

Canonization for Bounded and Dihedral Color Classes in Choiceless Polynomial Time

Moritz Lichter

TU Kaiserslautern, Germany
lichter@cs.uni-kl.de

Pascal Schweitzer

TU Kaiserslautern, Germany
schweitzer@cs.uni-kl.de

Abstract

In the quest for a logic capturing PTIME the next natural classes of structures to consider are those with bounded color class size. We present a canonization procedure for graphs with dihedral color classes of bounded size in the logic of Choiceless Polynomial Time (CPT), which then captures PTIME on this class of structures. This is the first result of this form for non-abelian color classes.

The first step proposes a normal form which comprises a “rigid assemblage”. This roughly means that the local automorphism groups form 2-injective 3-factor subdirect products. Structures with color classes of bounded size can be reduced canonization preservingly to normal form in CPT.

In the second step, we show that for graphs in normal form with dihedral color classes of bounded size, the canonization problem can be solved in CPT. We also show the same statement for general ternary structures in normal form if the dihedral groups are defined over odd domains.

2012 ACM Subject Classification Theory of computation \rightarrow Finite Model Theory; Theory of computation \rightarrow Complexity theory and logic

Keywords and phrases Choiceless polynomial time, canonization, relational structures, bounded color class size, dihedral groups

Digital Object Identifier 10.4230/LIPIcs.CSL.2021.31

Related Version A full version [21] is available at <https://arxiv.org/abs/2010.12182>.

Funding Received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (EngageS: agreement No. 820148).

1 Introduction

One of the central open questions in the field of descriptive complexity theory asks about the existence of a logic that captures polynomial time (PTIME) [11]. This question goes back to Chandra and Harel [5]. They ask whether there is a logic within which we can define exactly the polynomial-time computable properties of relational structures. For the complexity class NP such a logic is known, namely existential second order logic. This was shown by Fagin in his famous theorem [7]. However, for the class PTIME , the question has been open now for more than 35 years. A fundamental difficulty at its heart is a mismatch between logics and Turing machines. An input has to be written onto a tape to provide it to a Turing machine. So all inputs are necessarily ordered by the position of each character on the tape. This is the case even when there is no natural order to begin with, which for example happens with the vertices of a graph that is encoded. In contrast to this, such an order is typically not given for a logic. In fact, if an order is given a priori then there is a logic capturing PTIME , for example on totally ordered structures. Indeed, IFP (first order logic enriched with a fixed-point operator) is such a logic as shown by the Immerman-Vardi Theorem [18].



© Moritz Lichter and Pascal Schweitzer;
licensed under Creative Commons License CC-BY

29th EACSL Annual Conference on Computer Science Logic (CSL 2021).

Editors: Christel Baier and Jean Goubault-Larrecq; Article No. 31; pp. 31:1–31:18



Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In the ongoing search for a logic for unordered structures, one of the most promising candidates is the logic Choiceless Polynomial Time (CPT). It manages to capture an important aspect demanded from a “reasonable logic” in the sense of Gurevich [17], namely that such a logic cannot make arbitrary choices. Whenever there are multiple indistinguishable elements, a logic can either process all or none of them. It is impossible to pick one element arbitrarily and process just that. This is common for many algorithms executed on Turing machines exploiting the order given by the tape. In the form originally defined by Blass, Gurevich, and Shelah [3], CPT has a pseudocode-like syntax for processing hereditarily finite sets. Most importantly there is a construct to process all elements of a set in parallel, because we cannot choose one to process first. Subsequently, there were definitions by Rossman [27] and Grädel and Grohe [9] in a more “logical” way using iteration terms or fixed points.

The question of whether a logic capturing PTIME on a class of structures exists is closely linked to the problem of canonization. Suppose it is possible to canonize input structures from a particular class in a logic (i.e., to define an isomorphic copy enriched by a total order). Then the logic (extended by IFP) captures PTIME on this class by the already mentioned Immerman-Vardi Theorem. This yields a general approach to show that some logic captures PTIME on a class of structures: proving that canonization of the structures is definable in this logic. This approach has been the method of choice for numerous results in descriptive complexity theory. To this end, it was shown that canonization is IFP+C (IFP with counting) definable on interval graphs [20], graphs with excluded minors [10, 12, 13, 14], and graphs with bounded rank width [15]. Thus IFP+C captures PTIME on these classes. Regarding CPT, all CPT-definable properties and transformations (e.g., canonization) are in particular polynomial-time computable. If canonization is CPT-definable for a graph class then CPT captures PTIME on this class because CPT subsumes IFP+C.

Closely linked to the problem of canonization is the problem of isomorphism testing. A polynomial-time canonization algorithm implies a polynomial-time isomorphism test. While we do not know of a formal reduction the other way around, we usually have efficient canonization algorithms for all classes for which an efficient isomorphism test is known (see [28] for an overview). This statement can even be proven unconditionally for classes of vertex-colored structures with a CPT-definable isomorphism problem [16]. Accepting for the moment that canonization and isomorphism testing are algorithmically very related, we arrive at the following observation: if isomorphism testing is polynomial-time solvable on a class of structures then, to capture PTIME, we must “solve” the isomorphism problem in the logic anyway. If we do so in CPT we (almost) immediately obtain a logic capturing PTIME. In summary, it appears the question of a logic for PTIME boils down to isomorphism testing within a logic.

There is a notable class for which we have polynomial-time isomorphism testing and canonization algorithms, but for which we do not know how to canonize them in CPT. This is the class of structures with bounded color class size. Specifically, for an integer q , a q -bounded structure is a vertex-colored structure, where at most q vertices have the same color and a total order on the colors is given. Structures with bounded color class size can be canonized in polynomial time with group-theoretic techniques (see [8, 1, 2]). The introduction of group-theoretic techniques marks an important step for the design of canonization algorithms. The use of algorithmic group theory turned out to be very fruitful and subsequently lead to Luks’s famous polynomial-time isomorphism test for graphs of bounded degree [22]. It uses a more general and more complicated machinery than needed for bounded color classes.

For the purpose of isomorphism testing, these group-theoretic techniques inherently rely on choosing generating sets, and it is not clear how this can be done in a choiceless logic. A well-known construction of Cai, Fürer, and Immerman [4] shows that IFP+C does not

provide us with a PTIME logic for 2-bounded structures. Finding a natural, alternative logic for q -bounded structures is still an open problem. Studying structures with bounded color class size is a reasonable next step, because the canonization algorithm for them makes use of comparatively easy group theory but we still do not know how to transfer these techniques into logics.

A first result towards the canonization of structures with bounded color class size in CPT was the canonization of structures with abelian colors, that is the automorphism group of every color class is abelian, due to Abu Zaid, Grädel, Grohe, and Pakusa [29]. They use a certain class of linear equation systems to encode the group-theoretic structure of abelian color classes and solve these systems in CPT. In particular, they show that CPT captures PTIME on 2-bounded structures. Considering dihedral groups is a next natural step because dihedral groups are extensions of abelian groups by abelian groups.

Contribution. This paper presents a canonization procedure in CPT for finite q -bounded structures with dihedral colors. A color class is dihedral (resp. cyclic) if it induces a substructure whose automorphism group is dihedral (resp. cyclic). A dihedral group is the automorphism group of a regular n -gon consisting of rotations and reflections and we call it odd if n is odd. Dihedral groups are non-abelian for $n > 2$. We thereby provide the first canonization procedure for a class of q -bounded structures with non-abelian color classes and in particular show that CPT captures PTIME on it. Overall, we prove the following theorem:

► **Theorem 1.** *The following structures can be canonized in CPT:*

1. q -bounded relational structures of arity at most 3 with odd dihedral or cyclic colors
2. q -bounded graphs with dihedral or cyclic colors.

Our approach consists of two steps. As a first step, we propose a normal form for arbitrary finite q -bounded structures. Then, in a second step, we use group-theoretic arguments to canonize structures with dihedral colors given in the aforementioned normal form.

Concretely, the first step is a reduction transforming the input structure into a normal form, which ensures that a color class and its adjacent color classes form a “rigid assemblage”. That is, locally the automorphism groups form 2-injective 3-factor subdirect products or are quotient groups of other color classes. In the case of 2-injective 3-factor subdirect products, the automorphisms of three adjacent color classes are not independent of each other. This means that every nontrivial automorphism of the substructure induced by these three color classes is never constant on two of them. More precisely, we prove the following theorem (formal definitions and proofs are given later in the paper).

► **Theorem 2.** *For every q and signature τ there is a q' and another signature τ' , such that relational q -bounded τ -structures of arity r can be reduced canonization preservingly in CPT to q' -bounded 2-injective quotient τ' -structures.*

It was not necessary to consider 2-injective groups for abelian colors yet, but it is for non-abelian colors. Towards a reduction step, a purely group-theoretic analysis of 2-injective groups is given in [23]. The main insight is basically that such groups decompose naturally into structurally simpler parts which are related via a common abelian normal subgroup. We extend the techniques to canonize abelian color classes and show how they can be combined with the analysis of 2-injective groups to obtain a canonization procedure for said structures with dihedral colors in CPT. That is, we provide new methods to integrate group-theoretic reasoning, which is at the core of canonizing q -bounded structures algorithmically, into logics.

Our Technique. The strategy of our canonization procedure is to reduce the dihedral groups in some way to abelian groups and then exploit the canonization procedure of [29].

Since the automorphism groups of the color classes are restricted to be dihedral or cyclic, we can characterize all occurring 2-injective 3-factor subdirect products. Using this characterization we show that we can partition the input structure into parts we call reflection components. These reflection components have the property that automorphisms either simultaneously reflect the points in all color classes of the component or in none. In the latter case they rotate the points in all color classes. We use this property to force all groups in a reflection component to become abelian: we prohibit reflections in one color class of the reflection component and this automatically prohibits reflections in all other classes of the component, too. Once all reflections are removed, the remaining groups are abelian. Then we apply the canonization procedure for structures with bounded abelian color classes from [29] to the entire reflection component.

Not limiting ourselves to dihedral groups but also allowing cyclic groups has the benefit that the class of occurring groups is closed under quotients and subgroups. Quotient and subgroups of the input color classes occur naturally in our reduction process to the normal form. For dihedral groups it turns out that odd dihedral groups are easier to handle than the other ones. In fact, reflection components are not really independent but can have “global” dependencies. We show that for “odd” dihedral groups, reflection components can only be related via color classes with abelian automorphism groups, that is their global dependencies are abelian. For “even” dihedral groups, there is a single non-abelian exception that can connect reflection components, which complicates matters. For even dihedral color classes this restricts us to the treatment of graphs (see Theorem 1 above).

Towards generalization, it unfortunately becomes cumbersome to exploit the group structure theory in CPT, which is heavily required to execute the approach. Extending the treatment of linear equation systems, which is a subroutine in [29], to dihedral groups requires significant work already. We still follow the strategy of [29] and use a certain class of equation systems to encode the global dependencies. However, we need to generalize the equation systems. Consequently, we have to adapt all operations used on these equations systems to work in the more general setting (e.g. the check for consistency). This becomes technically even more involved than the techniques of [29] already are.

Related Work. There already exist various results for CPT regarding structures with bounded color class size in addition to the ones mentioned above: Cai, Fürer, and Immerman introduced the so-called CFI graphs. From every base graph, a pair of two non-isomorphic CFI graphs is derived. The isomorphism problem on these pairs of graphs is used to separate IFP+C from PTIME [4]. Dawar, Richerby, and Rossman showed in [6] that the isomorphism problem for the CFI graphs can be solved in CPT for base graphs of color class size 1.

This result was strengthened by Pakusa, Schalthöfer, and Selman to base graphs with logarithmic color class size [26]. The techniques of [6] and [26] are used in [29] to solve the mentioned equation systems.

The logic IFP+C has a strong connection to the higher dimensional Weisfeiler-Leman algorithm and to an Ehrenfeucht–Fraïssé-like game, the so-called bijective pebble game. They are often used to show that IFP+C identifies graphs in a given graph class, i.e., that for any two non-isomorphic graphs of the class there is an IFP+C formula distinguishing them. It was shown by Otto [24] that if IFP+C captures PTIME on a graph class then IFP+C also identifies all graphs in this class. The converse direction is open [12]. The capabilities of IFP+C to detect graph decompositions were recently investigated in [19].

Structure of this Paper. We begin with the characterization of 2-injective 3-factor subdirect products of dihedral and cyclic groups in Section 3. Then we turn to structures and to permutation groups. We begin with the reduction to the above mentioned normal form in Section 4 and then preprocess structures with dihedral colors in Section 5. In Section 6 we introduce tree-like cyclic linear equation systems (TCES) and show that a certain subclass of them can be solved in CPT. Finally, we define and analyze reflection components in Section 7 and give the CPT-definable canonization procedure for dihedral colors. Full formal proofs can be found in [21].

2 Preliminaries

Bounded Relational Structures. A (relational) signature $\tau = \{R_1, \dots, R_k\}$ is a set of relation symbols with associated arities $r_i \in \mathbb{N}$ for all $i \in [k] := \{1, \dots, k\}$. We consider signatures containing a binary relation symbol \preceq . A τ -structure \mathcal{H} is a tuple $\mathcal{H} = (H, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ where $R_i^{\mathcal{H}} \subseteq H^{r_i}$ for all $i \in [k]$ and $\preceq \subseteq H^2$ is a total preorder. Unless said otherwise, all structures considered in this paper will be finite. The preorder \preceq partitions H into equivalence classes, which we call *color classes*, and induces a total order on them. We denote the set of \mathcal{H} -color classes by $\mathbb{C}_{\mathcal{H}}$. For a set $I \subseteq H$ we denote with $\mathcal{H}[I]$ the substructure induced by I . If $I = C$ is a color class, we just write \mathcal{C} for $\mathcal{H}[C]$, if the structure \mathcal{H} is clear from the context. If $I \subseteq \mathbb{C}_{\mathcal{H}}$ we also write $\mathcal{H}[I]$ for $\mathcal{H}[\bigcup I]$. Two colors classes $C, C' \in \mathbb{C}_{\mathcal{H}}$ are *related*, if there is some tuple containing a vertex from C and C' in some $R_i^{\mathcal{H}}$.

A relation $R_i^{\mathcal{H}}$ is *homogeneous* if $R_i^{\mathcal{H}} \subseteq C^k$ for some $C \in \mathbb{C}_{\mathcal{H}}$ and $k \in \mathbb{N}$, otherwise it is *heterogeneous*. A structure \mathcal{H} is of *arity* r if the largest arity of a heterogeneous relation is r . A structure is *q-bounded*, if $|C| \leq q$ for all $C \in \mathbb{C}_{\mathcal{H}}$ and the arity of every homogeneous relation is bounded by q . We write $\text{Aut}(\mathcal{H})$ for the automorphism group of \mathcal{H} . For two structures \mathcal{H}_1 and \mathcal{H}_2 we write $\text{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of isomorphisms between \mathcal{H}_1 and \mathcal{H}_2 .

An ordered copy of \mathcal{H} is a pair $(\mathcal{H}', <)$, such that $\mathcal{H}' = (H', R_1^{\mathcal{H}'}, \dots, R_k^{\mathcal{H}'}, \preceq')$, $\mathcal{H}' \cong \mathcal{H}$, and $<$ is a total order that refines \preceq' . A canonical copy $\text{can}(\mathcal{H})$ is an ordered copy of \mathcal{H} obtained in a canonical way, i.e., defined in CPT in the following. For a canonical copy $\text{can}(\mathcal{H})$ we call the set $\text{Iso}(\mathcal{H}, \text{can}(\mathcal{H}))$ the *canonical labellings*.

Choiceless Polynomial Time. To give a concise definition of CPT, we follow the definition of [9] and use the same idea of e.g. [25] to enforce polynomial bounds.

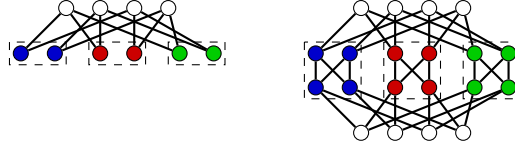
For a set of atoms A we denote with $\text{HF}(A)$ the *hereditarily finite sets over A*. This is the inclusion-wise smallest set with $A \subseteq \text{HF}(A)$ and $a \in \text{HF}(A)$ for every $a \subseteq \text{HF}(A)$. A set $a \in \text{HF}(A)$ is called *transitive*, if $c \in a$ whenever there is some b with $c \in b \in a$. We denote with $\text{TC}(a)$ the *transitive closure* of a , that is the least transitive set b with $a \subseteq b$.

Let τ be a signature. We extend τ by adding set-theoretic function symbols to obtain $\tau^{\text{HF}} := \tau \uplus \{\emptyset, \text{Atoms}, \text{Pair}, \text{Union}, \text{Unique}, \text{Card}\}$. For a τ -structure \mathcal{H} , the *hereditarily finite expansion* $\text{HF}(\mathcal{H})$ is a τ^{HF} -structure over the universe $\text{HF}(H)$ defined as follows: all relations in τ are interpreted as in \mathcal{H} . The other function symbols have the expected interpretation:

- $\emptyset^{\text{HF}(\mathcal{H})} = \emptyset$ and $\text{Atoms}^{\text{HF}(\mathcal{H})} = H$,
- $\text{Pair}^{\text{HF}(\mathcal{H})}(a, b) = \{a, b\}$ and $\text{Union}^{\text{HF}(\mathcal{H})}(a) = \{b \mid \exists c \in a. b \in c\}$,
- $\text{Unique}^{\text{HF}(\mathcal{H})}(a) = \begin{cases} b & \text{if } a = \{b\} \\ \emptyset & \text{otherwise} \end{cases}$ and $\text{Card}^{\text{HF}(\mathcal{H})}(a) = \begin{cases} |a| & \text{if } a \notin H \\ \emptyset & \text{otherwise} \end{cases}$,

where the number $|a|$ is encoded as a von Neuman ordinal.

A *BGS term* is composed as usual from variables and function symbols from τ^{HF} . There are two additional constructs: if $s(\bar{x}, y)$ and $t(\bar{x})$ are terms and $\varphi(\bar{x}, y)$ is a formula then



■ **Figure 1** Two vertex-colored graphs whose automorphism groups are the CFI group (left) and the double CFI group (right).

$r(\bar{x}) = \{s(\bar{x}, y) \mid y \in t(\bar{x}), \varphi(\bar{x}, y)\}$ is a *comprehension term*. If $s(x)$ is a term with a single free variable x , then s^* is an *iteration term*. *BGS formulas* are composed of terms t_1, \dots, t_k as $R(t_1, \dots, t_k)$ (for $R \in \tau$ of arity k), $t_1 = t_2$, and the usual boolean connectives.

Let \mathcal{H} be a τ structure. BGS terms and formulas are interpreted over $\text{HF}(H)$. We define the denotation $\llbracket t \rrbracket^{\mathcal{H}}: \text{HF}(H)^k \rightarrow \text{HF}(\mathcal{H})$ that for a term $t(\bar{x})$ with free variables $\bar{x} = (x_1, \dots, x_k)$ maps $\bar{a} = (a_1, \dots, a_k) \in \text{HF}(\mathcal{H})^k$ to the value of t if we replace x_i with a_i . For a formula $\varphi(\bar{x})$ we define $\llbracket \varphi \rrbracket^{\mathcal{H}}$ to be the set of all $\bar{a} = (a_1, \dots, a_k) \in \text{HF}(\mathcal{H})^k$ satisfying φ .

For the comprehension term r as above, the denotation is defined as follows: $\llbracket r \rrbracket^{\mathcal{H}}(\bar{a}) = \{\llbracket s \rrbracket^{\mathcal{H}}(\bar{a}b) \mid b \in \llbracket t \rrbracket^{\mathcal{H}}(\bar{a}), (\bar{a}b) \in \llbracket \varphi \rrbracket^{\mathcal{H}}\}$, where $\bar{a}b = (a_1, \dots, a_k, b)$. For an iteration term s^* we define a sequence of sets via $a_0 := \emptyset$ and $a_{i+1} := \llbracket s \rrbracket^{\mathcal{H}}(a_i)$. Let $\ell := \ell(s^*, \mathcal{H})$ be the least number with $a_{i+1} = a_i$. We set $\llbracket s^* \rrbracket^{\mathcal{H}} := a_i$ if such an ℓ exists and $\llbracket s^* \rrbracket^{\mathcal{H}} := \emptyset$ otherwise.

A CPT term (or formula respectively) is a tuple (t, p) (or (φ, p) respectively) of a BGS term (or formula) and a polynomial $p(n)$. CPT has the same semantics as BGS by replacing t with (t, p) everywhere (or φ with (φ, p)) with an exception for iteration terms: We set $\llbracket (s^*, p) \rrbracket^{\mathcal{H}} := \llbracket s^* \rrbracket^{\mathcal{H}}$ if $\ell(s^*, \mathcal{H}) \leq p(|H|)$ and $|\text{TC}(a_i)| \leq p(|H|)$ for all i , where the a_i are defined as above. Otherwise, we set $\llbracket (s^*, p) \rrbracket^{\mathcal{H}} := \emptyset$. We use $|\text{TC}(a_i)|$ as a measure of the size of a_i , because by transitivity of $\text{TC}(a_i)$, whenever there is a set $b_k \in \dots \in b_1 \in a_i$, then also $b_k \in \text{TC}(a_i)$ and thus $\text{TC}(a_i)$ counts all sets occurring somewhere in the structure of a_i .

3 Classification of 2-Injective Subdirect Products of Dihedral Groups

We begin with the classification of 2-injective subdirect products of dihedral groups. A group $\Gamma \leq G_1 \times G_2 \times G_3$ is called a (*3-factor*) *subdirect product* if the projection to each factor is surjective. It is called *2-injective* if $\ker(\bar{\pi}_i(\Gamma)) = \{1\}$ for all $i \in [3]$, where $\bar{\pi}_i$ is the projection on the two factors apart the i -th. Another way of looking at this is that two components of an element of Γ determine the third one uniquely.

For $n \geq 3$, the *dihedral group* D_n of order $2n$ is the automorphism group of a regular n -gon in the plane. It consists of n rotations and n reflections and acts naturally on the set of n vertices of the polygon. We regard the identity 1 as rotation and write $\text{Rot}(D_n)$ for the rotation subgroup consisting only of rotations. It is isomorphic to the cyclic group C_n of order n . Only in the degenerate cases D_1 and D_2 the dihedral group is abelian. It holds that $D_1 \cong C_2$ and $D_2 \cong C_2^2$. Elements in the direct product $D_{n_1} \times \dots \times D_{n_k}$ are rotations (resp. reflections) if all components are rotations (resp. reflections). For $n_i \geq 2$ it contains mixed elements that are neither a reflection nor a rotation. Subgroups of such a group may or may not contain mixed elements: A group $\Gamma \leq D_{n_1} \times \dots \times D_{n_k}$ with $n_i > 2$ for all $i \in [k]$ is called a *rotate-or-reflect* group if every $g \in \Gamma$ is a rotation or a reflection. Our classification involved many technical proofs and we only state its result here (proofs can be found in [21]). Roughly speaking, almost all 2-injective subdirect products of cyclic and dihedral groups are abelian or rotate-or-reflect groups. There are precisely two exceptions involving the double CFI group:

► **Definition 3** (CFI Groups). We call the group $\Gamma_{\text{CFI}} := \{(g_1, g_2, g_3) \in D_1^3 \mid g_1 g_2 g_3 = 1\} < D_1^3$ the CFI group and the wreath product $\Gamma_{2\text{CFI}} := \Gamma_{\text{CFI}} \wr C_2$ the double CFI group (cf. Figure 1).

► **Theorem 4.** Let $\Gamma \leq D_{n_1} \times D_{n_2} \times D_{n_3}$ be a 2-injective subdirect product. Then exactly one of following holds:

1. $n_i > 2$ for all $i \in [3]$ and Γ is a rotate-or-reflect group.
2. $n_i = 4$ for all $i \in [3]$ and Γ is isomorphic to the double CFI group $\Gamma_{2\text{CFI}}$.
3. $n_i \leq 2$, $n_j = n_k > 2$ for $\{i, j, k\} = [3]$, and $\bar{\pi}_i(\Gamma)$ is a rotate-or-reflect group.
4. $n_i \leq 2$ for all $i \in [3]$ and Γ is abelian.

► **Theorem 5.** Let $\Gamma \leq C_{n_1} \times D_{n_2} \times D_{n_3}$ be a 2-injective subdirect product. Then exactly one of the following holds:

1. $n_1 \leq 2$, $n_2, n_3 > 2$, and $\bar{\pi}_1(\Gamma)$ is a rotate-or-reflect group.
2. $n_1 = n_2 = n_3 = 4$ and $\Gamma \cong \Gamma_{2\text{CFI}} \cap (\text{Rot}(D_4) \times D_4 \times D_4)$.
3. $n_1, n_2, n_3 \leq 2$ and Γ is abelian.

Furthermore, there are no 2-injective subdirect products of $D_n \times G_2 \times G_3$ for $n > 2$ if G_2 and G_3 are abelian groups.

The classification is later used in the canonization of structures with bounded dihedral colors to analyze how color classes can be connected to others. But first, we make the local automorphism groups, which form 2-injective 3-factor subdirect products, explicit.

4 Normal Forms for Structures

In this section we describe a normal form for relational structures. We sketch how it can be obtained in CPT. It is also important that we have means within CPT to translate a canonical form of the normal form back into a canonical form of the original structure. A structure \mathcal{H} can be reduced to another structure \mathcal{H}' *canonization preservingly* in CPT, if we can define the reduction in CPT from \mathcal{H} to \mathcal{H}' and if we can define a canonical copy of \mathcal{H} whenever we are given \mathcal{H} and a canonical copy of \mathcal{H}' . Formally, the reduction is between classes of structures:

► **Definition 6.** A canonization-preserving CPT-reduction from a class of structures \mathcal{A} to a class of structures \mathcal{B} is a pair of CPT-interpretations (Φ, Ψ) with the following properties:

- Φ is a CPT-interpretation from \mathcal{A} -structures to \mathcal{B} -structures.
- Ψ is a CPT-interpretation from pairs of an \mathcal{A} -structure and an ordered \mathcal{B} -structure to ordered \mathcal{A} -structures.
- Given a CPT-interpretation Θ from \mathcal{B} -structures to ordered \mathcal{B} structures, i.e., a CPT-definable canonization procedure, then $\Psi((\mathcal{H}, \Theta(\Phi(\mathcal{H}))))$ is an ordered copy of \mathcal{H} for every \mathcal{A} -structure \mathcal{H} .

We also say that \mathcal{A} can be reduced canonization preservingly in CPT to \mathcal{B} if there is a canonization-preserving CPT-reduction from \mathcal{A} to \mathcal{B} .

As a first step, we are interested in structures whose color classes cannot be refined by “local” properties:

- We call a relational structure \mathcal{H} *transitive on s color classes* if for every $I \subseteq \mathbb{C}_{\mathcal{H}}$ satisfying $|I| \leq s$ the group $\text{Aut}(\mathcal{H}[I])|_C$ is transitive for every $C \in I$.
- Let $\mathcal{H} = (H, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ be a structure of arity r . We call (C_1, \dots, C_ℓ) the *type* of $R_i^{\mathcal{H}}$ if $R_i^{\mathcal{H}} \subseteq C_1 \times \dots \times C_\ell$ and $C_i \neq C_j$ for all $i \neq j$. We denote with $\mathbb{T}_{\mathcal{H}}$ the set of types of all relations that have a type. We say that \mathcal{H} *has typed relations* if every relation is either homogeneous or has a type.

We can define a CPT reduction from q -bounded structures to q -bounded typed structures transitive on s color classes, because the additional properties can be checked on substructures of constant size. If they are violated, we split the affected color classes and relations. Additionally, we ensure that all color classes are *regular*, i.e., their automorphism groups are regular (a color class C is regular if $|\text{Aut}(C)| = |C|$ and $\text{Aut}(C)$ is transitive, so has only one C -orbit). This is achieved by replacing a color class C by a certain ℓ -orbit of $\text{Aut}(C)$ (for a sufficiently large ℓ). Then we modify the relations to preserve the automorphism groups of the color classes and the connections between them. The said properties simplify the following constructions designed to gain more control on local automorphism groups.

We want to reduce certain local automorphism groups to 2-injective subdirect products. Recall from Section 3 the condition $\ker(\bar{\pi}_i(\Gamma)) = \{1\}$ for 2-injective products: If the condition is violated, we want to factor out a normal subgroup $N \triangleleft \text{Aut}(C)$ (the kernel above) of a color class C . By factoring N out of C , we obtain a quotient color class:

► **Definition 7** (Quotient Color Class). *Let $\mathcal{H} = (H, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ be a structure and the automorphism group of $C \in \mathcal{C}_{\mathcal{H}}$ be regular. Let $N \triangleleft \text{Aut}(C)$. We say that another color class C' is an N -quotient of C if $\text{Aut}(C') \cong \text{Aut}(C)/N$ and there is a function $R_j^{\mathcal{H}} \subseteq C \times C'$ determining the orbit partition of N acting on C , i.e., a vertex in C' corresponds to an N -orbit and the vertices of C are adjacent to its orbit vertices via $R_j^{\mathcal{H}}$. The relation $R_j^{\mathcal{H}}$ is called the orbit-map (of C).*

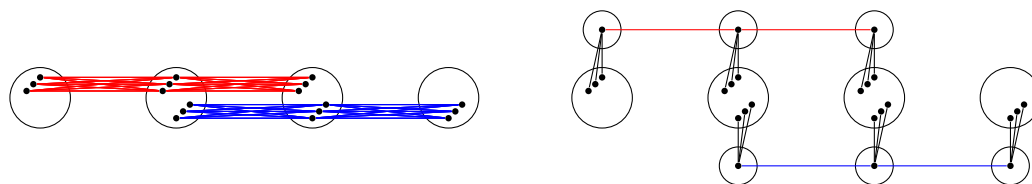
Quotient color classes can always be defined: the N -orbits on C become the vertices of the quotient color class and the orbit-map is given by containment of a C -vertex in an N -orbit.

Now we turn to structures with 2-injective subdirect products as local automorphism groups. The structures we are aiming for consist of two different kinds of color classes. The group color classes form 2-injective subdirect products. They are quotient color classes of extension color classes, which connect different group color classes via the orbit-maps (cf. Figure 4). Formally:

► **Definition 8.** *Let $\mathcal{H} = (H_{\text{gr}} \uplus H_{\text{ex}}, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ be a structure, where H_{gr} and H_{ex} are unions of color classes. We call a color class $C \subseteq H_{\text{gr}}$ (respectively $C \subseteq H_{\text{ex}}$) a group color class (respectively an extension color class). We define the group types $\mathbb{T}_{\text{gr}}^{\mathcal{H}} \subseteq \mathbb{T}^{\mathcal{H}}$ to be the set of all types only consisting of group color classes. Let $T = (C_1, \dots, C_j) \in \mathbb{T}_{\text{gr}}^{\mathcal{H}}$. We set $\Gamma_T^{\mathcal{H}} := \text{Aut}(\mathcal{H}[\bigcup_{i \in [j]} C_i]) \leq \bigotimes_{i \in [j]} \text{Aut}(C_i)$. Finally, we call \mathcal{H} an $(r-1)$ -injective quotient structure if it satisfies the following (cf. Figure 4):*

- a) \mathcal{H} is of arity r , has typed relations, and all color classes are regular.
- b) Every group color class $C \subseteq H_{\text{gr}}$ is an N -quotient of exactly one extension color class $C' \subseteq H_{\text{ex}}$, where $N \triangleleft \text{Aut}(C')$, and not related to any other extension color class apart from C' . Moreover, C is only related by the orbit-map to C' .
- c) All relations only between group color classes are of arity exactly r and every group color class occurs in only one group type.
- d) For every $T \in \mathbb{T}_{\text{gr}}^{\mathcal{H}}$ the group $\Gamma_T^{\mathcal{H}}$ is an $(r-1)$ -injective r -factor subdirect product (for a straightforward generalization of 2-injective 3-factor subdirect products).

Proof sketch of Theorem 2. We now consider a structure \mathcal{H} of arity 3. First, apply the previous reductions to obtain typed structures that are transitive on 3 color classes and which have regular color classes. Then consider a type $T = (C_1, C_2, C_3) \in \mathbb{T}_{\mathcal{H}}$. Let $\Gamma_T := \text{Aut}(\mathcal{H}[C_1 \cup C_2 \cup C_3])$ and $N_i^T := \ker(\bar{\pi}_i^{\Gamma_T})$ for all $i \in [3]$. Then the group $\Gamma_T/N_1^T/N_2^T/N_3^T$ is a 2-factor subdirect product. We realize this product in the structure by constructing the N_i^T -quotient color class of C_i and adjusting the relations as required. We perform this operation for all types T . The constructed quotient color classes form the group color classes of the output structure and the original ones the extension color classes (cf. Figure 2). ◀



■ **Figure 2** The situation in Theorem 2: On the left the input structure. Each circle represents one color class with drawn tuples of two relations (red and blue) between two $\Gamma_T/N_1^T/N_2^T/N_3^T$ -orbits of each color class (where T is different for the red and blue relation). On the right the altered structure: For the types of the red and blue relations there are new group color classes (on the top for red and on the bottom for blue), where the orbits are contracted to a single vertex. The “old” color classes became extension color classes.

For structures of arity $r > 3$, we also need to reduce the arity. For this, we insert color classes dividing a relation into two new relations of lower arity. In particular, if the input structure is of arity 3, the automorphism group of an output color class is a section (a subgroup of a quotient) of the automorphism group of some input color class (cf. Theorem 38 in [21]). We reduce to arity 3, because arity 2 does not simplify the problem further.

5 Structures with Dihedral Colors

A structure $\mathcal{H} = (H, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ has *dihedral colors* if for every \mathcal{H} -color class C the group $\text{Aut}(C)$ is a dihedral or cyclic group. We allow cyclic groups to ensure closure under taking subgroups and quotients. We want to make the dihedral groups explicit as follows: For cyclic groups we require two relations only to avoid case distinctions. Regular and dihedral color classes can always be brought in standard form. In fact, it is always possible to pick two 2-orbits of the color class, that serve as the two required relations. The standard form cycles will help us later to prohibit the reflections in a color class.

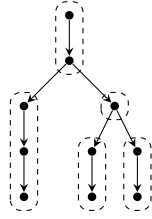
6 Cyclic Linear Equation Systems in CPT

Before we begin to canonize structures with dihedral colors, we need to discuss a special class of linear equation systems. These systems are later used in the canonization procedure to encode the canonical labellings. Let V be a set of variables and \preceq be a preorder on V . The *variable classes* are the \preceq -equivalence classes and are totally ordered by \preceq . A *cyclic constraint* on $W \subseteq V$ is a consistent set of linear equations over \mathbb{Z}_q containing for each pair $u, v \in W$ an equation of the form $u - v = d$ for $d \in \mathbb{Z}_q$.

► **Definition 9** (TCES). A tree-like cyclic linear equation system (TCES) over \mathbb{Z}_q (q a prime power) is a tuple (V, S, \preceq) with the following properties:

- The variable classes form a rooted tree with respect to being a direct successor in \preceq .
- S is a linear equation system on V containing for every variable class a cyclic constraint.
- For every constraint $\sum_i a_i u_i = d$ with $u_i \in V$ and $a_i, d \in \mathbb{Z}_q$ in S every pair of variables u_i, u_j is \preceq -comparable.

In CPT, a system of linear equations S is represented by a set of constraints, which itself are encoded as sets. TCESs generalize cyclic linear equations systems (CES) from [29], where \preceq must be total.



■ **Figure 3** The variable tree (a vertex represents a variable class) of a TCES with all local components. Reordering variable classes inside a local component does not affect the tree structure.

An important operation on CES is the check for consistency. We sketch the check from [29] and its extension to TCES only very roughly, since it requires many technical details (see Section 6 in [21]). In principle, one would like to choose one variable per variable class and eliminate the others using the $u - v = d$ equations. If this was done, we would be left with a totally ordered system (in case of CESs). Of course, choosing the variables is not possible, but the system can be encoded in an equation system of hyperterms, which in some way encode all possible choices of variables and allow arithmetic manipulation. In these hyperterms, the variable classes “became the variables” for which we can apply a variant of Gaussian elimination using a certain kind of hermite normal form. This variant takes care of the issue that we are working over rings and not over fields: it reorders the variable classes based on their coefficients in the equations. Only if they have the same, the variable classes are ordered by the original order.

The translation to hyperterms can be done with TCES exactly as for CES. But the Gaussian elimination does not work anymore: Reordering a tree by the order of the coefficients does not necessarily result in a tree anymore (which causes further difficulties). To overcome this problem, we restrict to TCES, where the reordering does not harm the tree structure but only happens inside so-called local components (cf. Figure 3). Now, we can apply another variant of Gaussian elimination, which does not require a total order on the variable classes. It processes local components from leafs to the root and handles local and global variables differently. We now define local components and these two kinds of variables.

A *local component* is a maximal and (in the variable tree) connected set of variable classes, in which the tree does not branch (see Figure 3). On the local components the preorder \preceq induces a tree in which every local component has degree > 1 or is a leaf. A variable is *local* if in every equation in which it occurs, i.e. it has non-zero coefficient, only variables of the same local component occur, too. Other variables are called *global*. An equation is *local* if it contains at least one local variable and *global* otherwise. For the subsequent canonization the rings \mathbb{Z}_{2^ℓ} will be of special interest and thus treated differently.

► **Definition 10.** A TCES $\mathcal{T} = (V, S, \preceq)$ over \mathbb{Z}_q is called *weakly global*, if

- q is a power of an odd prime and every equation (equivalently every variable) is local or
- $q = 2^\ell$ is a power of 2 and for every global variable $u \in V$ there is an equation $2u = 0 \in S$.

The definition states that only values of order at most 2 are candidates for solutions of global variables. The adaption of Gaussian elimination strongly depends on this property, in particular this restriction guarantees that reordering is only required inside local components.

► **Theorem 11.** *Solvability of weakly global TCESs over \mathbb{Z}_q is CPT-definable.*

The proof follows the same strategy as [25, 29] to solve CESs, but significant adaptations were required throughout the whole procedure.

Finally, we need to form the union of two TCESs. In general, a naive union is not a TCES anymore if the variable structures of the two systems are incompatible. For a working solution, we need the following notion: Let $\mathcal{T} = (V, S, \preceq)$ be a TCES. We say that V' are the *topmost variables* of \mathcal{T} if V' is the set of all variables of the local component containing the root class of the variable tree L_r (formally $V' = \bigcup L_r$). Let \mathcal{T}_1 and \mathcal{T}_2 be two TCESs over \mathbb{Z}_q which are CPT distinguishable. That is, there is a CPT term defining the ordered tuple $(\mathcal{T}_1, \mathcal{T}_2)$ or equivalently an order $\mathcal{T}_1 < \mathcal{T}_2$ (e.g. the two TCESs are defined by different CPT terms or can be ordered by their structure). If their common variables are topmost in both TCESs and whose order is compatible in them, we can define a TCES $\mathcal{T}_1 \uplus \mathcal{T}_2$ by joining them together at the topmost variables (cf. Definition 60 in[21]):

► **Lemma 12.** *If \mathcal{T}_1 and \mathcal{T}_2 (with variables V_i and topmost variables V'_i) are compatible, then $\mathcal{T}_1 \uplus \mathcal{T}_2$ is again a TCES with topmost variables $V'_1 \cup V'_2$ and it satisfies $L(\mathcal{T}_1 \uplus \mathcal{T}_2) = \bigcap_{i \in [2]} \text{ext}_{V_1 \cup V_2}(L(\mathcal{T}_i))$. If both \mathcal{T}_1 and \mathcal{T}_2 are weakly global then $\mathcal{T}_1 \uplus \mathcal{T}_2$ is weakly global, too.*

Here, $\text{ext}_{V_1 \cup V_2}(L(\mathcal{T}_i))$ denotes the set $N \subseteq \mathbb{Z}_q^{V_1 \cup V_2}$ whose projection to $\mathbb{Z}_q^{V'_i}$ is equal to $L(\mathcal{T}_i)$. We write $\mathcal{T} = (\mathcal{T}_{p_1}, \dots, \mathcal{T}_{p_k})$ for a sequence of TCESs over pairwise coprime prime powers p_i and $L(\mathcal{T})$ for the solution space of \mathcal{T} . Lemma 12 generalizes to series of TCESs by making the assumptions of the lemma for TCESs $\mathcal{T}_{p_i}, \mathcal{T}_{q_j}$ with p_i and q_j powers of the same prime.

7 Canonization of Structures with Dihedral Color Classes

Recall that for our canonization problem the reduction to normal forms (Theorem 2) shows that we can assume the input structure to be a dihedral 2-injective quotient structure. Our further strategy is as follows. We want to reduce canonization of dihedral 2-injective quotient structures to that of structures with abelian color classes and then apply the canonization procedure for abelian color classes. The main idea is to artificially prohibit reflections in one color class and then hope that this prohibits reflections in other color classes as well. For this, we want to exploit the classification of 2-injective subdirect products of dihedral groups (Theorems 4 and 5) saying that most 2-injective subdirect products are rotate-or-reflect groups. In particular, if we prohibit reflections in one color class of a rotate-or-reflect group then reflections in the other color classes are prohibited, too. This effect of prohibiting reflections continues through most 2-injective subdirect products and quotient color classes. However, it does not have to reach all color classes since some 2-injective subdirect product are not rotate-or-reflect groups (for example if one factor is abelian). We call the parts of the structure in which reflections are linked in this way and can only occur simultaneously *reflection components*. We analyze how reflection components can depend on each other. It will turn out, that different reflection components can indeed only be connected through abelian color classes. We call these color classes *border color classes*. Overall, we will follow a two-leveled approach: on the top level, we deal with the dependencies between the border (and all other abelian) color classes, and on the second level we consider each reflection component on its own and how it is embedded in its border color classes.

To ensure that the border color classes are indeed all abelian we have to forbid the single exception in Theorem 4, which is not a rotate-or-reflect group, namely the double CFI group.

► **Definition 13 (Double-CFI-Free Structure).** *We call a 2-injective dihedral quotient structure double-CFI-free, if for every $T \in \mathbb{T}_{\text{gr}}$ the group Γ_T is neither isomorphic to the double CFI group $\Gamma_{2\text{CFI}}$ nor to $\Gamma_{2\text{CFI}} \cap (\text{Rot}(\mathbb{D}_4) \times \mathbb{D}_4 \times \mathbb{D}_4)$.*

There are two natural classes of structures that are double-CFI-free after applying the preprocessing: graphs with dihedral colors and structures of arity 3 which are *odd dihedral*, that is, for every non-abelian $C \in \mathbb{C}_{\mathcal{H}}$ there is an odd k such that $\text{Aut}(C) \cong D_k$.

7.1 Reflection Components

Let $\mathcal{H} = (H_{\text{gr}} \uplus H_{\text{ex}}, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ be an arbitrary dihedral 2-injective double-CFI-free quotient structure. Whenever we construct a CPT term in the following, it does not depend on \mathcal{H} but gets \mathcal{H} as input and in particular satisfies the claimed properties for all dihedral 2-injective double-CFI-free quotient structures. We use the set $\mathbb{O} := \{\uparrow, \downarrow\}$ to denote orientations. For an orientation $o \in \mathbb{O}$ we set $\bar{o} := o'$ as the reverse orientation, so that $\mathbb{O} = \{o, o'\}$.

► **Definition 14** (Orientation). *We say that a structure $\mathcal{H}' = (H_{\text{gr}} \uplus H_{\text{ex}}, R_1^{\mathcal{H}'}, \dots, R_k^{\mathcal{H}'}, \preceq')$ is an orientation of \mathcal{H} if \preceq' refines \preceq with the following property: Let $C \in \mathbb{C}_{\mathcal{H}}$ be a color class that is split by \preceq' , then $\text{Aut}(\mathcal{H}[C])$ is a non-abelian dihedral group and C is split into two color classes C^{\uparrow} and C^{\downarrow} , such that each of the two classes contains one of the two oriented cycles inducing the standard form in C . We say that \mathcal{H}' orients C .*

By splitting the color class C in the above manner, we precisely prohibit the reflections in C . Because an orientation modifies only the preorder of the structure, defining an orientation of \mathcal{H} is always canonization preserving. For a color class C with dihedral automorphism group we can define in CPT two orientations \mathcal{H}_C^o for $o \in \mathbb{O}$ that only orient C (by the two possible orders $C^o \prec' C^{\bar{o}}$). Of course, we cannot choose one orientation canonically. But the orientation of C can canonically be propagated to other color classes in the following cases:

- a) Whenever C is part of a rotate-or-reflect group (because once we cannot reflect in one component, we cannot do so in the others), and
- b) whenever C' is a quotient of C (because once we remove reflections from C we can also remove remaining reflections from quotient groups).

To prove Case a) we use the classification of 2-injective subdirect products of dihedral groups (Theorems 4 and 5). We obtain an equivalence relation on the color classes: two classes are equivalent if an orientation of one color class can be propagated to the other. We define *reflection components* as the equivalence classes of said relation, which consists of color classes with dihedral automorphism group (cf. Definitions 65 and 70 in the full version).

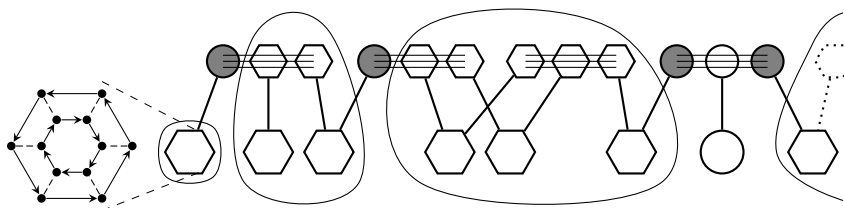
Because all color classes of a reflection component D can be oriented by orienting only a single color class, we can speak of the two orientations of a reflection component D . We now analyze how a reflection components can be connected to another one:

► **Definition 15** (Color Class in Standard Form). *Let \mathcal{H} be a structure and $C \in \mathbb{C}_{\mathcal{H}}$. We say that a color class $C \in \mathbb{C}_{\mathcal{H}}$ is in standard form if the following holds:*

- *If $\text{Aut}(C) \cong C_{|C|}$ then there are relations $R_i^{\mathcal{H}}, R_j^{\mathcal{H}} \subseteq C^2$ of arity 2 each forming a directed cycle of length $|C|$ on C .*
- *Otherwise $\text{Aut}(C) \cong D_{|C|/2}$ and there are two relations $R_i^{\mathcal{H}}, R_j^{\mathcal{H}} \subseteq C^2$ such that $R_j^{\mathcal{H}}$ defines two directed and disjoint cycles of length $|C|/2$ and $R_i^{\mathcal{H}}$ connects them by a perfect matching such that the two cycles are directed into opposite directions (cf. Figure 4).*

We say that the relations $R_i^{\mathcal{H}}$ and $R_j^{\mathcal{H}}$ induce the standard form of C . The color classes of \mathcal{H} are in standard form, if every color class is in standard form.

► **Definition 16** (Border Color Class). *Let $D \subseteq \mathbb{C}_{\mathcal{H}}$ be a reflection component. We call a color class $C \in \mathbb{C}_{\mathcal{H}}$ a border color class of D if $C \notin D$ and C is related to a color class contained in D . We denote with $B(D)$ the set of all border color classes of D .*



■ **Figure 4** A 2-injective quotient structure with dihedral colors: an abelian color class is drawn as circle, a non-abelian one as hexagon. The group color classes are at the top, the extension classes at the bottom. An edge between a group class C and an extension class C' denotes an orbit-map and C is a quotient of C' . Edges between group color classes indicate relations of arity 3. The reflection components are encircled and border color classes are gray. On the left a dihedral color class in standard form with automorphism group D_6 .

► **Lemma 17.** *Let $D \subseteq \mathbb{C}_{\mathcal{H}}$ be a reflection component and $C \in B(D)$ a border color class of D . Then $\text{Aut}(C)$ is isomorphic to one of $\{C_1, C_2, D_2\}$ and C is a group color class.*

This lemma is also proven using Theorems 4 and 5. So the border color classes of a reflection component D are all abelian group color classes. That is, the reflection components are embedded in a global abelian part of the structure (an example is shown in Figure 4). We define $\mathcal{H}_D := \mathcal{H}[B(D) \cup \bigcup D]$ and denote the two CPT-definable (abelian) orientations of \mathcal{H}_D with \mathcal{H}_D^o , $o \in \mathbb{O}$. Let $\text{can}(\mathcal{H}_D^o)$ be canonizations for all $o \in \mathbb{O}$. We denote with $\overline{\text{can}}(\mathcal{H}_D^o)$ the structure obtained from $\text{can}(\mathcal{H}_D^o)$ by undoing the orientation. Then $\mathcal{H}_D \cong \overline{\text{can}}(\mathcal{H}_D^o)$. Let $<$ be the lexicographical order on canonizations. We define the canonization $\text{can}(\mathcal{H}_D)$ to be the $<$ -minimal canonization $\overline{\text{can}}(\mathcal{H}_D^o)$ with $o \in \mathbb{O}$. We analyze the canonical labellings of D .

► **Lemma 18.** *If $\text{can}(\mathcal{H}_D^o) < \text{can}(\mathcal{H}_D^{\bar{o}})$, then $\text{Iso}(\mathcal{H}_D, \text{can}(\mathcal{H}_D)) = \text{Iso}(\mathcal{H}_D^o, \text{can}(\mathcal{H}_D^o))$.*

► **Lemma 19.** *If $\text{can}(\mathcal{H}_D^{\uparrow}) = \text{can}(\mathcal{H}_D^{\downarrow})$, then $\text{Iso}(\mathcal{H}_D, \text{can}(\mathcal{H}_D)) = \bigcup_{o \in \mathbb{O}} \text{Iso}(\mathcal{H}_D^o, \text{can}(\mathcal{H}_D^o))$.*

7.2 Canonizing Abelian Structures

Our canonization procedure strongly depends on the canonization procedure for q -bounded structured with *abelian* color classes. This procedure not only outputs a canonization, but also a CES *encoding* the canonical labellings. The automorphism group of a color class is decomposed into a direct sum of cyclic groups, which are used to define variables and cyclic constraints for this color class. In particular, if the automorphism group of a color class is the direct product of cyclic groups of prime power order q , then all variables for this color class range over \mathbb{Z}_q . A formal statement of the canonization procedure and the notion of encoding isomorphisms can be found in [29] and in Theorem 81 in the full version of this paper. It is also possible to start the canonization procedure for abelian color classes with a TCES that encodes an initial set of allowed labellings (Lemma 83 in [21]).

7.3 Canonization Procedure

For dihedral colors we want to maintain an equation system encoding all canonical labellings of all abelian color classes (and hence including all border color classes) that extend to canonical labellings of the input structure. This suffices to encode the dependencies between different reflection components because – as we have seen in the previous section – they can only be connected via abelian color classes. As initialization step, we apply the canonization procedure for abelian colors to all abelian color classes. Then we want to inductively add one

```

1 Compute the set  $A \subseteq \mathbb{C}_{\mathcal{H}}$  of abelian color classes;
2 Compute all reflection components  $D_1 < \dots < D_m$  of  $\mathcal{H}$ ;
3 Compute  $\text{can}(\mathcal{H}_0) := \text{can}(\mathcal{H}[A])$  and  $\Phi_0 := \text{Iso}(\mathcal{H}[A], \text{can}(\mathcal{H}[A]))$  using the
   canonization procedure for abelian colors;
4 for  $i \in [m]$  do
5   | Set  $D := D_i$  and define the two orientations  $\mathcal{H}_D^o$ ;
6   | Compute  $\text{can}(\mathcal{H}_D^o)$  and  $\Phi^o := \text{Iso}(\mathcal{H}_D^o, \text{can}(\mathcal{H}_D^o))$  such that  $\Phi_{i-1} \cap \text{ext}_A(\Phi^o) \neq \emptyset$ 
   | with the canonization procedure for abelian colors;
7   | if  $\text{can}(\mathcal{H}_D^o) < \text{can}(\mathcal{H}_D^{\bar{o}})$  for some  $o \in \mathbb{O}$  then
8   |   |  $\text{can}(\mathcal{H}_i) := \text{can}(\mathcal{H}_{i-1}) \cup \overline{\text{can}}(\mathcal{H}_D^o)$ ;
9   |   |  $\Phi_i := \Phi_{i-1} \cap \text{ext}_A(\Phi^o|_A)$ ;
10  | else
11  |   |  $\text{can}(\mathcal{H}_i) := \text{can}(\mathcal{H}_{i-1}) \cup \overline{\text{can}}(\mathcal{H}_D^\uparrow) \cup \overline{\text{can}}(\mathcal{H}_D^\downarrow)$ ;
12  |   |  $\Phi_i := \Phi_{i-1} \cap \text{ext}_A((\Phi^\uparrow \cup \Phi^\downarrow)|_A)$ ;
13  $\text{can}(\mathcal{H}) := \text{can}(\mathcal{H}_m)$ ;

```

■ **Figure 5** Canonizing a 2-injective double-CFI-free structure \mathcal{H} with dihedral colors in CPT.

reflection component in each step (possibly restricting the canonical labellings of the border color classes). To do so, we want to define a canonical copy of the reflection component D by taking the existing partial canonization into account. That is, given an equation system encoding all canonical labellings of the partial canonization computed so far, we want to increase both, the equation system and the canonization, by D in one step.

From now, we assume that the abelian color classes of a structure \mathcal{H} are smaller than the non-abelian ones (according to \preceq). The canonization procedure is given in Figure 5, where we use $\text{ext}_A(\Phi)$ as shorthand for $\text{ext}_{\cup_A}(\Phi)$. We fix the input structure $\mathcal{H} = (H, R_1^{\mathcal{H}}, \dots, R_k^{\mathcal{H}}, \preceq)$ in the following (again, our CPT terms will not depend on \mathcal{H}). The algorithm maintains canonizations $\text{can}(\mathcal{H}_i)$ of $\mathcal{H}_i := \mathcal{H}[A \cup \bigcup_{j \in [i]} D_j]$ and sets Φ_i of canonical labellings.

► **Lemma 20.** *For $i \leq m$ the following holds: $\mathcal{H}_i \cong \text{can}(\mathcal{H}_i)$ and $\Phi_i = \text{Iso}(\mathcal{H}_i, \text{can}(\mathcal{H}_i))|_A$.*

Proof Sketch. The canonization procedure for abelian color classes yields the desired set of canonical labellings. By Lemmas 18 and 19 the canonical labellings of the (unoriented) reflection component are computed correctly. ◀

We cannot compute with the sets Φ_i directly in CPT because they can be exponentially large. So we encode them with sequences of weakly global TCESs \mathcal{T}_i . We maintain that the variables V_A of the abelian color classes $A \subseteq \mathbb{C}_{\mathcal{H}}$ are contained in the topmost variables of the \mathcal{T}_i and thus the occurring TCESs will all be compatible. With what we have seen so far, the canonization procedure can be expressed in CPT apart one exception in Line 12: We have to show how to define a TCES encoding $\text{ext}_A((\Phi^\uparrow \cup \Phi^\downarrow)|_A) = \text{ext}_A(\text{Iso}(\mathcal{H}_D, \text{can}(\mathcal{H}_D))|_{B(D)})$.

7.4 Equation Systems for Reflection Components

Let $\mathcal{T} := \mathcal{T}_{i-1}$ for some $1 < i \leq m$ be in the series of weakly global TCESs for the canonization constructed so far, $D := D_i$ be the next reflection component to canonize (cf. Figure 5), and $\text{can}(\mathcal{H}_D^\uparrow) = \text{can}(\mathcal{H}_D^\downarrow)$. Let \mathcal{S}^o be the series of CESs encoding the sets $\Phi^o = \text{Iso}(\mathcal{H}_D^o, \text{can}(\mathcal{H}_D^o))$ and the variables of the (abelian) border color classes of D be $B = B_1 < \dots < B_k$. These variables are equal for \mathcal{S}^\uparrow and \mathcal{S}^\downarrow and are contained in the topmost variables V_A of \mathcal{T} .

We cannot fix an isomorphism in $\text{Iso}(\mathcal{H}_D^\uparrow, \mathcal{H}_D^\downarrow) = \text{Iso}(\mathcal{H}_D^\downarrow, \mathcal{H}_D^\uparrow)$ canonically, but one isomorphism contained in $\text{Iso}(\mathcal{H}_D^\uparrow, \mathcal{H}_D^\downarrow)|_{B(D)}$: Note that by Lemma 17 the border color classes have automorphism groups C_2^ℓ for $\ell \in \{0, 1, 2\}$. Hence, all variables for the border color classes range over \mathbb{Z}_2 . We rename the variables of the two series of CESs, such that both use different variables, but we still can identify a variable of a border color class of \mathcal{S}^o with a variable of a border color class of $\mathcal{S}^{\bar{o}}$. Hence, for two vectors $x^o \in \mathbb{Z}_2^{B_i^o}$ we still can write $x^\uparrow = x^\downarrow$. We denote with V^o (and B^o respectively) the changed variables for $o \in \mathbb{O}$.

► **Lemma 21.** *There is a CPT term defining two vectors $x^o = (x_1^o, \dots, x_k^o) \in \mathbb{Z}_2^{B_1^o} \times \dots \times \mathbb{Z}_2^{B_k^o}$ for both $o \in \mathbb{O}$ such that if $y^o \in L(\mathcal{S}^o)$, then there is a $y^{\bar{o}} \in L(\mathcal{S}^{\bar{o}})$ such that $y^o|_{B^o} + x^o = y^{\bar{o}}|_{B^{\bar{o}}}$.*

Proof Sketch. Assume we have defined x^o for the first i border color classes. We define a TCES that is consistent if and only if there are $y^o \in L(\mathcal{S}^o)$ for both $o \in \mathbb{O}$ that have different values for exactly the B_j ($j \in [i+1]$) with $j = i+1$ or $j \leq i$ and $x^o(u) = 1$ for all $u \in B_j$. If the TCES is consistent, we set all entries for B_{i+1} in x^o to 1 and otherwise to 0. ◀

We now use the vectors x^o to represent the canonical labellings of the border color classes, which additionally extend to canonical labellings of the reflection component, as a TCES.

► **Lemma 22.** *There is a CPT term defining a series of weakly global TCESs \mathcal{T}_D with the following properties: B is contained in the topmost variables of \mathcal{T}_D , \mathcal{T}_D encodes the set $\text{Iso}(\mathcal{H}_D, \text{can}(\mathcal{H}_D))|_{B(D)}$, and the size of \mathcal{T}_D is polynomial in $|D|$.*

Proof Sketch. Let x^o be the two vectors given by Lemma 21. We define a set of two variables $B_\alpha := \{\alpha^\uparrow, \alpha^\downarrow\}$ (and set $\alpha^o := \mathcal{H}_D^o$), $V_D := B \cup B_\alpha \cup V^\uparrow \cup V^\downarrow$, and \preceq_D such that it respects the orders on B and V^o and $B \prec B_\alpha \prec V^o$ for all $o \in \mathbb{O}$. The variable sets V^\uparrow and V^\downarrow are incomparable. We want to define a TCES \mathcal{T}_D enforcing that if $z \in L(\mathcal{T}_D)$, then there is an $o \in \mathbb{O}$ and a $y^o \in L(\mathcal{S}^o)$ such that $z = y^o|_B$. To do so, we guess two solutions $y^o \in L(\mathcal{S}^o)$ (one for each $o \in \mathbb{O}$) with the property that $y^o|_B + x^o = y^{\bar{o}}|_B$ (Lemma 21). Then we want to ensure that $z = y^\uparrow|_B$ or $z = y^\downarrow|_B$. To allow that one equality does not hold, we use the additional variables α^\uparrow to express the constraints $z = y^o|_B + \alpha^o \cdot x^o$. By enforcing that exactly one of α^\uparrow and α^\downarrow is 1, we obtain the desired system. Finally, to make the system linear, we encode the multiplication $\alpha^o \cdot x^o$. This is possible, because x^o does not depend on y^o and can be defined before defining the following TCES:

$$\begin{aligned} y^\uparrow \in L(\mathcal{S}^\uparrow), \quad y^\downarrow \in L(\mathcal{S}^\downarrow), \quad 1 = \alpha^\uparrow + \alpha^\downarrow \\ z(u) = y^\uparrow(u) = y^\downarrow(u) & \quad \text{if } x^\uparrow(u) = x^\downarrow(u) = 0, u \in B \\ z(u) = y^\uparrow(u) + \alpha^\uparrow = y^\downarrow(u) + \alpha^\downarrow & \quad \text{if } x^\uparrow(u) = x^\downarrow(u) = 1, u \in B \end{aligned}$$

where y^o is indexed by V^o and z is indexed by B and ranges over \mathbb{Z}_2 . If the variable α^o is assigned to 1, then $z = y^o|_{B^o} + x^o$ and $z = y^{\bar{o}}|_{B^o}$ otherwise. Because of the cyclic constraint $1 = \alpha^\uparrow + \alpha^\downarrow$, we add the vector x^o to a solution y^o of \mathcal{S}^o for exactly one orientation $o \in \mathbb{O}$. One verifies the construction with Lemmas 19 and 21. ◀

Now, we defined all operation on TCESs needed and conclude:

► **Theorem 23.** *Canonization of 2-injective double-CFI-free q -bounded gadget quotient structures is CPT-definable.*

This proves **Theorem 1**, because the involved classes of structures are double-CFI-free after applying Theorem 2. In particular CPT captures PTIME on these classes.

8 Conclusion

We separated a relational structure into 2-injective subdirect products and quotients, gave a classification of all 2-injective subdirect products of dihedral and cyclic groups, and used this classification to canonize relational structures with bounded dihedral colors of arity at most 3. We showed that the structure decomposes into reflection components and that in these components either all color classes have to be reflected or none. If we exclude a single 2-injective subdirect product, namely the double CFI group, the reflection components can only have abelian dependencies. This is always true for graphs, because the said group cannot be realized by graphs with dihedral colors. In fact, we demonstrated the increase of complexity when considering structures of arity 3 instead of 2. Apart from the fact that the double CFI group does not appear, a classification of 1-injective 2-factor subdirect products of dihedral groups is much easier. Considering higher arity, already 3-injective 4-factor subdirect products of dihedral groups cannot be classified to be (almost) abelian or reflect-or-rotate groups. If one instead tries to reduce the arity of the structures, one needs not only to work with a class of groups closed under taking quotients and subgroups (which is the case for dihedral and cyclic groups), but also closed under taking direct products.

One natural way to exclude the double CFI group is a restriction to odd dihedral colors. The difficulty with even dihedral groups might indicate that looking at odd (non-dihedral) groups could be a reasonable next step. A natural graph class with odd automorphism groups are tournaments. Since such groups are solvable there is hope for an inductive approach exploiting the abelian case. It could be possible that the techniques developed in this paper transfer to this case. Just like dihedral groups, odd groups are closed under taking quotients and subgroups. However, they are also closed under direct products (and are solvable), which would allow a reduction of the arity. Thus, it is possible to apply our reduction to quotients and 2-injective groups. As a next step, one could try to follow a similar strategy as for dihedral colors: identify components of the graph, in which the complexity of all color classes decreases simultaneously, when a single color class is made easier (similar to reflection components). This might not immediately result in abelian groups, but recursion on the complexity of the groups could be a reasonable option, e.g. on the length of the composition series or on the nilpotency class. All the mentioned avenues remain as future work.

References

- 1 László Babai. Monte carlo algorithms in graph isomorphism testing. Technical Report D.M.S. No. 79-10, Université de Montréal, 1979.
- 2 László Babai and Eugene M. Luks. Canonical labeling of graphs. In David S. Johnson, Ronald Fagin, Michael L. Fredman, David Harel, Richard M. Karp, Nancy A. Lynch, Christos H. Papadimitriou, Ronald L. Rivest, Walter L. Ruzzo, and Joel I. Seiferas, editors, *Proceedings of the 15th Annual ACM Symposium on Theory of Computing, 25-27 April, 1983, Boston, Massachusetts, USA*, pages 171–183. ACM, 1983. doi:10.1145/800061.808746.
- 3 Andreas Blass, Yuri Gurevich, and Saharon Shelah. Choiceless polynomial time. *Ann. Pure Appl. Log.*, 100(1-3):141–187, 1999. doi:10.1016/S0168-0072(99)00005-6.
- 4 Jin-yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12(4):389–410, 1992. doi:10.1007/BF01305232.
- 5 Ashok K. Chandra and David Harel. Structure and complexity of relational queries. *J. Comput. Syst. Sci.*, 25(1):99–128, 1982. doi:10.1016/0022-0000(82)90012-5.

- 6 Anuj Dawar, David Richerby, and Benjamin Rossman. Choiceless polynomial time, counting and the Cai-Fürer-Immerman graphs. *Ann. Pure Appl. Logic*, 152(1-3):31–50, 2008. doi:10.1016/j.apal.2007.11.011.
- 7 Ronald Fagin. Generalized first-order spectra and polynomial-time recognizable sets. In *Complexity of computation (Proc. SIAM-AMS Sympos. Appl. Math., New York, 1973)*, pages 43–73. SIAM-AMS Proc., Vol. VII, 1974.
- 8 Merrick L. Furst, John E. Hopcroft, and Eugene M. Luks. Polynomial-time algorithms for permutation groups. In *21st Annual Symposium on Foundations of Computer Science, Syracuse, New York, USA, 13-15 October 1980*, pages 36–41. IEEE Computer Society, 1980. doi:10.1109/SFCS.1980.34.
- 9 Erich Grädel and Martin Grohe. Is polynomial time choiceless? In Lev D. Beklemishev, Andreas Blass, Nachum Dershowitz, Bernd Finkbeiner, and Wolfram Schulte, editors, *Fields of Logic and Computation II - Essays Dedicated to Yuri Gurevich on the Occasion of His 75th Birthday*, volume 9300 of *Lecture Notes in Computer Science*, pages 193–209. Springer, 2015. doi:10.1007/978-3-319-23534-9_11.
- 10 Martin Grohe. Isomorphism testing for embeddable graphs through definability. In F. Frances Yao and Eugene M. Luks, editors, *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*, pages 63–72. ACM, 2000. doi:10.1145/335305.335313.
- 11 Martin Grohe. The quest for a logic capturing PTIME. In *Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science, LICS 2008, 24-27 June 2008, Pittsburgh, PA, USA*, pages 267–271. IEEE Computer Society, 2008. doi:10.1109/LICS.2008.11.
- 12 Martin Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. In *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science, LICS 2010, 11-14 July 2010, Edinburgh, United Kingdom*, pages 179–188. IEEE Computer Society, 2010. doi:10.1109/LICS.2010.22.
- 13 Martin Grohe. *Descriptive Complexity, Canonization, and Definable Graph Structure Theory*. Cambridge University Press, 2017.
- 14 Martin Grohe and Julian Mariño. Definability and descriptive complexity on databases of bounded tree-width. In Catriel Beeri and Peter Buneman, editors, *Database Theory - ICDT '99, 7th International Conference, Jerusalem, Israel, January 10-12, 1999, Proceedings*, volume 1540 of *Lecture Notes in Computer Science*, pages 70–82. Springer, 1999. doi:10.1007/3-540-49257-7_6.
- 15 Martin Grohe and Daniel Neuen. Canonisation and definability for graphs of bounded rank width. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–13. IEEE, 2019. doi:10.1109/LICS.2019.8785682.
- 16 Martin Grohe, Pascal Schweitzer, and Daniel Wiebking. Deep Weisfeiler Leman, 2020. arXiv:arXiv:2003.10935.
- 17 Yuri Gurevich. Logic and the challenge of computer science. In Egon Boerger, editor, *Current Trends in Theoretical Computer Science*, pages 1–57. Computer Science Press, 1988.
- 18 Neil Immerman. Languages that capture complexity classes. *SIAM J. Comput.*, 16(4):760–778, 1987. doi:10.1137/0216051.
- 19 Sandra Kiefer and Daniel Neuen. The power of the Weisfeiler-Leman algorithm to decompose graphs. In *MFCS*, volume 138 of *LIPICs*, pages 45:1–45:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
- 20 Bastian Laubner. Capturing polynomial time on interval graphs. In *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science, LICS 2010, 11-14 July 2010, Edinburgh, United Kingdom*, pages 199–208. IEEE Computer Society, 2010. doi:10.1109/LICS.2010.42.

- 21 Moritz Lichter and Pascal Schweitzer. Canonization for bounded and dihedral color classes in choiceless polynomial time, 2020. [arXiv:arXiv:2010.12182](https://arxiv.org/abs/2010.12182).
- 22 Eugene M. Luks. Isomorphism of graphs of bounded valence can be tested in polynomial time. *J. Comput. Syst. Sci.*, 25(1):42–65, 1982.
- 23 Daniel Neuen and Pascal Schweitzer. Subgroups of 3-factor direct products. *Tatra Mt. Math. Publ.*, 73:19–38, 2019.
- 24 Martin Otto. *Bounded variable logics and counting - a study in finite models*, volume 9 of *Lecture Notes in Logic*. Springer, 1997.
- 25 Wied Pakusa. *Linear Equation Systems and the Search for a Logical Characterisation of Polynomial Time*. PhD thesis, RWTH Aachen, 2015.
- 26 Wied Pakusa, Svenja Schalthöfer, and Erkal Selman. Definability of Cai-Fürer-Immerman problems in choiceless polynomial time. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*, volume 62 of *LIPICs*, pages 19:1–19:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPICs.CSL.2016.19.
- 27 Benjamin Rossman. Choiceless computation and symmetry. In Andreas Blass, Nachum Dershowitz, and Wolfgang Reisig, editors, *Fields of Logic and Computation, Essays Dedicated to Yuri Gurevich on the Occasion of His 70th Birthday*, volume 6300 of *Lecture Notes in Computer Science*, pages 565–580. Springer, 2010. doi:10.1007/978-3-642-15025-8_28.
- 28 Pascal Schweitzer and Daniel Wiebking. A unifying method for the design of algorithms canonizing combinatorial objects. In Moses Charikar and Edith Cohen, editors, *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*, pages 1247–1258. ACM, 2019. doi:10.1145/3313276.3316338.
- 29 Faried Abu Zaid, Erich Grädel, Martin Grohe, and Wied Pakusa. Choiceless polynomial time on structures with small abelian colour classes. In Erzsébet Csuhaj-Varjú, Martin Dietzfelbinger, and Zoltán Ésik, editors, *Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part I*, volume 8634 of *Lecture Notes in Computer Science*, pages 50–62. Springer, 2014. doi:10.1007/978-3-662-44522-8_5.