Coboundary and Cosystolic Expansion from Strong Symmetry

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Abstract

Coboundary and cosystolic expansion are notions of expansion that generalize the Cheeger constant or edge expansion of a graph to higher dimensions. The classical Cheeger inequality implies that for graphs edge expansion is equivalent to spectral expansion. In higher dimensions this is not the case: a simplicial complex can be spectrally expanding but not have high dimensional edge-expansion. The phenomenon of high dimensional edge expansion in higher dimensions is much more involved than spectral expansion, and is far from being understood. In particular, prior to this work, the only known bounded degree cosystolic expanders were derived from the theory of buildings that is far from being elementary.

In this work we study high dimensional complexes which are *strongly symmetric*. Namely, there is a group that acts transitively on top dimensional cells of the simplicial complex [e.g., for graphs it corresponds to a group that acts transitively on the edges]. Using the strong symmetry, we develop a new machinery to prove coboundary and cosystolic expansion.

It was an open question whether the recent elementary construction of bounded degree spectral high dimensional expanders based on coset complexes give rise to bounded degree cosystolic expanders. In this work we answer this question affirmatively. We show that these complexes give rise to bounded degree cosystolic expanders in dimension two, and that their links are (two-dimensional) coboundary expanders. We do so by exploiting the strong symmetry properties of the links of these complexes using a new machinery developed in this work.

Previous works have shown a way to bound the co-boundary expansion using strong symmetry in the special situation of "building like" complexes. Our new machinery shows how to get coboundary expansion for *general* strongly symmetric coset complexes, which are not necessarily "building like", via studying the (Dehn function of the) presentation of the symmetry group of these complexes.

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Extended abstract

High dimensional expansion is a vibrant emerging field that has found applications to PCPs [5] and property testing [10], to counting problems and matroids [1], to list decoding [4], and recently to a breakthrough construction of decodable quantum error correcting codes that outperform the state-of-the art previously known codes [7]. We refer the reader to [16] for a recent (but already outdated) survey.

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The term high dimensional expander means a simplicial complex that has expansion properties that are analogous to expansion in a graph. Nevertheless, the question of what is a high dimensional expander is still unclear. There is a spectral definition of high dimensional expanders that generalizes the spectral definition of expander graphs and a geometrical/topological definition that generalizes the notion of edge expansion (or Cheeger constant) of a graph. For a graph the spectral and the geometric definitions of expansion are known to be equivalent (via the celebrated Cheeger inequality) while in high dimensions the spectral and geometric definitions are known to be NOT equivalent (see [9, Theorem 4] and [19]).

The aim of this paper is to present elementary constructions of new families of 2-dimensional simplicial complexes with high dimensional edge expansion, and in particular, of new elementary *bounded degree* families of cosystolic expanders (see exact definition below). The question of giving an elementary construction of a family of bounded degree spectrally high dimensional expanders got recently a satisfactory answer. Namely, it was understood that such a family needs to obey a specific local spectral criterion and in [11] we used this understanding in order to construct elementary families of high dimensional *spectrally* expanding families (prior non-elementary constructions were known). Here, we further study the examples of [11], and show that they also give rise to bounded degree cosystolic expanders:

▶ **Theorem 1** (New cosystolic expanders, Informal, see also Theorem 22)**.** *For every large enough odd prime power q, the family of 2-skeletons of the 3-dimensional local spectral expanders constructed in [11] using elementary matrices over $\mathbb{F}_q[t]$ is a family of bounded degree cosystolic expanders.*

Prior to this work, the known examples of bounded degree cosystolic expanders arose from the theory of Bruhat-Tits buildings and were far from being elementary.

Relying on the work of the first named author and Evra (see Theorem 11 below), the proof of this Theorem boils down to proving that the links of our construction are coboundary expanders and that their coboundary expansion can be bounded independently of q (i.e., that the coboundary does not deteriorate as q increases). Thus, the real problem is bounding the coboundary expansion of the links. This goal is achieved utilizing the fact that the links are strongly symmetric coset complexes.

Coboundary expansion for strongly symmetric (coset) complexes

We call a simplicial complex strongly symmetric if it has a symmetry group acting transitively on top dimensional simplices. As noted above, our problem is to show that the links in our examples are coboundary expanders. In the graph setting, there is a classical Theorem (see Theorem 12 below) stating that for a strongly symmetric graph the Cheeger constant can be bounded from below by $\frac{1}{2D}$, where D denotes the diameter of the graph.

We generalize this idea: we define a high dimensional notion of radius and show that for strongly symmetric complexes, this radius can be used to bound the coboundary expansion. We then show that this radius can be bounded using filling constants of the complex. These ideas of bounding the coboundary expansion for symmetric complexes using filling constants already appeared implicitly in Gromov's work [8] and in the work of Lubotzky, Meshulam and Mozes [17]. However, these previous works considered the setting of spherical buildings and "building-like complexes" and thus bounding the filling constants in these examples were relatively simple due to the existence of apartments in the building (or "apartment-like" sub-complexes in "building-like" complexes). In our setting, we consider a more general situation (not assuming "apartment-like" sub-complexes) and thus bounding the filling constants becomes a much harder task.

What helps to solve this harder problem of bounding the filling constants is working with strongly symmetric *coset complexes* (see Definition 16). We note that this is not a very restrictive assumption - under some mild assumptions, every strongly symmetric complex is a coset complex (see proof in [13]). For a coset complex one can fully reconstruct the complex via its symmetry group and its subgroup structure. Thus every geometrical/topological property of a coset complex (including coboundary expansion) is encoded in some way in the presentation of its symmetry group. Using this philosophy, we are able to prove a bound for filling constants for two dimensional coset complex in terms of the presentation of its symmetry group (namely, in terms on its Dehn function - see definition below). Thus, for two dimensional coset complexes, we get a bound on the coboundary expansion in terms of presentation-theoretic properties of the symmetry group.

Coboundary expansion of the links in our construction

If follows from our work described above that in order to show that the links in our construction are coboundary expanders, we should verify a presentation-theoretic property for their symmetry group (namely, to bound its Dehn function). Luckily for us, the symmetry group of the links in our construction is a generalization of the group of unipotent groups over finite fields. For the finite field case, the presentation of these unipotent groups was studied by Biss and Dasgupta [2]. Using their ideas, we are able to show that the symmetry groups of links in our construction fulfil the presentation-theoretic condition that allows us to bound their coboundary expansion. Namely, we prove the following:

▶ **Theorem 2** (New coboundary expanders, Informal, see also Theorem 21)**.** *For every odd prime power q, the links of the 3-dimensional local spectral expanders constructed in [11] using elementary matrices over $\mathbb{F}_q[t]$ are coboundary expanders and their coboundary expansion can be bounded from below independent of q.*

Simplicial complexes

An n-dimensional simplicial complex X is a hypergraph whose maximal hyperedges are of size $n+1$, and which is closed under containment. Namely, for every hyperedge τ (called a face) in X, and every $\eta \subset \tau$, it must be that η is also in X. In particular, $\emptyset \in X$. For example, a graph is a 1-dimensional simplicial complex. Let X be a simplicial complex, we fix the following terminology/notation:

1. X is called *pure n-dimensional* if every face in X is contained in some face of size $n+1$.
2. The set of all k-faces (or k-simplices) of X is denoted $X(k)$, and we will be using the convention in which $X(-1) = \{\emptyset\}$.
3. For $0 \leq k \leq n$, the k-skeleton of X is the k-dimensional simplicial complex $X(0) \cup X(1) \cup ... \cup X(k)$. In particular, the 1-skeleton of X is the graph whose vertex set is $X(0)$ and whose edge set is $X(1)$.
4. For a simplex $\tau \in X$, the link of τ, denoted X_τ is the complex
$$\{\eta \in X : \tau \cup \eta \in X, \tau \cap \eta = \emptyset\}.$$
We note that if $\tau \in X(k)$ and X is pure n-dimensional, then X_τ is pure $(n-k-1)$-dimensional.
5. A family of pure n-dimensional simplicial complexes $\{X^{(s)}\}_{s \in \mathbb{N}}$ is said to have bounded degree if there is a constant $L > 0$ such that for every $s \in \mathbb{N}$ and every vertex v in $X^{(s)}$, v is contained in at most L n-dimensional simplices of $X^{(s)}$.

The coboundary/cosystolic expansion and high order Cheeger constants

Let us recall the geometric notion of expansion in graphs known as the edge expansion or Cheeger constant of a graph:

▶ **Definition 3** (Cheeger constant of a graph). *For a graph $X = (V, E)$:*

$$h(X) := min_{A \neq \emptyset, V} \frac{|E(A, \bar{A})|}{min\{w(A), w(\bar{A})\}},$$

where for a set of vertices $U \subsetneq V$, $w(U)$ denotes is the sum of the degrees of the vertices in U.

The generalization of the Cheeger constant to higher dimensions originated in the works of Linial, Meshulam and Wallach ([15], [18]) and independently in the work of Gromov ([8]) and is now known as *coboundary expansion*. Later, a weaker variant of high dimensional edge expansion known as *cosystolic expansion* arose in order to answer questions regarding topological overlapping.

In order to define coboundary and cosystolic expansion, we also need some terminology. Let X be an n-dimensional simplicial complex. Fix the following notations/definitions:

1. The space of k-cochains denoted $C^k(X) = C^k(X, \mathbb{F}_2)$ is the $\mathbb{F}_2$-vector space of functions from $X(k)$ to $\mathbb{F}_2$.
2. The *coboundary map* $d_k : C^k(X, \mathbb{F}_2) \rightarrow C^{k+1}(X, \mathbb{F}_2)$ is defined as:
$$d_k(\phi)(\sigma) = \sum_{\tau \subset \sigma, |\tau| = |\sigma| - 1} \phi(\tau),$$
3. The spaces of k-coboundaries and k-cocycles are subspaces of $C^k(X)$ defined as:
$B^k(X) = B^k(X, \mathbb{F}_2) = \text{Image}(d_{k-1}) =$ the space of k-coboundaries.
$Z^k(X) = Z^k(X, \mathbb{F}_2) = \text{Ker}(d_k) =$ the space of k-cocycles.
4. The function $w : \bigcup_{k=-1}^{n} X(k) \rightarrow \mathbb{R}_+$ is defined as
$$\forall \tau \in X(k), w(\tau) = \frac{|\{\sigma \in X(n) : \tau \subseteq \sigma\}|}{\binom{n+1}{k+1}|X(n)|}.$$
We note that $\sum_{\tau \in X(k)} w(\tau) = 1$.
5. For every $\phi \in C^k(X)$, $w(\phi)$ is defined as
$$w(\phi) = \sum_{\tau \in \text{supp}(\phi)} w(\tau).$$
6. For every $0 \leq k \leq n - 1$, define the following k-expansion constants:
$$\text{Exp}_b^k(X) = \min \left\{ \frac{w(d_k \phi)}{\min_{\psi \in B^k(X)} w(\phi + \psi)} : \phi \in C^k(X) \setminus B^k(X) \right\}.$$
$$\text{Sys}^k(X) = \min \left\{ w(\psi) : \psi \in Z^k(X) \setminus B^k(X) \right\},$$
and
$$\text{Exp}_z^k(X) = \min \left\{ \frac{w(d_k \phi)}{\min_{\psi \in Z^k(X)} w(\phi + \psi)} : \phi \in C^k(X) \setminus Z^k(X) \right\}.$$

After these notations, we can define coboundary/cosystolic expansion:

▶ **Definition 4** (Coboundary expansion). *Let $\varepsilon > 0$ be a constant. We say that X is an ε-coboundary expander if for every $0 \leq k \leq n-1$, $\text{Exp}_b^k(X) \geq \varepsilon$.*

▶ Remark 5. We leave it for the reader to verify that in the case where X is a graph, i.e., the case where $n = 1$, $\text{Exp}_b^0(X)$ is exactly the Cheeger constant of X. Thus, we think of $\text{Exp}_b^k(X)$ as the k-dimensional Cheeger constant of X.

▶ **Definition 6** (Cosystolic expansion). *Let $\varepsilon > 0, \mu > 0$ be constants and X an n-dimensional simplicial complex X. We say that X is a (ε, μ)-cosystolic expander if for every $0 \leq k \leq n-1$, $\text{Exp}_z^k(X) \geq \varepsilon$ and $\text{Sys}^k(X) \geq \mu$.*

▶ Remark 7. We note that if $\text{Exp}_b^k(X) > 0$, then it can be shown that $B^k(X) = Z^k(X)$ and thus $\text{Exp}_b^k(X) = \text{Exp}_z^k(X)$. However, there are examples of simplicial complexes with $\text{Exp}_b^k(X) = 0$ and $\text{Exp}_z^k(X) > 0, \text{Sys}^k(X) > 0$.

As in expander graphs, we are mainly interested in a family of bounded degree cosystolic expanders (and not a single complex that is a cosystolic expander):

▶ **Definition 8** (A family of bounded degree cosystolic expanders). *A family of n-dimensional simplicial complexes $\{Y^{(s)}\}_{s \in \mathbb{N}}$ is a family of bounded degree cosystolic expanders if:*
- *The number of vertices of $Y^{(s)}$ tends to infinity with s.*
- *$\{Y^{(s)}\}_{s \in \mathbb{N}}$ has bounded degree.*
- *There are universal constants $\varepsilon > 0, \mu > 0$ such that for every s, $Y^{(s)}$ is a (ε, μ)-cosystolic expander.*

▶ Remark 9. The motivation behind the definition of a family of cosystolic expanders is to proved a family of bounded degree complexes that have the topological overlapping property (see [13] for the exact definition).

The Evra-Kaufman criterion for cosystolic expansion

In [6], Evra and the first named author gave a criterion for cosystolic expansion. In order to state this criterion, we will need the following definition:

▶ **Definition 10** (Local spectral expansion). *For $\lambda \geq 0$, a pure n-dimensional simplicial complex X is called a (one-sided) λ-local spectral expander if for $-1 \leq k \leq n-2$ and every $\tau \in X(k)$, the one-skeleton of X_τ is a connected graph and the second largest eigenvalue of the random walk on the one-skeleton of X_τ is less or equal to λ.*

The idea behind the Evra-Kaufman criterion for cosystolic expansion is the following: We can deduce cosystolic expansion from local spectral expansion and local coboundary expansion (i.e., coboundary expansion in the links) given that the local spectral expansion is "strong enough" so it "beats" the local coboundary expansion. More formally:

▶ **Theorem 11** ([6, Theorem 1] Evra-Kaufman criterion for cosystolic expansion). *For every $\varepsilon' > 0$ and $n \geq 3$ there are $\mu(n, \varepsilon') > 0, \varepsilon(n, \varepsilon') > 0$ and $\lambda(n, \varepsilon') > 0$ such that for every pure n-dimensional simplicial complex if*
- *X is a λ-local spectral expander.*
- *For every $0 \leq k \leq n-2$ and every $\tau \in X(k)$, X_τ is a ε'-coboundary expander.*

Then the $(n-1)$-skeleton of X is a (ε, μ)-cosystolic expander.

Thus, in order to prove cosystolic expansion in examples, we should verify two things: local spectral expansion and coboundary expansion in the links. In our examples from [11] described below, local spectral expansion is already known and we are left with proving coboundary expansion for the links. In order to do so, we will develop machinery to prove coboundary expansion for symmetric complexes of a special type called coset complexes.

Coboundary expansion for strongly symmetric simplicial complexes

As noted above, unlike the case of graphs, in simplicial complexes a high dimensional version of Cheeger inequality does not hold. Thus, there is a need to develop machinery in order to prove coboundary expansion that does not rely on spectral arguments. For graphs such machinery is available, under the assumptions that the graph has a large symmetry group. A discussion regarding the Cheeger constant of symmetric graphs appear in [3, Section 7.2] and in particular, the following Theorem is proven there:

▶ **Theorem 12** ([3, Theorem 7.1]). *Let X be a finite connected graph such that there is a group G acting transitively on the edges of X. Denote $h(X)$ to be the Cheeger constant of X and D to be the diameter of X. Then $h(X) \geq \frac{1}{2D}$.*

▶ Remark 13. Note that the inequality stated in the Theorem does not hold without the assumption of symmetry. For instance, let X_N be the graph that is the ball of radius N in the 3-regular infinite tree. Then the diameter of X is $2N+1$ and $h(X_N)$ is of order $O(\frac{1}{2^N})$.

In this paper, using the ideas of [8] and [17], we prove a generalization of Theorem 12 to the setting of (strongly) symmetric simplicial complexes. We first define the notion of strongly symmetric simplicial complexes.

▶ **Definition 14** (Strongly symmetric complex). *A simplicial complex X is called* strongly symmetric *if there is a group that acts simply transitively on its top dimensional faces. E.g., For graphs (one dimensional complexes) we require a group that acts simply transitively on the edges.*

We then define a high dimensional notion of radius which we call a cone radius, but this definition is a little technical and thus appears in Section 2 (see Definition 28). We then prove the following:

▶ **Theorem 15** (Informal, see Theorem 30 for the formal statement). *Let X be a strongly symmetric simplicial complex. If the k-dimensional (cone) radius of X is bounded by D, then $\text{Exp}_b^k(X) \geq \frac{1}{\binom{n+1}{k+1} D}$, i.e., the k-coboundary expansion is bounded from below as a function of the k-th radius.*

Bounding the high dimensional radius for coset complexes

By Theorem 15, in order to prove coboundary expansion for strongly symmetric complexes, it is enough to bound their high dimensional radius. Following the ideas of Gromov [8], we bound the radius by bounding certain filling constants, that we will not define here. In order to bound these filling constants and thus the high dimensional radius, we will assume that our strongly symmetric complex is of a special type, namely that it is a coset complex:

▶ **Definition 16** (Coset complex). *Given a group G with subgroups $K_{\{i\}}, i \in \mathrm{I}$, where I is a finite set. The coset complex $X = X(G, (K_{\{i\}})_{i \in \mathrm{I}})$ is a simplicial complex defined as follows:*

1. *The vertex set of X is composed of disjoint sets $S_i = \{gK_{\{i\}} : g \in G\}$.*
2. *For two vertices $gK_{\{i\}}, g'K_{\{j\}}$ where $i, j \in \mathrm{I}, g, g' \in G$, $\{gK_{\{i\}}, g'K_{\{j\}}\} \in X(1)$ if $i \neq j$ and $gK_{\{i\}} \cap g'K_{\{j\}} \neq \emptyset$.*
3. *The simplicial complex X is the clique complex spanned by the 1-skeleton defined above, i.e., $\{g_0 K_{\{i_0\}}, ..., g_k K_{\{i_k\}}\} \in X(k)$ if for every $0 \leq j, j' \leq k$, $g_j K_{\{i_j\}} \cap g_{j'} K_{\{i_{j'}\}} \neq \emptyset$.*

Although this Definition may seem daunting at first, we note that it is very natural in examples. Namely, in [13] we shows that under some mild assumptions, strongly symmetric simplicial complexes are actually coset complexes.

As noted above, for coset complexes, every property of the complex should be reflected in some way in its symmetry group and its subgroup structure. Following this philosophy, we prove that for coset complexes, the 0-th and 1-th dimensional coboundary expansion can be bounded using the presentation of the group from which the complex arose.

In order to describe our result, we recall some definitions from group theory. Given a group G, a generating set $S \subseteq G$ is a set of elements of G such that every element in G can be written as a finite product (or sum if G is commutative) of elements of S. One can always take $S = G$, but usually one can make due with a smaller set. For example, for the group G of addition of integers modulo n, $G = (\mathbb{Z}/n\mathbb{Z}, +)$, one can take $S = \{\pm 1\}$. Given a group G with a generating set S, a word with letters in S is called trivial if it equal to the identity. For example, in $G = (\mathbb{Z}/n\mathbb{Z}, +)$ with $S = \{\pm 1\}$, the words $1 + 1 + (-1) + (-1)$ and $1 + ... + 1$ (n summands) $= n \cdot 1$ are trivial.

We say that a group G has a presentation $G = \langle S | R \rangle$, if S is a generating set of G and R is a set of trivial words called relations such that every trivial word in G can be written using the words in $R \cup \{ss^{-1}, s^{-1}s : s \in S\}$ (allowing products, conjugations and inverses). Again, one can always take $S = G \setminus \{e\}$ and R to be the entire multiplication table of G, i.e., all the words of the form $g_1 g_2 g_3^{-1} = e$, where $g_1 g_2 = g_3$. However, in concrete examples, one can usually make due with fewer generators and relations. For example, for the group $G = (\mathbb{Z}/n\mathbb{Z}, +)$ it is sufficient to take $S = \{\pm 1\}$ and the single relation $n \cdot 1$. We note that it is not always easy to determine if a set of relations gives a presentation of G.

Given a presentation $G = \langle S | R \rangle$, the Dehn function for this presentation is a function $\text{Dehn} : \mathbb{N} \to \mathbb{N}$ such that $\text{Dehn}(m)$ describes how many elements of $R \cup \{ss^{-1}, s^{-1}s : s \in S\}$ does one need to write a trivial word in G of length $\leq m$. With this terminology, we prove the following:

▶ **Theorem 17.** *Let G be a finite group with subgroups $K_{\{i\}}, i \in \{0, 1, 2\}$. Denote $X = X(G, (K_{\{i\}})_{i \in \{0,1,2\}})$. Assume that G acts strongly transitively on X.*

For every $i \in \{0, 1, 2\}$, denote R_i to be all the non-trivial relations in the multiplication table of $K_{\{i\}}$, i.e., all the relations of the form $g_1 g_2 g_3 = e$, where $g_1, g_2, g_3 \in K_{\{i\}} \setminus \{e\}$. Assume that $G = \langle \bigcup_i K_{\{i\}} | \bigcup_i R_i \rangle$ and let Dehn *denote the Dehn function of this presentation.*

Then:

1. *For*

$$N_0' = 1 + \max_{g \in G} \min \left\{ l : g = g_1 ... g_l \text{ and } g_1, ..., g_l \in \bigcup_i K_{\{i\}} \right\},$$

it holds that $\text{Exp}_b^0(X) \geq \frac{1}{3N_0'}$.

2. *There is a universal polynomial $p(x, y)$ independent of X such that*

$$\text{Exp}_b^1(X) \geq \frac{1}{3p(2N_0' + 1, \text{Dehn}(2N_0' + 1))}.$$

Our construction

So far, we described general tools that we developed in order to prove coboundary and cosystolic expansion. Now we will describe our construction from [11] on which we aim to apply these tools.

In [11], we used coset complexes to construct n-dimensional spectral expanders. Below, we only describe the construction for $n = 3$: Fix $s \in \mathbb{N}, s > 4$ and q be a prime power. Denote $G_q^{(s)}$ to be the group of 4×4 matrices with entries in $\mathbb{F}_q[t]/\langle t^s \rangle$ generated by the set

$$\{e_{1,2}(a + bt), e_{2,3}(a + bt), e_{3,4}(a + bt), e_{4,1}(a + bt) : a, b \in \mathbb{F}_q\}.$$

For $0 \leq i \leq 2$, define $H_{\{i\}}$ to be the subgroup of $G_q^{(s)}$ generated by

$$\{e_{j,j+1}(a+bt), e_{4,1}(a+bt) : a,b \in \mathbb{F}_q, 1 \leq j \leq 3, j \neq i+1\}$$

and define $H_{\{3\}}$ to be the subgroup of $G^{(s)}$ generated by

$$\{e_{1,2}(a+bt), e_{2,3}(a+bt), e_{3,4}(a+bt) : a,b \in \mathbb{F}_q\}.$$

Denote $X_q^{(s)} = X(G_q^{(s)}, (H_{\{i\}})_{i\in\{0,\dots,3\}})$ to be the coset complex as defined above.

The main result of [11] applied to $\{X_q^{(s)}\}_{s>4}$ above can be summarized as follows:

1. The family $\{X_q^{(s)}\}_{s>4}$ has bounded degree (that depends on q).
2. The number of vertices of $X_q^{(s)}$ tends to infinity with s.
3. For every s, $X_q^{(s)}$ is $\frac{1}{\sqrt{q}-3}$-local spectral expander.

In light of Theorem 17, we will also need some facts regrading the links of $X_q^{(s)}$. We give the following explicit description of the links in our construction: We note that for every fixed q it holds that there is a complex X such that for every $s > 4$ and every vertex $v \in X_q^{(s)}$, there is a coset complex denoted $X_{link,q}$ such that the link of v is isomorphic to $X_{link,q}$ (all the links are isomorphic).

The complex $X_{link,q}$ can be described explicitly as follows: Denote the group $G_{link,q}$ to be a subgroup of 4×4 invertible matrices with entries in $\mathbb{F}_q[t]$ in generated by the set $\{e_{i,i+1}(a+bt) : a,b \in \mathbb{F}_q, 1 \leq i \leq 4\}$. More explicitly, an 4×4 matrix A is in $G_{link,q}$ if and only if

$$A(i,j) = \begin{cases} 1 & i=j \\ 0 & i>j \\ a_0 + a_1 t + \dots + a_{j-i}t^{j-i} & i<j, a_0,\dots,a_{j-i} \in \mathbb{F}_q \end{cases},$$

(observe that all the matrices in G are upper triangular).

For $0 \leq i \leq 3$, define a subgroup $K_{\{i\}} < G$ as

$$K_{\{i\}} = \langle e_{j,j+1}(a+bt) : j \in \{1,\dots,4\} \setminus \{i+1\}, a,b \in \mathbb{F}_q \rangle.$$

Define $X_{link,q}$ to be the coset complex $X_{link,q} = X(G_{link,q}, (K_{\{i\}})_{i\in\{0,1,2\}})$. As noted above, for every $s > 4$, all the 2-dimensional links of $X_q^{(s)}$ are isomorphic to $X_{link,q}$. Also,

▶ **Theorem 18** ([12, Theorems 2.4, 3.5])**.** *The complex $X_{link,q}$ above is strongly symmetric, namely the group $G_{link,q}$ of unipotent matrices described above acts transitively on the triangles of $X_{link,q}$.*

New coboundary and cosystolic expanders

Finally, we describe how the general machinery we developed can be applied in our construction.

First, by applying Theorem 11 on the family $\{X_q^{(s)}\}_{s\in\mathbb{N}}$ yields the following Corollary:

▶ **Corollary 19.** *Let $\{X_q^{(s)}\}_{s\in\mathbb{N}}$ be the family of n-dimensional simplicial complexes from [11]. Assume there is a constant $\varepsilon' > 0$ such that for every odd q, every s, every $0 \leq k \leq n-2$ and every $\tau \in X(k)$, X_τ is a ε'-coboundary expander. Denote $Y_q^{(s)}$ to be the $(n-1)$-skeleton of $X_q^{(s)}$. Then for any sufficiently large odd prime power q, the family $\{Y_q^{(s)}\}_{s\in\mathbb{N}}$ is a family of bounded degree cosystolic expanders.*

Thus, by this Corollary, in order to prove Theorem 1 it is enough to show that for every odd q, there is a constant $\varepsilon' > 0$ such that for every odd q and every $s \in \mathbb{N}$, the 2-skeleton of the link of every vertex v in $X_q^{(s)}$ is a ε' coboundary expander.

As we noted, the links are strongly transitive coset complexes which we denoted $X_{q,link}$ and described explicitly above. By Theorem 17, in order to bound the coboundary expansion of the links, we need to consider the presentation of their symmetry group $G_{link,q}$ defined above. Generalizing on the work of Biss and Dasgupta [2] we prove the following:

▶ **Theorem 20.** *For any prime power q denote $G_{q,link}, K_{\{0\}}, K_{\{1\}}, K_{\{2\}}$ as above and for every $i \in \{0,1,2\}$, denote R_i to be all the non-trivial relations in the multiplication table of $K_{\{i\}}$. Then*

- $$\sup_{q \text{ odd prime power}} \left(\max_{g \in G_{q,link}} \min \left\{ l : g = g_1 ... g_l \text{ and } g_1, ..., g_l \in \bigcup_i K_{\{i\}} \right\} \right) < \infty.$$
- *For every odd q it holds that*

  $$G_{q,link} = \langle \bigcup_i K_{\{i\}} | \bigcup_i R_i \rangle$$

  and the Dehn function of this presentation is bounded independently of q.

This Theorem combined with Theorem 17 gives:

▶ **Theorem 21** (First Main Theorem - new explicit two dimensional coboundary expanders)**.** *For every odd prime power q, $X_{link,q}$ is a coboundary expander and* $\mathrm{Exp}_0(X), \mathrm{Exp}_1(X)$ *are bounded from below by a constant that is independent of q.*

Applying Corollary 19 it follows that:

▶ **Theorem 22** (Second Main Theorem - elementary two dimensional bounded degree cosystolic expanders)**.** *Let $s \in \mathbb{N}, s > 4$ and q be a prime power and $X_q^{(s)}$ as above. For every s, let $Y_q^{(s)}$ be the 2-skeleton of $X_q^{(s)}$, i.e., the 2-dimensional complex $Y_q^{(s)} = X_q^{(s)}(0) \cup X_q^{(s)}(1) \cup X_q^{(s)}(2)$. For any sufficiently large odd prime power q, the family $\{Y_q^{(s)}\}_{s \in \mathbb{N}, s>4}$ is a family of bounded degree cosystolic expanders.*

Technical details of the paper

In Section 1, we give the precise definitions and notations regarding (co)homology. In Section 2, we define the cone radius of a simplicial complex and prove that for symmetric simplicial complexes, the cone radius can be used to bound the coboundary expansion.

The other technical details of the paper are available in its Arxiv version [13]. Namely, in [13] the reader can find the following:

- A precise definition of the filling constants of a simplicial complex and a proof that the filling constants of the complex can be used to bound the cone radius.
- A review the idea of coset complexes and a proof that our assumption of strong symmetry combined with some extra assumptions on a complex imply that it is a coset complex.
- A bound on the first two filling constants for a coset complex in terms of algebraic properties of the presentation of the group and subgroups from which it arises.
- New examples of coboundaries expanders arising from coset complexes of unipotent groups.
- New examples of bounded degree cosystolic and topological expanders.
- A proof that the existence of a cone function is equivalent to the vanishing of (co)homology.

1 Homological and Cohomological definitions and notations

The aim of this section is to recall a few basic definitions regarding homology and cohomology of simplicial complexes that we will need below.

Let X be an n-dimensional simplicial complex. A simplicial complex X is called *pure* if every face in X is contained in some face of size $n+1$. The set of all k-faces of X is denoted $X(k)$, and we will be using the convention in which $X(-1) = \{\emptyset\}$.

We denote by $C_k(X) = C_k(X, \mathbb{F}_2)$ the $\mathbb{F}_2$-vector space with basis $X(k)$ (or equivalently, the $\mathbb{F}_2$-vector space of subsets of $X(k)$), and $C^k(X) = C^k(X, \mathbb{F}_2)$ the $\mathbb{F}_2$-vector space of functions from $X(k)$ to $\mathbb{F}_2$.

The *boundary map* $\partial_k : C_k(X, \mathbb{F}_2) \to C_{k-1}(X, \mathbb{F}_2)$ is:

$$\partial_k(\sigma) = \sum_{\tau \subset \sigma, |\tau| = |\sigma| - 1} \tau,$$

where $\sigma \in X(k)$, and the *coboundary map* $d_k : C^k(X, \mathbb{F}_2) \to C^{k+1}(X, \mathbb{F}_2)$ is:

$$d_k(\phi)(\sigma) = \sum_{\tau \subset \sigma, |\tau| = |\sigma| - 1} \phi(\tau),$$

where $\phi \in C^k$ and $\sigma \in X(k+1)$.

For $A \in C_k(X)$ and $\phi \in C^k(X)$, we denote

$$\phi(A) = \sum_{\tau \in A} \phi(\tau),$$

Thus, for $\phi \in C^k(X)$ and $A \in C_{k+1}(X)$

$$(d_k\phi)(A) = \phi(\partial_{k+1} A)$$

We sometimes refer to k-chains as subsets of $X(k)$, e.g., the 0-chain $\{u\} + \{v\}$ will be sometimes referred to as the set $\{\{u\}, \{v\}\}$. For $A \in C_k(X)$, we denote $|A|$ to be the size of A as a set.

Well known and easily calculated equations are:

$$\partial_k \circ \partial_{k+1} = 0 \text{ and } d_{k+1} \circ d_k = 0 \tag{1}$$

Thus, if we denote: $B_k(X) = B_k(X, \mathbb{F}_2) = \text{Image}(\partial_{k+1}) =$ the space of k-boundaries.
$Z_k(X) = Z_k(X, \mathbb{F}_2) = \text{Ker}(\partial_{k+1}) =$ the space of k-cycles.
$B^k(X) = B^k(X, \mathbb{F}_2) = \text{Image}(d_{k-1}) =$ the space of k-coboundaries.
$Z^k(X) = Z^k(X, \mathbb{F}_2) = \text{Ker}(d_k) =$ the space of k-cocycles.
We get from (1)

$$B_k(X) \subseteq Z_k(X) \subseteq C_k(X) \text{ and } B^k(X) \subseteq Z^k(X) \subseteq C^k(X).$$

Define the quotient spaces $\widetilde{H}_k(X) = Z_k(X)/B_k(X)$ and $\widetilde{H}^k(X) = Z^k(X)/B^k(X)$, the k-homology and the k-cohomology groups of X (with coefficients in $\mathbb{F}_2$).

2 Cone radius as a bound on coboundary expansion

Below, we define a generalized notion of diameter (or more precisely radius) of a simplicial complex. We will later show that in symmetric simplicial complexes a bound on this radius yields a bound on the coboundary expansion of the complex.

▶ **Definition 23** (Cone function). *Let X be a pure n-dimensional simplicial complex. Let $-1 \leq k \leq n-1$ be a constant and v be a vertex of X. A k-cone function with apex v is a linear function $\text{Cone}_k^v : \bigoplus_{j=-1}^{k} C_j(X) \rightarrow \bigoplus_{j=-1}^{k} C_{j+1}(X)$ defined inductively as follows:*

1. *For $k=-1$, $\text{Cone}_{-1}^v(\emptyset) = \{v\}$.*
2. *For $k \geq 0$, $\text{Cone}_k^v|_{\bigoplus_{j=-1}^{k-1} C_j(X)}$ is a $(k-1)$-cone function with an apex v and for every $A \in C_k(X)$, $\text{Cone}_k^v(A) \in C^{k+1}(X)$ is a $(k+1)$-chain that fulfills the equation*

$$\partial_{k+1} \text{Cone}_k^v(A) = A + \text{Cone}_k^v(\partial_k A).$$

▶ **Observation 24.** *By linearity, the condition that*

$$\partial_{k+1} \text{Cone}_k^v(A) = A + \text{Cone}_k^v(\partial_k A), \forall A \in C^k(X)$$

is equivalent to the condition:

$$\partial_{k+1} \text{Cone}_k^v(\tau) = \tau + \text{Cone}_k^v(\partial_k \tau), \forall \tau \in X(k).$$

▶ Remark 25. We note that by linearity, a k-cone function is needs only to be defined on k-simplices, but it gives us homological fillings for every k-cycle in X: for every $A \in Z_k(X)$,

$$\partial_{k+1} \text{Cone}_k^v(A) = A + \text{Cone}_k^v(\partial_k A) = A + \text{Cone}_k^v(0) = A,$$

i.e., $\partial_{k+1} \text{Cone}_k^v(A) = A$. This might be computationally beneficial for other needs (apart from the results of this paper), since usually there are exponentially more k-cycles than k-simplices.

▶ **Example 26** (0-cone example)**.** Let X be an n-dimensional simplicial complex. Fix some vertex v in X. By definition, for every $\{u\} \in X(0)$, $\text{Cone}_0^v(\{u\})$ is a 1-chain such that $\partial_0 \text{Cone}_0^v(\{u\}) = \{u\} + \{v\}$.

If the 1-skeleton of X is connected, we can define $\text{Cone}_0^v(\{u\})$ to be a 1-chain that consists of a sum of edges that form a path between $\{u\}$ and $\{v\}$. If the 1-skeleton of X is not connected, a 0-cone function does not exist: for $\{u\} \in X(0)$ that is not in the connected component of $\{v\}$, $\text{Cone}_0^v(\{u\})$ cannot be defined. Assuming that the 1-skeleton of X is connected, we note that the construction of Cone_0^v is usually not unique: different choices of paths between $\{u\}$ and $\{v\}$ give different 0-cone functions.

▶ **Example 27** (1-cone example)**.** Let X be an n-dimensional simplicial complex. Assume that the 1-skeleton of X is connected and define a 0-cone function as in the example above and define Cone_2^v on $C_0(X)$ as that 0-cone function. We note that for every $\{u,w\} \in X(1)$, $\{u,w\} + \text{Cone}_0^v(\{u\}) + \text{Cone}_0^v(\{w\})$ forms a closed path, i.e., a 1-cycle, in X. If $\widetilde{H}_1(X) = 0$, we can deduce that $\{u,w\} + \text{Cone}_1^v(\{u\}) + \text{Cone}_1^v(\{w\})$ is a boundary. Therefore, for every $\{u,w\} \in X(1)$, we can choose $\text{Cone}_1^v(\{u,w\}) \in X(2)$ such that

$$\partial_2 \text{Cone}_1^v = \{u,w\} + \text{Cone}_1^v(\{u\}) + \text{Cone}_1^v(\{w\}).$$

▶ **Definition 28** (Cone radius). *Let X be an n-dimensional simplicial complex, $-1 \leq k \leq n-1$ and v a vertex of X. Given a k-cone function Cone_k^v define the volume of Cone_k^v as*

$$\mathrm{Vol}(\mathrm{Cone}_k^v) = \max_{\tau \in X(k)} |\mathrm{Cone}_k^v(\tau)|.$$

Define the k-th cone radius of X to be

$$\mathrm{Crad}_k(X) = \min\{\mathrm{Vol}(\mathrm{Cone}_k^v) : \{v\} \in X(0), \mathrm{Cone}_k^v \text{ is a } k\text{-cone function}\}.$$

If k-cone functions do not exist, we define $\mathrm{Crad}(X) = \infty$.

▶ Remark 29. The reason for the name "cone radius" is that in the case where $k = 0$, $\mathrm{Crad}_0(X)$ is exactly the (graph) radius of the 1-skeleton of X. Indeed, for $k = 0$, choose $\{v\} \in X(0)$ such that for every $\{v'\} \in X(0)$,

$$\max_{\{u\} \in X(0)} \mathrm{dist}(v, u) \leq \max_{\{u\} \in X(0)} \mathrm{dist}(v', u),$$

where dist denotes the path distance. For such a $\{v\} \in X(0)$, define $\mathrm{Cone}_0^v(\{u\})$ to be the edges of a shortest path between v and u. By our choice of v, it follows that $\mathrm{Vol}(\mathrm{Cone}_0^v)$ is the radius of the one-skeleton of X and we leave it to the reader to verify that this choice gives $\mathrm{Crad}_0(X) = \mathrm{Vol}(\mathrm{Cone}_0^v)$.

The main result of this section is that in a symmetric simplicial complex X, the k-th cone radius gives a lower bound on $\mathrm{Exp}_b^k(X)$:

▶ **Theorem 30.** *Let X be a pure finite n-dimensional simplicial complex. Assume that X is strongly symmetric, i.e., that there is a group G of automorphisms of X acting transitively on $X(n)$. For every $0 \leq k \leq n-1$, if $\mathrm{Crad}_k(X) < \infty$, then $\mathrm{Exp}_b^k(X) \geq \frac{1}{\binom{n+1}{k+1} \mathrm{Crad}_k(X)}$.*

Theorem 30 stated above generalizes a result of of Lubotzky, Meshulam and Mozes [17] in which coboundary expansion was proven for symmetric simplicial complexes given that they are "building-like", i.e., that they have have sub-complexes that behave (in some sense) as apartments in a Bruhat-Tits building.

We note that the notion of a cone function is already evident in Gromov's original work [8]. Gromov considered what he called "random cones", which was a probability over a family of cone functions and show that the expectancy of the occurrence of a simplex in the support of this family bounds $\mathrm{Exp}_b^k(X)$ (see also the work of Kozlv and Meshulam [14, Theorem 2.5]). Using Gromov's terminology, in the proof of the Theorem above, we show that under the assumption of symmetry a single cone function yields a family of random cones and the needed expectancy is bounded by the cone radius. In the sake of completeness, we will not prove the Theorem without using Gromov's results.

In order to prove Theorem 30, we will need some additional lemmas.

▶ **Lemma 31.** *For $-1 \leq k \leq n-1$ and a k-cone function Cone_k^v with apex v. Define the contraction operator $\iota_{\mathrm{Cone}_k^v}$,*

$$\iota_{\mathrm{Cone}_k^v} : \bigoplus_{j=-1}^{k} C^{j+1}(X) \to \bigoplus_{j=-1}^{k} C^j(X)$$

as follows: for $\phi \in C^{j+1}(X)$ and $A \in C_j(X)$, we define

$$(\iota_{\mathrm{Cone}_k^v} \phi)(A) = \phi(\mathrm{Cone}_k^v(A)).$$

Then for every $\phi \in C^k(X)$,

$$\iota_{\mathrm{Cone}_k^v} d_k \phi = \phi + d_{k-1} \iota_{\mathrm{Cone}_k^v} \phi.$$

Proof. Let $A \in C_k(X)$, then

$$\iota_{\text{Cone}_k^v} d_k \phi(A) = (d_k \phi)(\text{Cone}_k^v(A)) =$$
$$\phi(\partial_{k+1}(\text{Cone}_k^v(A))) = \phi(A + (\text{Cone}_k^v(\partial_k A))) =$$
$$\phi(A) + (\iota_{\text{Cone}_k^v} \phi)(\partial_k A) = \phi(A) + (d_{k-1} \iota_{\text{Cone}_k^v} \phi)(A),$$

as needed. ◀

Naively, it might seem that this Lemma gives a direct approach towards bounding the coboundary expansion: if one could find is some constant $C = C(n, k, \text{Crad}_k(X))$ such that $w(\iota_{\text{Cone}_k^v} d_k \phi) \leq C w(d_k \phi)$, then for every ϕ,

$$\frac{w(d_k \phi)}{\min_{\psi \in B^k(X)} w(\phi + \psi)} \geq \frac{1}{C} \frac{w(\iota_{\text{Cone}_k^v} d_k \phi)}{w(\phi + d_{k-1} \iota_{\text{Cone}_k^v} \phi)} = \frac{1}{C}.$$

However, by Remark 13, we note that without symmetry, the existence of a k-cone function cannot give an effective bound on the coboundary expansion.

Our proof strategy below is to improve on this naive idea by using the symmetry of X: we will show that for a group G that acts on X, the group G also acts on k-cone functions and we will denote this action by ρ. We then show that when G acts transitively on $X(n)$, we can average the action on the k-cone function that realizes the cone radius and deduce that

$$\frac{1}{|G|} \sum_{g \in G} w(\iota_{\rho(g). \text{Cone}_k^v} d_k \phi) \leq \binom{n+1}{k+1} \text{Crad}_k(X) w(d_k \phi).$$

Thus, using an averaged version of the naive argument above will get a bound on the coboundary expansion.

We start by defining an action on k-cone functions. Assume that G is a group acting simplicially on X. For every $g \in G$ and every k-cone function Cone_k^v define

$$(\rho(g). \text{Cone}_k^v)(A) = g.(\text{Cone}_k^v(g^{-1}.A)), \forall A \in \bigoplus_{j=-1}^{k} C_j(X).$$

▶ **Lemma 32.** *For $g \in G$, $-1 \leq k \leq n-1$ and a k-cone function Cone_k^v with apex v, $\rho(g). \text{Cone}_k^v$ is a k-cone function with apex $g.v$ and $\text{Vol}(g. \text{Cone}_k^v) = \text{Vol}(\text{Cone}_k^v)$. Moreover, ρ defines an action of G on the set of k-cone functions.*

Proof. If we show that $g. \text{Cone}_k^v$ is a k-cone function the fact that $\text{Vol}(g. \text{Cone}_k^v) = \text{Vol}(\text{Cone}_k^v)$ will follow directly from the fact that G acts simplicially.

The proof that $\rho(g). \text{Cone}_k^v$ is a k-cone function is by induction on k. For $k = -1$,

$$(\rho(g). \text{Cone}_{v,-1})(\emptyset) = g. \text{Cone}_{v,-1}(g^{-1}.\emptyset) = g. \text{Cone}_{v,-1}(\emptyset) = g.\{v\} = \{g.v\},$$

then $\rho(g). \text{Cone}_{v,-1}$ is a (-1)-cone function with an apex $g.v$.

Assume the assertion of the lemma holds for $k-1$. Thus, $\rho(g). \text{Cone}_k^v|_{\bigoplus_{j=-1}^{k-1} C_j(X)}$ is a $(k-1)$-cone function with an apex $g.v$ and, by Observation 24, we are left to check that for every $\tau \in X(k)$,

$$\partial_{k+1}(\rho(g). \text{Cone}_k^v)(\tau) = \tau + \rho(g). \text{Cone}_k^v(\partial_k \tau).$$

Note that the G acts simplicially on X and thus the action of G commutes with the ∂ operator. Therefore, for every $\tau \in X(k)$,

$$\partial_{k+1}(\rho(g).\operatorname{Cone}_k^v)(\tau) = \partial_{k+1}(g.(\operatorname{Cone}_k^v(g^{-1}.\tau))) = g.\left(\partial_{k+1}\operatorname{Cone}_k^v(g^{-1}.\tau)\right) = g.\left(g^{-1}.\tau + \operatorname{Cone}_k^v(\partial_k g^{-1}.\tau)\right) = \tau + g.\operatorname{Cone}_k^v(g^{-1}.\partial_k\tau) = \tau + \rho(g).\operatorname{Cone}_k^v(\partial_k\tau).$$

The fact that ρ is an action is straight-forward and left for the reader. ◀

Applying our proof strategy above, will lead us to consider the constant $\theta(\eta)$ defined in the Lemma below.

▶ **Lemma 33.** *Assume that G is a group acting simplicially on X and that this action is transitive on n-simplices. Let $0 \leq k \leq n-1$ and assume that $\operatorname{Crad}_k(X) < \infty$. Fix Cone_k^v to be a k-cone function such that $\operatorname{Crad}_k(X) = \operatorname{Vol}(\operatorname{Cone}_k^v)$. For every $\eta \in X(k+1)$, denote*

$$\theta(\eta) = \frac{1}{w(\eta)|G|}\sum_{g\in G}\sum\{w(\tau) : \tau \in X(k), \eta \in (\rho(g).\operatorname{Cone}_k^v)(\tau))\}.$$

Then for every $\eta \in X(k+1)$, $\theta(\eta) \leq \binom{n+1}{k+1}\operatorname{Crad}_k(X)$.

Proof. Fix some $\eta \in X(k+1)$. First, we note that G acts transitively on $X(n)$ and therefore $\bigcup_{gG_\eta}\{g.\sigma : \sigma \in X(n), \eta \subseteq \sigma\} = X(n)$. This yields that

$$|X(n)| \leq \frac{|G|}{|G_\eta|}|\{\sigma \in X(n) : \eta \subseteq \sigma\}|,$$

and therefore $|G_\eta| \leq |G|\binom{n+1}{k+1}w(\eta)$.

Second, we note that for every $g \in G$, and every $\eta \in X(k+1)$,

$$\eta \in (\rho(g).\operatorname{Cone}_k^v)(\tau) \Leftrightarrow \eta \in g.(\operatorname{Cone}_k^v(g^{-1}.\tau)) \Leftrightarrow g^{-1}.\eta \in \operatorname{Cone}_k^v(g^{-1}.\tau).$$

Thus,

$$\begin{aligned}
\theta(\eta) &= \frac{1}{w(\eta)|G|}\sum_{g\in G}\sum\{w(\tau) : \tau \in X(k), \eta \in (\rho(g).\operatorname{Cone}_k^v)(\tau))\} = \\
&\frac{1}{w(\eta)|G|}\sum_{g\in G}\sum\{w(\tau) : \tau \in X(k), g^{-1}.\eta \in \operatorname{Cone}_k^v(g^{-1}.\tau)\} = \\
&\frac{1}{w(\eta)|G|}\sum_{g\in G}\sum\{w(g^{-1}.\tau) : \tau \in X(k), g^{-1}.\eta \in \operatorname{Cone}_k^v(g^{-1}.\tau)\} = \\
&\frac{1}{w(\eta)|G|}\sum_{g\in G}\sum\{w(\tau) : \tau \in X(k), g^{-1}.\eta \in \operatorname{Cone}_k^v(\tau)\} = \\
&\frac{1}{w(\eta)|G|}\sum_{g\in G}\sum_{\tau\in X(k)}\sum_{g^{-1}.\eta\in\operatorname{Cone}_k^v(\tau)} w(\tau) = \\
&\frac{1}{w(\eta)}\sum_{\tau\in X(k)} w(\tau)\sum_{g\in G}\sum_{g^{-1}.\eta\in\operatorname{Cone}_k^v(\tau)}\frac{1}{|G|} \leq \\
&\frac{1}{w(\eta)}\sum_{\tau\in X(k)} w(\tau)|\operatorname{Cone}_k^v(\tau)|\frac{|G_\eta|}{|G|} \leq \\
&\frac{1}{w(\eta)}\sum_{\tau\in X(k)} w(\tau)\operatorname{Crad}_k(X)\binom{n+1}{k+1}w(\eta) = \\
&\binom{n+1}{k+1}\operatorname{Crad}_k(X)\sum_{\tau\in X(k)} w(\tau) = \binom{n+1}{k+1}\operatorname{Crad}_k(X),
\end{aligned}$$

as needed. ◀

We turn now to prove Theorem 30:

Proof. Assume that G is a group acting simplicially on X such that the action is transitive on $X(n)$. Let $0 \leq k \leq n-1$ and assume that $\mathrm{Crad}_k(X) < \infty$. For $\phi \in C^k(X)$, we denote

$$[\phi] = \{\phi + \psi : \psi \in B^k(X)\}, \text{ and } w([\phi]) = \min_{\phi' \in [\phi]} w(\phi').$$

Thus, we need to prove that for every $\phi \in C^k(X)$,

$$\frac{1}{\binom{n+1}{k+1} \mathrm{Crad}_k(X)} w([\phi]) \leq w(d_k \phi),$$

or equivalently,

$$|G| w([\phi]) \leq |G| w(d_k \phi) \left(\binom{n+1}{k+1} \mathrm{Crad}_k(X) \right).$$

Fix Cone_k^v to be a k-cone function such that $\mathrm{Crad}_k(X) = \mathrm{Vol}(\mathrm{Cone}_k^v)$. By Lemma 32, for every $g \in G$, $\rho(g).\mathrm{Cone}_k^v$ is a k-cone function.

In the notation of Lemma 31, for every $g \in G$, denote $\iota_g = \iota_{\rho(g).\mathrm{Cone}_k^v}$. By Lemma 31, for every $g \in G$, $w([\phi]) \leq w(\iota_g d_k \phi)$ and therefore

$$\begin{aligned} |G| w([\phi]) \leq \sum_{g \in G} w(\iota_g d_k \phi) = \sum_{g \in G} \sum \{w(\tau) : \tau \in \mathrm{supp}(\iota_g d_k \phi)\} = \\ \sum_{g \in G} \sum \{w(\tau) : \tau \in X(k), d_k \phi(\rho(g).\mathrm{Cone}_k^v(\tau)) = 1\} \leq \\ \sum_{g \in G} \sum \{w(\tau) : \tau \in X(k), \mathrm{supp}(d_k \phi) \cap \rho(g).\mathrm{Cone}_k^v(\tau) \neq \emptyset\} \leq \\ \sum_{\eta \in \mathrm{supp}(d_k \phi)} \sum_{g \in G} \sum \{w(\tau) : \tau \in X(k), \eta \in \rho(g).\mathrm{Cone}_k^v(\tau)\} = \\ \sum_{\eta \in \mathrm{supp}(d_k \phi)} |G| w(\eta) \theta(\eta) \leq^{\text{Lemma 33}} |G| w(d_k \phi) \left(\binom{n+1}{k+1} \mathrm{Crad}_k(X) \right), \end{aligned}$$

as needed. ◀

The converse of Theorem 30 is also true, i.e., the existence of a cone function is equivalent to vanishing of (co)homology and thus to coboundary expansion (for a proof see [13]).

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