{D}|$ and we assumed D to be minimum. Thus $L \setminus \{a\}$ is a constraint core for (r, c)-Dominating Set in G. We iterate this procedure until $|L| < \mathsf{c}_{c,2r,r,2}^{\mathrm{lily}} \mathbf{dom}_r^c(G)$ and end up with a linear constraint core. ◁

In the following, let $L \subseteq V(G)$ be a constraint core for (G, k) with $|L| \leq \mathsf{c}^{\mathrm{lily}} \mathbf{dom}_r^c(G)$ and let $O = V(G) \setminus L$. If $|L| > \mathsf{c}^{\mathrm{lily}} k$, we can conclude that $k > \mathbf{dom}_r^c(G)$ and output a trivial no-instance, thus assume from now on that $|L| \leq \mathsf{c}^{\mathrm{lily}} k$. We apply Lemma 11 with $X = L$ and r, c as here to obtain a projection kernel $\hat{G}$ with $|\hat{G}| \leq \mathsf{c}_{r,c}^{\mathrm{total}} |L| = O(k)$ which a) preserves $\leq r$-neighbourhoods in L and b) realizes every r-projection onto L that is realized p times in G at least $\min\{c, p\}$ times. We claim that (G, k) is equivalent to the annotated instance $(\hat{G}, L, k)$.

Assume that D is an (r, c)-dominating set of G, clearly it is also a solution to the annotated instance (G, L, k). Partition D into $D_L = D \cap L$ and $D_O = D \setminus L$. Consider $x \in D_O$ and note that $|[x] \cap D_O| < c$ for the r-neighbourhood class $[x] \in O/\sim_L^r$ since otherwise we could remove a vertex from $[x] \cap D_O$ from D and still (r, c)-dominate all of L. With this observation, construct the set $\hat{D}_O$ as follows: for every vertex $x \in D_O$ we include $|[x] \cap D_O|$

vertices from $O \cap V(\hat{G})$ in $\hat{D}_O$, by property b) of the projection kernel $\hat{G}$ we know that at least c such vertices are available. Then the set $\hat{D} := D_L \cup \hat{D}_O$ (r, c)-dominates all of L in $\hat{G}$, by property a) of $\hat{G}$, and we are done. In the other direction, let $\hat{D}$ be an (r, c)-dominator of L in $\hat{G}$. By property a) and b) of $\hat{G}$ the set $\hat{D}$ therefore also (r, c)-dominates L in G, and since L is a constraint core of G it then (r, c)-dominates all of G. We conclude that $(\hat{G}, L, k)$ is equivalent to (G, k) and $|\hat{G}| = O(k)$. ◀

Annotated (r, c)-Scattered Set parametrised by k
Input: A graph G, a set $U \subseteq V(G)$ and an integer k.
Problem: Is there a set $I \subseteq U$ of size at least k such that $|N^r[v] \cap I| \leq c$ for all $v \in V(G)$?

The following proof makes use of the pad equivalence $\sim_\nu$ defined in Section 4: recall two centres u, v of a water lily (R, C) satisfy $u \sim_\nu v$ if they have the same projection-types onto R at the same distance (for distances smaller than the lily's depth) inside their respective pads.

▶ **Theorem 18.** *(r, c)-Scattered Set over a hereditary BE-class $\mathcal{G}$ admits a linear bikernel into Annotated (r, c)-Scattered Set over the same class $\mathcal{G}$. Moreover, the resulting graph is an (r, c)-projection kernel of the original graph.*

Proof. Let (G, k) be an instance of (r, c)-Scattered Set where G is taken from a BE class. As a first step, we deal with the case that $\mathbf{sct}_r^c(G)$ is large. We compute an $\mathsf{c}_r^{\mathrm{dvrk}}$-approximate r-dominating set D using Theorem 3. If $|D| > \mathsf{c}_r^{\mathrm{dvrk}} \operatorname{wcol}_{2r}^2(G) \cdot k$, we conclude by Theorems 1 and 2 that $\mathbf{sct}_r^c(G) \geq \mathbf{sct}_r(G) > k$ and we output a trivial yes-instance. Otherwise, assume $|D| \leq \mathsf{c}_r^{\mathrm{dvrk}} \operatorname{wcol}_{2r}^2(G) \cdot k$ and define $\mathsf{c}^{\mathrm{lily}} := \mathsf{c}_{1,2r,r,2,\nu}^{\mathrm{lily}}$. We first show that (r, c)-Scattered Set admits a linear solution core.

▷ Claim. (r, c)-Scattered Set has a linear solution core in BE classes.

Proof. Let $U \subseteq V(G)$ be solution core of G with $|U| \geq \mathsf{c}^{\mathrm{lily}} \mathbf{dom}_r(G)$. Using Corollary 16, we find in polynomial time a ν-uniform water lily (R, C), $C \subseteq U$, $|R| \leq \mathsf{c}^{\mathrm{lily}}$ of depth r, radius $2r$, adhesion 1 and ratio 2. Let $a \in C$ be an arbitrary centre, we claim that $U \setminus \{a\}$ is still a solution core, i.e. there exists an optimal (r, c)-scattered set that does not contain a.

To that end, let I be a minimum (r, c)-scattered set and assume $a \in I$. We claim that there exists an (r, c)-scattered set I' of the same size which excludes a. First observe that every vertex that lives in a pad $N^{2r}[a']$, $a' \in C$, has at least c neighbours in R at distance $\leq r$. Therefore $|N_{G-R}^{2r}[C] \cap I| \leq |R|$ as otherwise we would find a vertex in R whose r-neighbourhood contains more than c vertices of I. Since $|C| \geq 2|R|$ there are at least $|R|$ centres $C' \subseteq C$ such that their pads $N_{G-R}^{2r}[C']$ do not intersect I. Since (R, C) is uniform and $a \in I$, we know that $|N^r[a'] \cap I| = |N^r[a] \cap I| < c$ for every centre $a \in C$.

Take $a' \in C'$ and let $I' := I \setminus \{a\} \cup \{a'\}$. To see that I' is (r, c)-scattered, consider any vertex $u' \in N^r[a']$ (note that vertices at distance $> r$ from a' are not affected by the exchange of a by a'). By ν-uniformity, there exists a vertex $u \in N^r[a]$ with $\pi_R^r[u] = \pi_R^r[u']$. In particular, $P_R^r(u) \cup S_R^r(u) = P_R^r(u') \cup S_R^r(u')$; therefore $(N^r[u] \cap I) \setminus \{a\} = (N^r[u'] \cap I') \setminus \{a'\}$ and we conclude that $|N^r[a'] \cap I'| \leq c$. It follows that $U \setminus \{a\}$ is a solution core. We iterate the above procedure until $|U| \leq \mathsf{c}^{\mathrm{lily}} \mathbf{dom}_r^c(G)$ and end up with a linear solution core. ◁

In the following, let $U \subseteq V(G)$ be a solution core for (G, k) with $|U| \leq \mathsf{c}^{\mathrm{lily}} \mathbf{dom}_r(G) \leq \mathsf{c}^{\mathrm{lily}} |D| = O(k)$. We apply Lemma 11 with $X = U$ and r, c as here to obtain a projection kernel $\hat{G}$ with $|\hat{G}| \leq \mathsf{c}_{r,c}^{\mathrm{total}} |U| = O(k)$ that a) preserves $\leq r$-neighbourhoods in U and b) realizes every r-projection onto U that is realized p times in G at least $\min\{c, p\}$ times. Since distances in $\hat{G}[U]$ are as in $G[U]$, it is easy to see that any set $I \subseteq U$ is (r, c)-scattered in $\hat{G}$ iff it is (r, c)-scattered in G. Since U is further a solution core for G, we conclude that (G, k) is equivalent to the annotated instance $(\hat{G}, U, k)$. ◀

Annotated $(r, [\lambda, \mu])$-Domination parametrised by k

Input: A graph G, sets $L, U \subseteq V(G)$ and an integer k.

Problem: Is there a set $D \subseteq U$ of size at most k such that $|N^r[v] \cap D| \geq \lambda$ for all $v \in L$ and $|N^r[v] \cap D| \leq \mu$ for all $v \in V(G)$?

▶ **Theorem 19.** *$(r, [\lambda, \mu])$-Domination over a hereditary BE-class $\mathcal{G}$ admits a linear bikernel into Annotated $(r, [\lambda, \mu])$-Domination over the same class $\mathcal{G}$. Moreover, the resulting graph is an (r, c)-projection kernel of the original graph.*

Proof. Since the cases where either $\mu = \infty$ or $\lambda = 0$ are equivalent to (r, c)-Dominating Set or (r, c)-Scattered Set and thus covered by Theorems 17 and 18, we here only consider the case of $\lambda \neq 0$ and $\mu \neq \infty$. Note that any solution to the problem is in particular an (r, μ)-dominating set. As a first step, we therefore deal with the case that $\mathbf{dom}^{\mu}_r(G)$ is too large by computing an (r, μ)-dominating set using the algorithm described in Theorem 13. If the algorithm returns a solution larger than $\mathsf{c}^{\mathrm{cdom}}_{r,c} k$, we conclude that $\mathbf{dom}^{\mu}_r(G) > k$ and therefore that (G, k) must be a no-instance; in which case we output a trivial no-instance. Otherwise, let $\hat{D}$ be the resulting (r, c)-dominating set.

Let (G, L, U, k) be an instance of Annotated $(r, [\lambda, \mu])$-Domination with $L = U = V(G)$. Clearly, (G, L, U, k) is equivalent to (G, k). In the following, we gradually reduce the size of L and U while maintaining this equivalence. To that end, we will use the pad signature ν which is to be understood to take the "vertex labels" L, U into account.

Assume that $|L| > (\mathsf{c}^{\mathrm{lily}} + 1)|\hat{D}|$ with $\mathsf{c}^{\mathrm{lily}} := \mathsf{c}^{\mathrm{lily}}_{r,2r,\mu+1,\nu}$. Then, using $\hat{D}$ in the construction used in the proof of Lemma 15, we find a ν-uniform water lily (R, C) with $C \subseteq L \setminus \hat{D}$ of depth r, radius $2r$ and ratio $(\mu + 1)$.

▷ Claim. Let $a \in C$. Then the instances (G, L, U, k) and $(G, L \setminus \{a\}, U, k)$ are equivalent.

Proof. Any solution for (G, L, U, k) is also a solution to $(G, L \setminus \{a'\}, U, k)$, therefore we only have to show the opposite direction. Let D be a solution for $(G, L \setminus \{a\}, U, k)$. Since $R \subseteq L \cap U$, the set D can intersect at most $\mu|R|$ pads or otherwise we would violate an upper constraint for at least one of the vertices in R. It follows that at least $|R|$ pads of (R, C) cannot contain any vertex of D; let the centres of these pads be $C' \subseteq C$. Choose $a' \in C'$ distinct from a (since $|C'| \geq |R| \geq \lambda > 1$ such a vertex exists). Note that $a' \in L$, therefore $|N^r[a'] \cap D| \geq \lambda$. But since $N^r_{G-R}[a'] \cap D = \emptyset$, these solution vertices must lie in $SP^r_R(a')$. Now simply observe that, by uniformity of (R, C), $SP^r_R(a) = SP^r_R(a')$ and therefore $|N^r[a'] \cap D| \geq |SP^r_R(a) \cap D| \geq \lambda$. Accordingly, D is also a solution for (G, L, U, k). ◁

We repeat the above procedure until $|L \setminus \hat{D}| \leq \mathsf{c}^{\mathrm{lily}} k$. Now assume that $|U \setminus (L \cup \hat{D})| > \mathsf{c}^{\mathrm{lily}} k$ and let (R, C) be a ν-uniform water lily with $C \subseteq U \setminus (L \cup \hat{D})$ of depth r, radius $2r$ and ratio $(\mu + 1)|R|$.

▷ Claim. Let $a \in C$. Then the instances (G, L, U, k) and $(G, L, U \setminus \{a\}, k)$ are equivalent.

Proof. By construction of (R, C), every vertex $x \in N^{2r}_{G-R}[C]$ is (r, μ)-dominated by $R \cap \hat{D}$. Importantly, $R \cap \hat{D} \subseteq R \cap U$, therefore any solution D of (G, L, U, k) can intersect $N^r[R]$ in at most $\mu|R|$ vertices. In particular, at most $\mu|R|$ pads of (R, C) can contain vertices of D, let us call the centres of these empty pads $C' \subseteq C$.

If $a \notin D$, clearly D is a solution of $(G, L, U \setminus \{a\}, k)$ and there is nothing to prove. Assume therefore that $a \in D$. Let $a' \in C'$ be an arbitrary centre of an empty pad. We claim that $D' := D \setminus \{a\} \cup \{a'\}$ is a solution to $(G, L, U \setminus \{a\}, k)$. To that end, consider any vertex $x \in N^r[a] \cup N^r[a']$, we will show that D' fulfils any constraints associated with x.

▶ Case 1. $x \in N^r_{G-R}[a]$. By ν-uniformity, there exists a vertex $x' \in N^r_{G-R}[a']$ such that $SP^r_{G-R}(x) = SP^r_{G-R}(x')$ and x' is contained in L (U) iff x is contained in L (U). For the special case that $x = a$ we let $x' = a'$. Assume $x \in L$, then $x' \in L$ and accordingly $|N^r[x'] \cap D| \geq \lambda$. Since $N^{2r}_{G-R}[a'] \cap D = \emptyset$, we have that $N^r[x'] \cap D = SP^r_R(x') \cap D = SP^r_R(x') \cap D' = SP^r_R(x) \cap D'$, therefore $|N^r[x] \cap D'| = |N^r[x'] \cap D| \geq \lambda$ and the lower-bound constraint for x is satisfied by D'. If $x \in R$, simply note that $|N^r[x] \cap D'| \leq |N^r[x] \cap D| \leq \mu$, hence the upper-bound constraint for x is satisfied by D'.

▶ Case 2. $x \in N^r_{G-R}[a']$. Again, by ν-uniformity, there exists a vertex $\hat{x} \in N^r_{G-R}[a]$ such that $SP^r_{G-R}(x) = SP^r_{G-R}(\hat{x})$ and $\hat{x}$ is contained in L (U) iff x is contained in L (U). For the special case that $x = a'$ we let $\hat{x} = a$. If $x \in L$, simply note that $|N^r[x] \cap D'| \geq |N^r[x] \cap D| \geq \lambda$, hence the lower-bound constraint for x is satisfied by D'. Assume $x \in R$. Then $\hat{x} \in R$ and accordingly $|N^r[\hat{x}] \cap D| \leq \mu$. More specifically, since $a \in N^r[\hat{x}] \cap D$, we know that $|SP^r_R[\hat{x}] \cap D| \leq \mu - 1$. Because $N^r[x] \cap D' = (SP^r_R[x] \cap D') \cup \{a'\} = (SP^r_R[\hat{x}] \cap D) \cup \{a'\}$ we conclude that $|N^r[x] \cap D'| \leq \mu$ and the upper-bound constraint for x is satisfied by D'.

▶ Case 3. $x \in SP^r_R(a) = SP^r_R(C)$. Simply note that by uniformity $|N^r[x] \cap D| = |N^r[x] \cap D'|$ and therefore D' satisfies all constraints for x.

Therefore D' is indeed a solution for $(G, L, U \setminus \{a\}, k)$ of equal size and we conclude that the instances (G, L, U, k) and $(G, L, U \setminus \{a\}, k)$ are equivalent, as claimed. ◁

We repeat the above procedure until $|U \setminus (L \cup \hat{D})| \leq \mathsf{c}^{\text{lily}} k$ and end up with an instance (G, L, U, k) which is equivalent to our initial instance (G, k) and further satisfies $|L| \leq \mathsf{c}^{\text{lily}} k$ and $|U| \leq |L| + |\hat{D}| + |U \setminus (L \cup \hat{D})| \leq (2\mathsf{c}^{\text{lily}} + \mathsf{c}^{\text{cdom}}_{r,c})k$.

Finally, let us construct the bikernel from this annotated instance. Note that, by construction, $L \subseteq U$. Let $\hat{U}$ be the shortest-path closure of U in G as per Lemma 9, then $|\hat{U}| \leq \mathsf{c}^{\text{pathcl}}_r |U|$ and $\hat{G} := G[\hat{U}]$ preserves all distances up to length r between vertices in U. In particular, $N^r_{\hat{G}}[v] \cap U = N^r_G[v] \cap U$. Since the annotated instance asks for solutions contained entirely in U and $L \subseteq U$, we conclude that the instance (G, L, U, k) and $(\hat{G}, L, U, k)$ are equivalent, therefore the latter is also equivalent to (G, k) which finally proves the claim. ◀

If we sacrifice the constraint to construct a (bi)kernel that is contained in the same hereditary graph class, we are able to construct BE-kernels by reducing from the annotated problem back into the original problems. In the following constructions, we usually tried to minimize the increase in the parameter k, not the increase of the expansion characteristics of the class.

▶ **Theorem 20.** *(r, c)-Dominating Set admits a linear BE-kernel.*

Proof. For an instance (G, k) of (r, c)-Dominating Set, where G is taken from a BE class, we first construct a bikernel $(\hat{G}, L, k)$ of Annotated (r, c)-Dominating Set according to Theorem 17. Recall that $\hat{G}$ is an (r, c)-projection kernel of (G, L).

First consider $r \geq 2$. We construct G' from $\hat{G}$ by adding new vertices $a_1, \ldots, a_c, b_1, b_2, b_3$ to the graph. We connect every a_i, $1 \leq i \leq c$ to both b_1 and b_2; then connect b_1 to every vertex in $O := V(\hat{G}) \setminus L$ via a path of length $r - 1$ and connect b_2 to b_3 by such a path as well. From the construction it is clear that G' has size $O(k)$, we are left with proving that the two instances (G, k) and $(G', k + c)$ are equivalent.

Assume that D' is a minimum (r, c)-dominating set for G' of size $\leq k + c$. By a simple exchange argument, we can assume that D' contains all vertices a_i in order to (r, c)-dominate b_3. These vertices already (r, c)-dominate all of O and the paths leading from b_1 to O. As such, we can assume that an optimal solution D' does not contain internal vertices of those paths (otherwise we might as well exchange an internal vertex for the path's endpoint in O). Then the set $\hat{D} := D' \setminus \{a_1, \dots a_c\}$ has size at most k and (r,c)-dominates all of L; thus $\hat{D}$ in particular is a solution to $(\hat{G}, L, k)$.

In the other direction, assume that $\hat{D}$ is a minimum solution for $(\hat{G}, L, k)$, that is, $\hat{D}$ (r,c)-dominates L in $\hat{G}$. Let $D' := \hat{D} \cup \{a_1, \dots, a_c\}$, it is easy to see that D' (r,c)-dominates G' and has size $|D'| = |D| + c$. For $r = 1$ we modify the construction as follows: we add vertices $a_1, \dots, a_c, b$ and connect all a_i to $O \cup \{b\}$. The argument for why the resulting instance is equivalent is very similar to the case $r \geq 2$ and we omit it here.

We conclude that $(\hat{G}, L, k)$ and $(G', k + c)$ are indeed equivalent, and thus also to (G, k). It is only left to show that the construction of G' increased the expansion characteristics by some arbitrary function independent of $|G|$. Simply note that we can construct G' from G by adding $c + 3$ apex-vertices (which increases the expansion characteristics only by an additive constant) and then remove or subdivide edges incident to them (which does not increase the expansion characteristics). ◀

▶ **Theorem 21.** *(r,c)-Scattered Set admits a linear BE-kernel.*

Proof. Let (G, k) be an input of (r,c)-Scattered Set where G is taken from a BE class. We first construct the annotated bikernel $(\hat{G}, U, k)$ according to Theorem 18 and then construct G' from $\hat{G}$ by adding vertices $a_1, a_2, b_1, \dots, b_c$ and edges $a_2 b_i$ for all $1 \leq i \leq c$. We further connect a_1 to all vertices in $O := V(\hat{G}) \setminus U$ via paths of length r and to a_2 via a path of length $r - 1$ (for $r = 1$ we identify a_1 and a_2). It is is clear that G' has size $O(k)$, we are left to prove that the instances $(\hat{G}, U, k)$ and $(G', k + c)$ are equivalent.

First, consider a maximal (r,c)-scattered set I' in G'. Since $O \cup \{b_1, \dots, b_c\} \subset N^r[a_1]$ we may assume, by a simple exchange argument, that $\{b_1, \dots, b_c\} \subseteq I'$. Accordingly, $O \cap I' = \emptyset$ and $I := I' \setminus \{b_1, \dots, b_c\}$ is an (r,c)-scattered set contained entirely in U. Therefore I is (r,c)-scattered in $\hat{G}$ as well and $|I| = |I'| + c$.

In the other direction, assume that $\hat{I} \subseteq U$ is a maximal (r,c)-scattered set in $\hat{G}$. Then $N^r_{G'}[a_1] \cap I = \emptyset$ and we can add up to c vertices from $N^r[a_1]$ to I. Since the vertices b_i all lie at distance $2r$ from O, we conclude that $I' := I \cup \{b_1, \dots, b_c\}$ is indeed (r,c)-scattered in G' and $|I'| = |I| + c$. We conclude that the instances $(\hat{G}, U, k)$ and $(G', k+c)$ are equivalent and hence (G, k) and $(G', k + c)$ are as well. The argument why the expansion characteristics only increase by a constant are similar to the arguments in Theorem 20. ◀

▶ **Theorem 22 ($\star$).** *Total r-Domination, r-Roman Domination, and r-Perfect Code admit a linear BE-kernel.*

6 Multikernels

The following results are applicable to *e.g.* planar graphs or graph classes defined by an excluded minor of minimum degree two. In the following, let $\mathbf{dom}_r^{\text{total}}(G)$ denotes the total r-domination number and $\mathbf{dom}_r^{\text{roman}}(G)$ the r-Roman domination number of G. We will also write $\mathbf{dom}_r(G, L)$, $\mathbf{dom}_r^{\text{total}}(G, L)$, and $\mathbf{dom}_r^{\text{roman}}(G, L)$ for the annotate domination numbers (where only the set $L \subseteq V(G)$ has to be dominated).

▶ **Theorem 23 ($\star$).** *Let $\mathcal{G}$ be a hereditary graph class that is further closed under adding pendant vertices. Given a graph $G \in \mathcal{G}$ and an integer r we can compute in polynomial time a graph $G' \in \mathcal{G}$ and an integer c with the following properties:*

- $|G'| = O(\mathbf{dom}_r(G)) = O(\mathbf{dom}_r^{total}(G)) = O(\mathbf{dom}_r^{roman}(G))$,
- $\mathbf{dom}_r(G') = \mathbf{dom}_r(G) + c$, $\quad \mathbf{dom}_r^{total}(G') = \mathbf{dom}_r^{total}(G) + c$ *and*
- $\mathbf{dom}_r^{roman}(G') = \mathbf{dom}_r^{roman}(G) + 2c$.

Recall that an r-scattered set is equivalent to a $2r$-independent set and in particular that $\mathbf{sct}_r(G) = \mathbf{ind}_{2r}(G)$.

▶ **Theorem 24** (⋆). *Let $\mathcal{G}$ be a hereditary graph class that is further closed under adding pendant vertices. Given a graph $G \in \mathcal{G}$ and integers $\lambda \leq \mu$ we can compute in polynomial time a graph $G' \in \mathcal{G}$ and integers $c_\lambda, \ldots, c_\mu$ with the following properties:*
- $|G'| = O(\mathbf{dom}_\lambda(G))$,
- *for all $\lambda \leq r \leq \mu$ it holds that $\mathbf{dom}_r(G') = \mathbf{dom}_r(G) + c_r$ and $\mathbf{ind}_{2r}(G') = \mathbf{ind}_{2r}(G) + c_r$.*

7 Conclusion

We defined the notion of *water lilies* and showed that in BE-classes these structures can be used to compute linear-sized cores, bikernels, and BE-kernels. These constructions are almost universal, to the point were we can combine them into “multikernels”. It stands to reason that there might be a general formulation for these types of kernels. As a technical step, we also prove that (r, c)-Dominating Set admits a constant-factor approximation in BE-classes.

We are certain that our techniques directly translate to nowhere dense classes but leave this endeavour as future work. Given that the problems treated here all have constraints whose boundaries form intervals, we ask whether the following artificial problem admits a polynomial kernel in BE-classes: find a set D of size at most k such that $|N^r[v] \cap D| \notin \{0, 2\}$.

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