Borel Sets in Reverse Mathematics

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Abstract

We present what is known about the reverse mathematical strength of weak theorems involving Borel sets.

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Theorems about Borel sets are often proved using arguments which appeal to some property of Borel sets, rather than recursing on the Borel structure of the set directly. For example, the statement "there is no Borel well-ordering of the reals" can be proved using either a measure or category argument. More generally, suppose we are given a theorem about Borel sets and a proof based on measure theory. Could the same theorem also be proved with a category argument? In principle, when the answer is "no", reverse mathematics provides a framework for proving this negative answer. However, early treatments of Borel sets in reverse mathematics (see [3]) used arithmetic transfinite recursion (ATR_0) as a base theory, and thus were not able to distinguish between a measure argument, a category argument, and a direct recursion.

Given a code for a Borel set B and an element x which may or may not be in B, a direct recursion on the structure of B is generally required to determine whether $x \in B$ or $x \notin B$. Therefore, in models of second order arithmetic which do not satisfy ATR_0, there are codes B and elements x for which the model is not powerful enough to see either $x \in B$ or $x \notin B$ [2]. To avoid giving artificial strength to theorems which simply assert that a Borel set has an element, in [1] we defined the notion of a *completely determined Borel set*. Roughly speaking, inside a given model, B is completely determined if for all x either $x \in B$ or $x \notin B$. This allows a separation between direct recursion on the one hand, and measure and category arguments on the other.

▶ **Theorem 1.** *The following principles are strictly weaker than* ATR_0*:*

1. *Every completely determined Borel set has the property of Baire. [1]*
2. *Every completely determined Borel set is measurable. [5]*

Furthermore, if $\mathcal{M}$ is an ω-model of (1), then for every $Z \in \mathcal{M}$ there is a $G \in \mathcal{M}$ that is $\Delta^1_1(Z)$-generic [1]. And if $\mathcal{M}$ is an ω-model of (2), then for every $Z \in \mathcal{M}$ there is an $R \in \mathcal{M}$ that is $\Delta^1_1(Z)$-random [5].

The hyperarithmetic sets, HYP, give the weakest ω-model of second order arithmetic in which the completely determined Borel sets are at all well behaved, being closed under countable unions and intersections. However, in HYP, one can construct some "Borel" sets by choice arguments.

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▶ **Theorem 2** ([4]). *In HYP:*
1. *There is a completely determined Borel well-ordering of the reals.*
2. *Every completely determined Borel n-regular acyclic graph has a completely determined Borel 2-coloring.*
3. *The prisoners have a completely determined Borel winning strategy in the infinite prisoner hat game.*

We present what is known about the reverse mathematical strength of these and other weak theorems involving Borel sets.

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