Proof Pearl : Playing with the Tower of Hanoi Formally

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Abstract

The Tower of Hanoi is a typical example that is used in computer science courses to illustrate all the power of recursion. In this paper, we show that it is also a very nice example for inductive proofs and formal verification. We present some non-trivial results that have been formalised in the Coq proof assistant.

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Supplementary Material The code associated with this paper can be found at
Software (Code Repository): `https://github.com/thery/hanoi`
archived at `swh:1:dir:6f8e383f8827c6fd688847c23fab91a497bd60c7`

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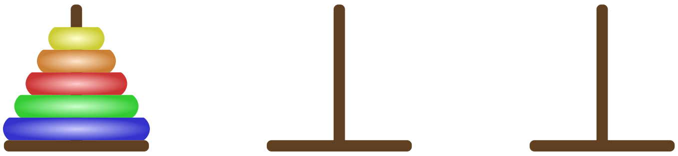
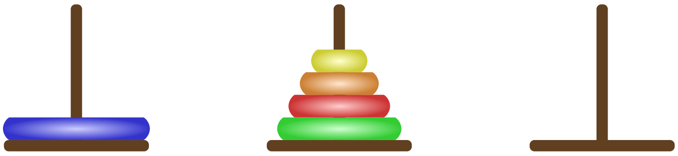
1 Introduction

The Tower of Hanoi is often used in computer science courses as example to teach recursion. The puzzle is composed of three pegs and some disks of different sizes. Here is a drawing of the initial configuration for five disks [1]:



Initially, all the disks are stacked in increasing size on the left peg. The goal is to move them to the right peg using the middle peg as an auxiliary peg. There are two rules. First, only one disk can be moved at a time and this disk must be on the top of its peg. Second, a larger disk can never be put on top of a smaller one.

A program $P^{3\mathrm{r}}$ that solves this puzzle can easily be written using recursion : one builds the program $P^{3\mathrm{r}}_{n+1}$ that solves the puzzle for $n+1$ disks using the program $P^{3\mathrm{r}}_{n}$ that solves the puzzle for n disks. The algorithm proceeds as follows. We first call $P^{3\mathrm{r}}_{n}$ to move the top-n disks to the middle peg using the right peg as the auxiliary peg.



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[1] We use macros designed by Martin Hofmann and Berteun Damman for our drawings.

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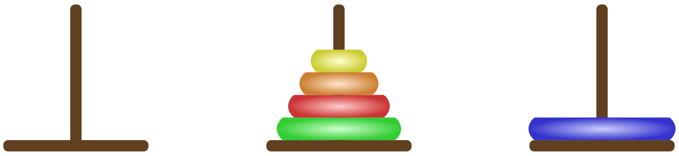
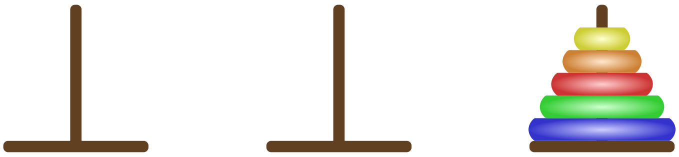
Then we move the largest disk to its destination.



Finally, we use P^{3r}_n to move the n disks on the intermediate peg to their destination using the left peg as the auxiliary peg.



This simple recursive algorithm is also optimal: it produces the minimal numbers of moves.

In this paper, we consider some variants of this puzzle (with three or four pegs, with some constraints on the moves one can perform) and explain how these puzzles and their optimal solution have been formalised.

2 General setting

We present the general settings of our formalisation that has been done in Coq using the SSReflect extension [5]. Then, we explain more precisely the variants of the puzzle that we have taken into consideration and how they have been formalised. We have tried as much as possible to be precise and present exactly what has been formalised. For this, we adopt the syntax of the Mathematical Component library [11] that we have been using. We also use the following typesetting convention. Notions that are present in the library are written using a `typewriter font` while our own definitions are written using a **roman font**.

Syntax Summary

We first give of an overview of the syntax of the Mathematical Component library. For a more detailed presentation, we refer the reader to [5]. The basic data structures we are using are natural numbers and lists. For natural numbers, they are implemented by an inductive type with two constructors : a zero (0) and a successor (`S`). There is a special notation to hide the application of the successor. So, n`.+1`, n`.+2` and n`.+3` represent (`S` n), (`S` (`S` n)) and (`S` (`S` (`S` n)) respectively. For example, the addition of two natural numbers is defined as

`Fixpoint` $m + n$ `:=` `if` m `is` m_1`.+1` `then` $(m_1 + n)$`.+1` `else` n.

For lists, `[::]` is the empty list, h `::` t is the list whose head is the element h and whose tail is the list t and l_1 `++` l_2 is the concatenation of the two lists l_1 and l_2. We also use the function `last` that returns the last element of a list. It has an extra argument to handle the case of empty list, `last` c `[::]` $= c$ and `last` c $(h$ `::` $t) =$ `last` h t. Finally, `[seq f` i `|` i `<-` l `]` represents the list built by applying the function f to all the elements of l.

In Coq, there is a distinction between a logical proposition (the type `Prop`) and a boolean expression (the type `bool`) but the library facilitates the bridge between the two by using the so-called small scale reflection [7]. As everything is finite in our application, we state most of our definitions in the boolean world. For example, we use the syntax `[rel` a b `|` P`]` to define a relation between two arbitrary objects a and b with P being a boolean expression where the variables a and b can occur. The boolean equality is written a `==` b, the inequality a `!=` b. The syntax for boolean operators is `!`b, b_1 `&&` b_2, b_1 `||` b_2 and b_1 `==>` b_2 for `NEG`, `AND`, `OR` and `IMP` respectively. There is a cumulative notation for the `AND` and the `OR` : `[&&` b_1, b_2, `...` `&` b_n`]` and `[||` b_1, b_2, `...` `|` b_n`]`. Finally, the syntax for the two boolean quantifications is `[forall` x `:` T, P`]` and `[exists` x `:` T, P`]`.

Disks

A disk is represented by its size. We use the type `I`$_n$ of natural numbers strictly smaller than n for this purpose.

`Definition` **disk** n := `I`$_n$.

In the following, we use the convention that a variable n will always represent a number of disks, and d a disk (an element of **disk** n).

As there is an implicit conversion from `I`$_n$ to natural number, the comparison of the respective size of two disks is simply written as $d_1 < d_2$. A minimal element (`ord0`) and a maximal element (`ord_max`) are defined for `I`$_n$ when n is not zero[2]. We use them to represent the smallest disk and the largest one.

`Definition` **sdisk** : **disk** n`.+1` := `ord0`.
`Definition` **ldisk** : **disk** n`.+1` := `ord_max`.

In particular, **ldisk** is the main actor of our proofs by simple induction on the number of disks, it represents the largest disk.

Pegs

We also use `I`$_n$ for pegs.

`Definition` **peg** k := `I`$_k$.

In the following, we use the convention that a variable k will always represent a number of pegs, and p a peg (an element of **peg** k).

We mostly use elements of **peg** 3 or **peg** 4 but some generic properties hold for **peg** k. An operation associated to pegs is the one that picks a peg that differs from an initial peg p_i and a destination peg p_j when possible[3]. It is written as $p[p_i, p_j]$. Generic and specific properties are derived from it. For example, we have:

---

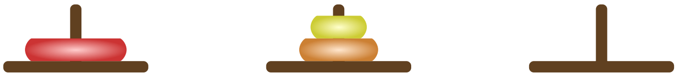
[2] `I`$_0$ is an empty type.
[3] If it is not possible (when working with **peg** 1 or **peg** 2), it returns `ord0`

```
Lemma opeg_sym (p1 p2 : peg k) : p[p1, p2] = p[p2, p1].
Lemma opegDl (p1 p2 : peg k.+3) : p[p1, p2] != p1.
Lemma opeg3Kl (p1 p2 : peg 3) : p1 != p2 → p[p[p1, p2], p1] = p2.
```

The symmetry is valid for every number of pegs. The property of being distinct is only fulfilled when we have more than two pegs. Finally, there is a version of the pigeon-hole principle for three pegs.

Configuration

If we start from the initial position and follow the rules, all the configurations we can encounter are such that on each peg, the disks are always ordered from the largest to the smallest. This means that just recording which disk is on which peg is enough to encode a configuration. To give an example, if we consider the following configuration with three disks and three pegs :



knowing that the small disk is on the second peg, the medium disk is on the second peg and the large disk is on the first peg is enough to recover the information that the small disk is on top of the medium one.

Configurations are then simply represented by finite functions from disks to pegs.

```
Definition configuration k n := {ffun disk n → peg k}.
```

From a technical point of view, using finite functions gives for free functional extensionality (which is not valid for usual functions in Coq). As a consequence, we can use the boolean equality `==` to test equality between two configurations.

A *perfect configuration* is a configuration where all the disks are on the same peg. So, it is encoded as the constant function:

```
  Definition perfect p := [ffun d ⇒ p].
```

It is written as $c[p]$ in the following, or $c[p, n]$ when the number of disks n is given explicitly in order to help typechecking.

Note that our encoding of configurations has the merit of covering exactly valid configurations. The price to pay is that we have to recover some natural notions. One of these notions is the predicate **on_top** d c that indicates that the disk d is on top of its peg in the configuration c. It is defined as follows:

```
Definition on_top (d : disk n) (c : configuration k n) :=
   [forall d1 : disk n, c d == c d1 ==> d ≤ d1].
```

It simply states that d is on top if every disk on the same peg as d has a size larger than d.

Most of the main results of the library are proved by some kind of inductive argument. In order to apply the inductive hypothesis formally, it is then central to be able to see a subset of a configuration composed of some disks and/or some pegs as a proper configuration with lesser disks and/or lesser pegs. As configurations are functions, it consists in restricting the domain and/or the codomain of the function. We call these operations transformations and usually also define their inverse. Here, we are only going to give an overview of the transformations we have been using : showing their type and what they are used for but not their definition. We refer to the library for the actual implementation.

The most common transformation that is used in proofs by simple induction is to see the configuration without the largest disk as a configuration with one disk less or conversely to extend a configuration with a new large disk that is put at the bottom on an arbitrary peg p to get a configuration with one disk more[4].

```
Definition cunliftr (c : configuration k n.+1) : configuration k n.
Definition cliftr (c : configuration k n) (p : peg k) : configuration k n.+1.
```

A dedicated notation ↓[c] is associated with **cunlift** c and a corresponding one, ↑[c]$_p$, for **clift** c p. A set of basic properties is derived for these operations. For example, we have:

```
Lemma cunliftrK (c : configuration k n.+1) : ↑[↓[c]]_(c ldisk) = c.
Lemma perfect_unliftr (p : peg k) : ↓[c[p, n.+1]] = c[p, n].
```

If we remove the largest disk then add it to the same peg, we get the same configuration. If we remove the last disk of a perfect configuration on peg p, we obtain a perfect configuration.

Similarly, in proofs by strong induction, one may need to take bigger piece of configuration. One way to do this is to be directed by the type (here the addition).

```
Definition clshift (c : configuration k (m + n)) : configuration k m.
Definition crshift (c : configuration k (m + n)) : configuration k n.
Definition cmerge (c1 : configuration k m) (c2 : configuration k n) :
   configuration k (m + n).
```

clshift builds a configuration with m disks taking the m largest disks of c while **crshift** builds a configuration with n disks taking the n smallest disks of c. As a matter of fact, ↓[c] and ↑[c]$_p$ are just defined as a special case of these operators for $m = 1$.

Other kinds of transformations that have been defined but not used frequently are:

```
Definition ccut (C : c ≤ n) (c : configuration k n) : configuration k c.
Definition ctuc (C : c ≤ n) (c : configuration k n) : configuration k (n - c).
Definition cset (s : {set (disk n)}) (c : configuration k n) : configuration k #|s|.
```

---

[4] Following, the `lift` and `unlift` operations for $\mathtt{I}_n$ of the Mathematical Component Library, we take the convention that lifting a configuration adds a disk.

```
Definition cset2 (sp : {set (peg k)}) (sd : {set (disk n)})
                 (c : configuration k n) : configuration #|sp| #|sd|.
```

ccut and **ctuc** are the equivalent of **crshift** and **clshift** but directed by the proof C rather than the type. **cset** considers a subset of disks but with the same number of peg (#$|s|$ is the cardinal of s). **cset2** is the most general and considers both a subset of disks and a subset of pegs.

Move

A move is defined as a relation between configurations. As we want to possibly add constraints on moves, it is parameterised by a relation r on pegs: $r\ p_1\ p_2$ indicates that it is possible to go from peg p_1 to peg p_2. Assumptions are usually added on the relation r (such as irreflexivity or symmetry) in order to prove basic properties of moves.

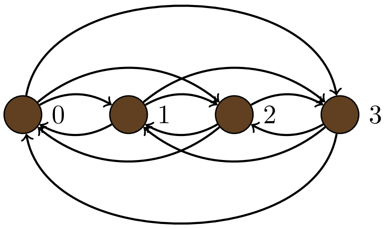
Here is the formal definition of a move:

```
Definition move : rel (configuration k n) :=
   [rel c1 c2 | [exists d1 : disk n,
     [&& r (c1 d1) (c2 d1), on_top d1 c1, on_top d1 c2 &
         [forall d2, d1 != d2 ==> c1 d2 == c2 d2]]]].
```

It simply states that there is a disk d_1 that fulfills 4 conditions:

- the move of d_1 from its peg in c_1 to its peg in c_2 is compatible with r;
- the disk d_1 is on top of its peg in c_1;
- the disk d_1 is on top of its peg in in c_2;
- it is the unique disk that has possibly moved.

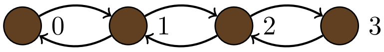
The standard puzzle has no restriction on the moves between pegs as long as we don't put a large disk on top of a small one. If we draw the possible moves as arrows between pegs, the picture for four pegs gives the following complete graph:



We call this version *regular*. It is denoted with the r exponent. For example, $P_5^{4\,\mathtt{r}}$ corresponds to the puzzle with four pegs and five disks with no restriction on the moves. Its associated relation **rrel** only enforces irreflexivity:

```
Definition rrel : rel (peg k) := [rel x y | x != y].
Definition rmove : rel (configuration k n) := move rrel.
```

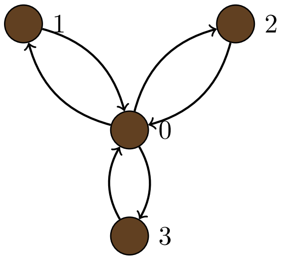
The first variant we consider is where one can only move from one peg to its neighbour. The picture for four pegs is the following:



This version is called *linear* and its associated infix is `l`. For example, the puzzle with four pegs and five disks with linear moves is written $P_5^{4\mathtt{l}}$. The corresponding relation **lrel** uses simple arithmetic to check neighbourhood :

```
Definition lrel : rel (peg k) := [rel x y | (x.+1 == y) || (y.+1 == x)].
Definition lmove : rel (configuration k n) := move lrel.
```

Finally the last variant we consider is the one where one central peg is the only one that can communicate with its outer pegs. A picture for four pegs gives



This version is called *star* and its associated exponent is `s`. For example, the puzzle with four pegs and five disks with star moves is written $P_5^{4\mathtt{s}}$. The corresponding relation **srel** uses multiplication to put the peg 0 in the center :

```
Definition srel : rel (peg k) := [rel x y | (x != y) && (x * y == 0)].
Definition smove : rel (configuration k n) := move srel.
```

Note that these categories may overlap when there are few pegs. For example, $P_n^{3\mathtt{l}}$ and $P_n^{3\mathtt{s}}$ correspond to the same puzzle.

2.1 Path and distance

Moves are defined as relations over configurations. So, we can see sequences of moves as paths on a graph whose nodes are the configurations. As configurations belong to a finite type, we can benefit from the elements of graph theory that are present in the Mathematical Component library. For example, (`rgraph move c`) returns the set of all the configurations that are reachable from c in one move, (`connect move` c_1 c_2) indicates that c_1 and c_2 can be connected through moves, or (`path move c cs`) gives that the sequence cs of configurations is a path that connects c with the last element of cs.

Now, all the transformations on configurations need to be lifted to paths. As distinct configurations may become identical when taking sub-parts, we first need to define an operation on sequences that removes repetitions.

```
Fixpoint rm_rep (A : eqType) (a : A) (s : seq A) :=
  if n is b :: s1 then
    if a == b then rm_rep b s1 else b :: rm_rep b s1
  else [::]
```

It is then possible to derive properties on paths. For example, we have:

```
Lemma path_clshift (c : configuration (m + n)) cs :
   path move c cs →
   path move (clshift c) (rm_rep (clshift c) [seq (clshift i) | i <- cs]).
```

As we want to show that some algorithms are optimal, the last ingredient we need is a notion of distance between configurations. Unfortunately, there is no built-in notion of distance in the Mathematical Component library, so we have to define one. For this, we first build recursively the function **connectn** that computes the set of elements that are connected with exactly n moves. Then, we can define the distance between two points x and y as the smallest n such as (**connectn** r n x y) holds. It is defined as (**gdist** r x y) and written as `d[`x`,` y`]`$_r$ in the following. From this definition, the triangle inequality is derived as :

```
Lemma gdist_triangular r x y z : d[x, y]_r ≤ d[x, z]_r + d[z, y]_r.
```

Finally, we introduce the notion of geodesic path: a path that realises the distance.

```
Definition gpath r x y p :=
   [&& path r x p, last x p == y & d[x, y]_r == size p].
```

Companion theorems are derived for these basic notions. For example, the following lemma shows that concatenation behaves well with respect to distances.

```
Lemma gdist_cat r x y p1 p2 :
   gpath r x y (p1 ++ p2) → d[x, y]_r = d[x, last x p1]_r + d[last x p1, y]_r.
```

3 Puzzles with three pegs

The proofs associated with the puzzles with three pegs are straightforward. They are done by induction on the number of disks inspecting the moves of the largest disk. What makes the simple induction work so well with three pegs is that when, from a configuration c, the largest disk moves from p_i to p_j, all the smaller disks in c are necessarily on the peg $p[p_i, p_j]$. So, they make a perfect configuration on which one can apply the inductive hypothesis using `↓[`c`]`.

3.1 Regular puzzle

It is easy to translate in Coq the algorithm described in the introduction. We write it as a recursive function that works on n disks and generates the sequence of configurations that goes from the configuration $c[p_1]$ to the configuration $c[p_2]$:

```
Fixpoint ppeg n p1 p2 :=
   if n is n1.+1 then
     let p3 := p[p1, p2] in
     [seq ↑[i]_p1 | i ← ppeg n1 p1 p3] ++ [seq ↑[i]_p2 | i ← c[p3] :: ppeg n1 p3 p2]
   else [::].
```

We pick an auxiliary peg p_3, appropriately lift the results of the two recursive calls and concatenate them to get the resulting path. It is easy to prove the basic properties of this function

```
Lemma size_ppeg n p1 p2 : size (ppeg n p1 p2) = 2^n − 1
Lemma last_ppeg n p1 p2 c : last c (ppeg n p1 p2) = c[p2].
Lemma path_ppeg n p1 p2 : p1 != p2 → path rmove c[p1] (ppeg n p1 p2).
```

Note that even for such simple theorems, there are some little subtleties. The **last_ppeg** does hold unconditionally even when the function returns the empty list. It is because this only happens when the configuration has no disk, so the theorem is an equality between elements of an empty type. Also, the **path_ppeg** needs the condition p_1 `!=` p_2 otherwise it will try to move the largest peg from p_1 to p_1 that is not valid since the relation **rmove** is irreflexive.

The key property of the **ppeg** function is that it builds a path of minimal size. As a matter of fact, we have proved something slightly stronger : it is the unique minimal path. In order to state this property, we make use of the extended comparison ($e_1 \leq e_2$ `?= iff` C) that is available in the library. It tells not only that e_1 is smaller than e_2 but also that the condition C indicates exactly when the comparison between e_1 and e_2 is an equality. This comparison comes with some algebraic rules. For example, the transitivity gives that if ($e_1 \leq e_2$ `?= iff` C_1) and ($e_2 \leq e_3$ `?= iff` C_2) hold, we have ($e_1 \leq e_3$ `?= iff` C_1 `&&` C_2). With this comparison, the uniqueness and minimality are stated as :

```
Lemma ppeg_min n p1 p2 cs :
   p1 != p2 → path rmove c[p1] cs → last c[p1] cs = c[p2] →
   2^n − 1 ≤ size cs ?= iff (cs == ppeg n p1 p2).
```

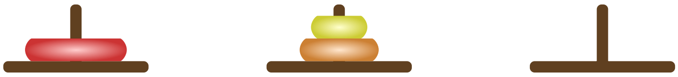
The proof is simply done by double induction (one on the size of cs and one on n) inspecting the moves of the largest disk in the sequence cs. Since p_1 differs from p_2, we know that the largest disk must move at least one time in cs. If it moves exactly once, the inductive hypothesis on n let us conclude directly. If it moves at least twice, the equality never holds. If the first two moves are on different pegs, adding the inductive hypothesis on $n-1$ twice plus the two moves of the largest disk gives us a path of size at least $2 \times (2^{n-1} - 1) + 2 = 2^n$. If the largest disk moves on one peg and then returns to the peg p_1, the path has some repetition, so the inductive hypothesis on the size of cs let us conclude.

From this theorem, we easily derive the corollary on the distance.

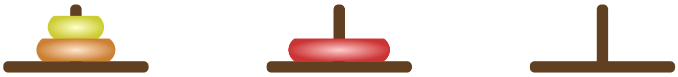
```
Lemma gdist_rhanoi3p n (p1 p2 : peg 3) :
   d[c[p1, n], c[p2, n]]_rmove = (2^n − 1) × (p1 != p2)
```

Note that we have used the automatic conversion from boolean to integer (`true` is 1 and `false` is 0) to include the case where p_1 and p_2 are the same peg.

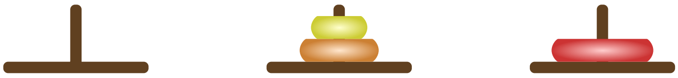
The last theorem only talks about going from a perfect configuration to another one. What about the distance between two arbitrary configurations c_1 and c_2? It seems natural to apply the same greedy strategy always trying to move the largest disk to its destination. The greedy algorithm would proceed as follows : If c_1 **ldisk** is equal to c_2 **ldisk**, we just perform the recursive call for smaller disks. If they are different, we perform a recursive call to move the smaller disks in c_1 to an intermediate peg, $p[c_1\ \mathbf{ldisk}, c_2\ \mathbf{ldisk}]$, then move the largest disk to its position in c_2, and finally perform another recursive call to move the smaller disks to their position in c_2. Unfortunately, this natural strategy is not optimal anymore. We can illustrate this with an example with 3 disks. Let us suppose that we try to go from the initial configuration:



to the final position:



The greedy strategy would move the largest disk only once and would have size seven, one for the largest disk, plus two times moving the two small disks. The optimal strategy instead moves the largest disks twice. It moves it first to the right peg:



and now it follows the greedy strategy and makes four extra moves to reach the target configuration. So, we have a solution of size five.

We have formalised exactly what the optimal solutions are :

- between an arbitrary configuration and a perfect configuration, the greedy strategy is always optimal;
- between two arbitrary configurations c_1 and c_2, the optimal strategy just needs to compare the one-jump solution with the two-jump solution only for the largest disk d such that c_1 d differs from c_2 d.

3.2 Linear puzzle

Implementing the greedy strategy for the linear puzzle is slightly more complicated. As we can move only between pegs that are neighbours, the largest disk may need to jump twice to reach its destination. But from the optimality point of view the situation is much simpler, the greedy strategy is always optimal. This is formally proved and we get the expected theorem about distance between perfect configurations:

`Lemma` **gdist_lhanoi3p** n $(p_1\ p_2\ :\ \mathbf{peg}\ 3)$:
$d[c[p_1, n], c[p_2, n]]_{\mathbf{lmove}} =$
`if` **lrel** p_1 p_2 `then` $(3^n - 1)/2$ `else` $(3^n - 1) \times (p_1$ `!=` $p_2)$

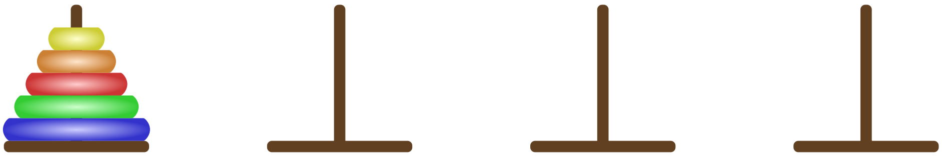
Note that as there are n disks and 3 pegs, there are 3^n possible configurations. This last theorem tells us that the solution that goes from the perfect configuration where all the disks are on the left peg to the perfect configuration where they are on the right peg visits all the configurations!

4 Puzzles with four pegs

Adding a peg changes completely the situation. If the previous simple recursive algorithm still works, it does not give anymore an optimal solution. The new strategy is implemented by the so-called Frame-Stewart algorithm. We explain it using the regular puzzle with four pegs. Then, we explain how the proofs about the distances for $P^{4\mathrm{r}}_n$ and $P^{4\mathrm{s}}_n$ have been formalised in Coq.

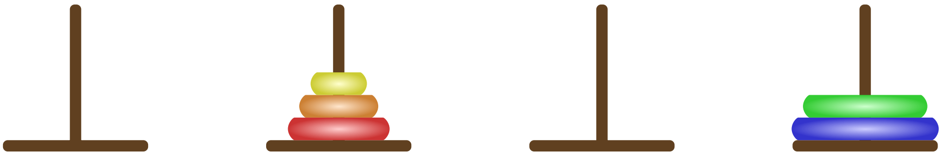
4.1 The Frame-Stewart Algorithm

Let us build, $P^{4\mathrm{r}}_n$, an algorithm that moves the disks from the leftmost peg to the rightmost one for the regular puzzles with four pegs.



We choose an arbitrary m smaller than n and use $P^{4\mathrm{r}}_m$ to move the top-m disks to an intermediate peg.

The remaining $n-m$ disks can now freely move except on this intermediate peg, so we can use P^{3r}_{n-m} to move them to their destination.



and reuse $P_{4\mathrm{r}k}$ to move the top-m disks to their destination.

Now, we can choose the parameter m as to minimise the number of moves. This means that we have $|P^{4\mathrm{r}}_n| = \min\limits_{m<n} 2|P^{4\mathrm{r}}_m| + (2^{n-m}-1)$. This strategy can be generalised to an arbitrary

number k of pegs, leading to the recurrence relation:

$$|P_n^{k\mathtt{r}}| = \min_{m<n} 2|P_m^{k\mathtt{r}}| + |P_{n-m}^{k-1\mathtt{r}}|.$$

Knowing if this general program is optimal is an open question but it has been shown optimal for $P_n^{4\mathtt{r}}$ and $P_n^{4\mathtt{s}}$. It is what we have formalised. Note that, from the proving point of view, the new strategy just seems to move from simple induction to strong induction with this new parameter m. As a matter of fact, if we look closely, this new strategy is just a generalisation of the previous one. If we apply it to three pegs, taking the minimum is trivial : there is only one way we can move $n-m$ pegs for the $P_{n-m}^{2\mathtt{r}}$ puzzle; it is by taking $m = n-1$.

4.2 Regular puzzle

The proof given in [2] that shows that the Frame-Stewart algorithm is optimal for the regular puzzle with four pegs is rather technical. As a matter of fact, this technicality was a motivation of our formalisation. The proof is very well written and very convincing but contains several cases which makes it difficult to assess its correctness. A formalisation ensures that no detail has been overlooked. As an anecdote, the WIKIPEDIA page [13] about the Tower of Hanoi in March 2020 was indicating that there was actually a journal paper [6] with a proof of the optimality of the Frame-Stewart algorithm for the regular puzzle with k pegs. When we started formalising this proof, we quickly got stuck on the formalisation of Corollary 3. We contacted the author that told us that it was a known flaw in the proof as documented in [4].

In what follows, we are only going to highlight the overall structure of the proof of optimality of $P_n^{4\mathtt{r}}$. We refer to the paper proof given in [2] for the details. The first step is to relate $|P_n^{4\mathtt{r}}|$ with triangular numbers. Following [2], we write $|P_n^{4\mathtt{r}}|$ as $\Phi(n)$. We introduce the notation Δn for the sum of the first n natural numbers :

$$\Delta n = \sum_{i \le n} i = \frac{n(n+1)}{2}.$$

A number n is triangular if it is a Δi for some i. By analogy to the square root, we introduce the triangular root ∇n :

$$\Delta(\nabla n) \le n < \Delta(\nabla(n+1)).$$

Now, we can give explicit formula for $\Phi(n)$:

$$\Phi(n) = \sum_{i<n} 2^{\Delta i}$$

It is relatively easy to show that the function Φ verifies the recurrence relation of the Frame-Stewart algorithm: $\Phi(n) = \min_{m<n} 2\Phi(m) + (2^{n-m} - 1)$. Then, what is left to be proved is that it is the optimal solution. The key ingredient of the proof is of course to find the right inductive invariant. This is done thanks to a valuation function Ψ that takes a finite set over the natural numbers and returns a natural number:

$$\Psi E = \max_{L \in \mathbb{N}} ((1-L)2^L - 1 + \sum_{n \in E} 2^{\min(\nabla n, L)})$$

The idea is that E will contain the disks we are interested in. If we consider the set $[n]$ of all the natural numbers smaller than n (i.e. we are interested by all the disks)

$$[n] = \{\mathbf{set}\ i \mid i < n\}$$

we get back the Φ function we are aiming at:

$$\Psi[n] = \frac{\Phi(n+1)-1}{2} = \frac{\sum_{i<n} 2^{\nabla(i+1)}}{2}$$

Now we can present the central theorem. We consider two configurations u and v of n disks and the four pegs p_0, p_1, p_2 and p_3. If v is such that there is no disk on pegs p_0 and p_1 and E is defined as

$$E = \{\textbf{set } i \mid \textbf{the disk } i \textbf{ is on the peg } p_0 \textbf{ in } u\}$$

the invariant is:

$$d[u,v]_{\textbf{rmove}} \geq \Psi E$$

The proof proceeds by strong induction on the number of disks. It examines a geodesic path p from u to v. If p_2 is the peg where the disk **ldisk** is in v (it cannot be p_0 nor p_1), it considers T the largest disk that was initially on the peg p_0 and visits at least one time the peg p_3. If such a disk does not exist, the inequality easily holds. Then, it considers inside the path p the configuration x_0 before which the disk T leaves the peg p_0 for the first time and the configuration x_3 in which the disk T reaches the peg p_3 for the first time. Similarly, it considers the configuration z_0 before which the disk n leaves the peg p_0 for the first time and the configuration z_2 before which the disk n reaches the peg p_2 for the last time. Examining the respective positions of x_0, x_3, z_0 and z_2 in p and applying some surgery on the configurations of the path p in order to fit the inductive hypothesis it concludes that the inequality holds in every cases.

The six-page long proof of the main theorem 2.9 in [2] translates to a 1000-line long Coq proof.

```
Lemma gdist_le_psi (u v : configuration 4 n) (p0 p2 p3 : peg 4) :
   [∧ p3 != p2, p3 != p0 & p2 != p0] → (codom v) \subset [:: p2 ; p3] →
   Ψ [set i | u i == p0] ≤ d[u, v]rmove.
```

From which, we easily derive the expected theorem:

```
Lemma gdist_rhanoi4 (n : nat) (p1 p2 : peg 4) :
   p1 != p2 → d[c[p1, n], c[p2, n]]rmove = Φ n.
```

5 Star puzzle

We first recall how to apply the Frame-Stewart algorithm to the star puzzle. Let us build the program $P_n^{4\mathrm{s}}$ that generates the moves between two perfect configurations: one on an outer peg p_i ($p_i \neq 0$) and the other on another outer peg p_j ($p_j \neq p_i \neq 0$). We first choose a parameter m and use $P_m^{4\mathrm{s}}$ to move the top-m disk to the third outer peg p_k ($p_k \neq p_j \neq p_i \neq 0$). Now, we use $P_{n-m}^{3\mathrm{s}}$ (which is identical to P_{n-m}^{31}) to move the $n-m$ from peg p_i to peg p_j and avoiding p_k. Finally, we use $P_k^{4\mathrm{s}}$ to move the top-m disk from peg p_k to peg p_j. This leads to the recurrence relation:

$$|P_n^{4\mathrm{s}}| = \min_{m<n} 2|P_m^{4\mathrm{s}}| + (3^{n-m} - 1).$$

Now, we have to find a mathematical object that verifies this recurrence relation. This time it is not the triangular numbers but the increasing sequence α_1 of the elements $2^i 3^j$. The first elements of this sequence are 1, 2, 3, 4, 6, 8, 9. If we define $S_1(n) = \sum_{i<n} \alpha_1(i)$, it is relatively easy to prove that

$$S_1(n) = \min_{m<n} 2S_1(m) + (3^{n-m} - 1)/2.$$

It follows that $2S_1$ verifies the recurrence relation.

Here, the proof of optimality is even more intricate, we just give an idea of the inductive invariant and how the proof proceeds. We refer to [3] for a detailed and very clear exposition of the proof. The first generalisation is to consider distance not only between two configurations but between $l+1$ configurations (i.e. a distance between u_0 and u_l passing through u_1, …, u_{l-1}): $\sum_{i<l} d[u_i, u_{i+1}]$. These intermediate configurations ($0 < i < l$) are alternating. If p_1, p_2 and p_3 are the outer pegs, the configuration u_i is supposed to have its disks on pegs p_2 and $a[p_1, p_3](i)$ where alternation is defined as

$$a[p_i, p_j](0) = p_i \qquad a[p_i, p_j](n+1) = a[p_j, p_i](n)$$

Taking into account this new parameter l, we need to lift S_1 to a parametrised function S_l.

$$S_l(n) = \min_{m<n} 2S_1(m) + l(3^{n-m} - 1)/2$$

and α_1 to $\alpha_l(n) = S_l(n+1) - S_l(n)$. Finally, we introduce the penality function β defined as

$$\beta_{n,l}(k) = \texttt{if } 1 < l \texttt{ and } k+1 = n \texttt{ then } \alpha_l(k) \texttt{ else } 2\alpha_1(k)$$

Given these definitions, the inductive invariant looks like:

```
Lemma main (p1 p2 p3 : peg 4) n l (u : {ffun I_l.+1 → configuration 4 n}) :
    p1 != p2 → p1 != p3 → p2 != p3
    p1 != p0 → p2 != p0 → p3 != p0 →
    (∀k, 0 < k < l → codom (u k) subset [:: p2; a[p1, p3] k]) →
    (S_[l] n).*2 ≤ sum_(i < l) d[u i, u i.+1]_smove +
                   sum_(k < n) (u ord0 k != p1) * β_[n, l] k +
                   sum_(k < n) (u ord_max k != a[p1, p3] l) * β_[n, l] k.
```

The proof is done by simple induction on n. It is then split in several cases depending on the number of elements i such that $u_i(\mathbf{ldisk}) = a[p_1, p_3](i)$. Furthermore, the inductive invariant has this alternating assumption. So, often, one needs to split the path in a bunch of alternating sub-paths in order to apply the inductive hypothesis on each of these sub-paths. This leads to a lower-bound where various values of $S_i(m)$ appear. Key properties of $S_l(n)$ (i.e. its convexity[5] in n and concavity[6] in l) are then used to derive simpler lower-bounds. For example, the combination of these two theorems

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[5] A function on natural numbers is convex if f and $n \mapsto f(n+1) - f(n)$ are both increasing.

[6] A function on natural numbers is concave if f is increasing and $n \mapsto f(n+1) - f(n)$ is decreasing.

```
Lemma concaveEk f i k : concave f → f (i + k) + f (i − k) ≤ 2 × (f i)
Lemma concave_dsum_alphaL_l n : concave (fun l → S_[l] n).
```

are often used to simplify inequalities dealing with the S_l function.

The paper proof of the **main** lemma is 15-page long and translates into 3500 lines of Coq proof script. As it contains several crossed references between cases, the formal proof of the **main** lemma is composed of 3 separate sub-lemmas plus one use of the "without loss of generality" tactic [8].

Finally, the Frame-Stewart algorithm gives an upper-bound to the distance and the **main** lemma applied with $l = 1$ gives a lower-bound. Altogether we get the expected theorem:

```
Lemma gdist_shanoi4 (n : nat) (p1 p2 : peg 4) :
   p1 != p2 → p1 != p0 → p1 != p0 → d[c[p1, n], c[p2, n]]_smove = (S_[1] n).*2.
```

6 Conclusion

We have presented a formalisation of the Tower of Hanoi with three and four pegs. Of course, we have only scratched the surface of what can be proved. We refer the reader to [9, 10] for an account of all the mathematical objects and programming techniques this simple puzzle can be linked to.

We started this formalisation as a mere exercise. Then, we got addicted and tried to prove more difficult results. This development relies heavily on the graph library for the modeling part. In that respect, this work bears some similarity with the formalisation of another puzzle, the mini-Rubik, where it is proved that the Rubik cube 2x2x2 can be solved in a maximum of 11 moves by looking at the diameter of the Cayley graph of all its configurations [12]. An attractive aspect of this formalisation is also that it uses very elementary mathematics. So, it is heavily testing the capability to do various flavour of inductive proofs and to manipulate combinations of big operators [1] such as $\max_{i \le n} F$, $\min_{i \le n} F$ and $\sum_{i \le n} F$. To give only one example, in order to prove the concavity of the function S_l we had to revisit the merge sort algorithm. It is usually given as a beginner exercise as a way to merge two sorted lists. Here, it is used to "merge" two increasing functions f and g in an increasing function $\textbf{fmerge}(f, g)$[7] and we had to prove that

$$\textbf{fmerge}(f, g)(n) = \max_{i \le n} \min(f(i), g(n - i))$$

When doing it formally, it is very easy to get lost in this kind of proof.

The main contribution of this work is the formal proofs about the distance between two perfect configurations for the 4 pegs versions. These results are relatively recent. We believe that our formal proofs are a natural companion to the paper proofs. These paper proofs are very technical. We have mechanically checked all the details. As a matter of fact, we have been using very little automation, so our formal proofs follow very closely the paper proofs. The main difference is that our formalisation used natural numbers only. So, we have tried

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[7] this is used for example to build the increasing sequence of 2^i3^j

to avoid as much as possible subtraction. In our formal setting, $(m - n) + n = m$ is a valid theorem only if we add the assumption that $n \leq m$. So an expression such as $a - b \leq c - d$ in the paper proof is translated into $a + d \leq b + c$ in the formal development.

Our formal proofs can, of course, be largely improved. To cite only one example, in the proof of the star puzzle with 4 pegs, lots of subcases are proved by simple symbolic manipulations with applications of some concavity theorems like **concaveEk**. These are currently done by hand. There is clearly room for automating all these steps.

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