

Fast Mixing via Polymers for Random Graphs with Unbounded Degree

Andreas Galanis ✉

Department of Computer Science, University of Oxford, UK

Leslie Ann Goldberg ✉

Department of Computer Science, University of Oxford, UK

James Stewart ✉

Department of Computer Science, University of Oxford, UK

Abstract

The polymer model framework is a classical tool from statistical mechanics that has recently been used to obtain approximation algorithms for spin systems on classes of bounded-degree graphs; examples include the ferromagnetic Potts model on expanders and on the grid. One of the key ingredients in the analysis of polymer models is controlling the growth rate of the number of polymers, which has been typically achieved so far by invoking the bounded-degree assumption. Nevertheless, this assumption is often restrictive and obstructs the applicability of the method to more general graphs. For example, sparse random graphs typically have bounded average degree and good expansion properties, but they include vertices with unbounded degree, and therefore are excluded from the current polymer-model framework.

We develop a less restrictive framework for polymer models that relaxes the standard bounded-degree assumption, by reworking the relevant polymer models from the edge perspective. The edge perspective allows us to bound the growth rate of the number of polymers in terms of the total degree of polymers, which in turn can be related more easily to the expansion properties of the underlying graph. To apply our methods, we consider random graphs with unbounded degrees from a fixed degree sequence (with minimum degree at least 3) and obtain approximation algorithms for the ferromagnetic Potts model, which is a standard benchmark for polymer models. Our techniques also extend to more general spin systems.

2012 ACM Subject Classification Theory of computation → Generating random combinatorial structures; Theory of computation → Design and analysis of algorithms

Keywords and phrases Markov chains, approximate counting, Potts model, expander graphs, random graphs

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2021.36

Category RANDOM

Related Version *A full version with all proofs can be found at:* <https://arxiv.org/abs/2105.00524>

1 Introduction

The polymer model framework [21, 14] is a classical tool from statistical mechanics which has recently been used to obtain efficient approximation algorithms for analysing spin systems (such as the Potts model) in parameter regimes where standard algorithmic approaches are provably inefficient/inaccurate on general graphs. These algorithms apply to certain classes of graphs that typically have sufficiently strong expansion properties relative to their local growth rates. Typically, the local growth rate is restricted by a bounded-degree assumption. Examples of such classes include bounded-degree expanders [20, 22, 7, 5, 6, 2, 10, 15] and the d -dimensional grid [16, 4, 20, 17]. The purpose of this work is to expand the current framework for applying polymer models by relaxing the bounded-degree assumption and using alternative methods to capture the growth of the graph.



© Andreas Galanis, Leslie Ann Goldberg, and James Stewart;
licensed under Creative Commons License CC-BY 4.0

Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021).

Editors: Mary Wootters and Laura Sanità; Article No. 36; pp. 36:1–36:13



Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

To briefly review the current framework, we use as a running example the q -state ferromagnetic Potts model with parameter $\beta > 0$. For a graph $G = (V_G, E_G)$, the set $\Omega_{G,q}$ of configurations of the model is the set of all (not necessarily proper) q -colourings σ of V_G using the set of colours $[q] = \{1, \dots, q\}$ where $q \geq 3$. The weight of a colouring σ is $w(\sigma) = e^{\beta m_G(\sigma)}$ where $m_G(\sigma)$ is the number of monochromatic edges under σ . The so-called partition function $Z = Z_{G,q,\beta}$ is the aggregate weight of all σ and the Gibbs distribution $\mu = \mu_{G,q,\beta}$ is the probability distribution on the set of all σ , in which each σ has mass proportional to its weight, i.e., $\mu(\sigma) = w(\sigma)/Z$. We will study the computational problems of approximating the partition function and approximately sampling from the Gibbs distribution. In general, these problems are computationally hard ($\#$ BIS-hard) when the parameter β is sufficiently large [13, 12].

The recent works [16, 20] introduced a framework based on polymer models that bypasses the worst-case hardness, on classes of bounded-degree graphs with expansion properties. The rough intuition for the Potts model is that, for large β , due to the expansion properties, the colourings with non-negligible weight are close to the so-called ground-states of the model, i.e., the q configurations in which all vertices get the same colour. Polymer models capture the deviation of configurations from these ground states. Given a ground state with colour r , a polymer is a connected set of vertices, none of which is coloured with r , and a polymer configuration (with respect to the ground state r) corresponds to the set of all polymers (see Example 4 for more details). The Potts model can then be decomposed into q polymer models, each of which can be studied using relatively streamlined algorithmic methods (based on interpolation [1] and Markov chains). This framework has already found multiple algorithmic applications in far more general settings [16, 4, 17, 20, 19, 22, 7, 9, 5, 6, 10, 15].

Despite these advances, the current applications of polymer models rely crucially on the fact that the maximum degree of the underlying graph is bounded. This fact is used to control the number of polymers of a given size (which is crucially needed for the algorithmic analysis). As a result of this limitation, applications to several other interesting classes of graphs are ruled out, excluding for example sparse random graphs, which have bounded average degree and good expansion properties, but include vertices with unbounded degree.

1.1 Main Results

In this paper, we propose a framework for polymer models that overcomes the bounded-degree limitations of previous algorithms, by revisiting the Markov chain approach of [7]. We introduce a new condition which requires that the weight of each polymer decays exponentially in its total degree (the sum of the degrees of the vertices in the polymer) instead of decaying exponentially in the polymer's size. This new condition allows us to prove rapid mixing for a Markov chain which is an adapted edge-version of the so-called polymer dynamics of [7]. Crucially, the fact that the new condition is formulated in terms of the total degree of a polymer allows us to relax the assumption that the instance has bounded degree.

As an application of our method, we consider the q -state ferromagnetic Potts model on sparse random graphs of unbounded degree with a given degree sequence, as detailed below.

► **Definition 1.** *Let d be a positive real number and n be a positive integer. We define $\mathcal{D}_{n,d}$ to be the set of all degree sequences $\{x_1, x_2, \dots, x_n\}$ that satisfy*

1. *For all $i \in [n]$, $3 \leq x_i \leq n^\rho$ where $\rho = \frac{1}{50}$, and*
2. *$\sum_{i \in [n]} x_i^2 \leq dn$.*

We write $G \sim \mathcal{G}(n, \vec{x})$ to indicate that G is a graph chosen uniformly at random from the set of all simple n -vertex graphs with degree sequence \vec{x} . G satisfies a property with high probability (whp) if the probability that G satisfies the property is $1 - o(1)$, as a function of n (uniformly over \vec{x}).

Note that $\mathcal{D}_{n,d}$ is empty unless $d \geq 9$. The assumption that all degrees are greater than or equal to 3 (rather than 2) guarantees that the random graph G is connected and has good expansion properties. The degree lower bound also means that our results do not apply to Erdős-Rényi random graphs. The upper bound on the degrees is mild and can in fact be relaxed somewhat further (but in general cannot be made to be linear in n due to the sparsity assumption in Item 2).

We give an efficient algorithm for approximately sampling¹ from and approximating the partition function² of the ferromagnetic Potts model on random graphs with a given degree sequence for all sufficiently large β .

► **Theorem 2.** *Let d be a real number and $q \geq 3$ be an integer. For the ferromagnetic Potts model, there is β_0 such that for all $\beta \geq \beta_0$ there is a poly-time approximate sampling algorithm for $\mu_{G,q,\beta}$ and an FPRAS for $Z_{G,q,\beta}$ that work with high probability on random graphs $G \sim \mathcal{G}(n, \vec{x})$ for any degree sequence $\vec{x} \in \mathcal{D}_{n,d}$.*

► **Remark 3.** Note that β_0 depends on d and q , and our arguments later (see Remark 17) show that $\beta_0 = Cd \log d \log q$ for some $C > 0$ (independent of d or q). If the desired accuracy ε is at least e^{-n} then the running time of the sampling algorithm is $O(n \log \frac{n}{\varepsilon} \log \frac{1}{\varepsilon})$ and the running time of the FPRAS is $O(n^2 (\log \frac{n}{\varepsilon})^3)$.

We further remark that the bounded-degree assumption has also been relaxed in [17] for the ferromagnetic Potts model on lattice graphs; the approach therein however is tailored to a certain flow representation of the Potts model, which is used as a basis for the corresponding polymer models and therefore does not extend to general spin systems. Our approach applies to general polymer models as detailed in the next section and our focus on the ferromagnetic Potts model is mainly to illustrate the method without further technical overhead; the approach for example can be adapted to general spin systems on bipartite random graphs with a given degree sequence (analogously to [10]).

2 Polymers

The main technique that we use to prove Theorem 2 is to refine the polymer framework by introducing a new polymer sampling condition which requires that the weight of each polymer decays exponentially in its total degree. In order to state the new condition we first give some relevant definitions.

Let $G = (V_G, E_G)$ be a graph – we refer to G as the “host graph” of the polymer model. Let $[q] = \{1, \dots, q\}$ be a set of spins and $g = \{g_v\}_{v \in V_G}$ specify a set of ground-state spins for the vertices, i.e., $g_v \subseteq [q]$ for each $v \in V_G$. A polymer is a pair $\gamma = (V_\gamma, \sigma_\gamma)$ consisting of a connected set of vertices V_γ and an assignment $\sigma_\gamma: V_\gamma \rightarrow [q]$ such that $\sigma_\gamma(v) \in [q] \setminus g_v$. Let \mathcal{P}_G be the set of all polymers. A polymer model for the host graph G is specified by a subset of allowed polymers $\mathcal{C}_G \subseteq \mathcal{P}_G$, and a weight function $w_G: \mathcal{C}_G \rightarrow \mathbb{R}_{\geq 0}$. For polymers $\gamma, \gamma' \in \mathcal{P}_G$, we write $\gamma \sim \gamma'$ to denote that γ, γ' are compatible, i.e., that

¹ A polynomial-time approximate sampling algorithm for $\mu_{G,q,\beta}$ is an algorithm that, given an accuracy parameter $\varepsilon > 0$ and a graph $G = (V_G, E_G)$ as input, outputs a sample from a probability distribution that is within total variation distance ε of $\mu_{G,q,\beta}$, in time $\text{poly}(|V_G|, 1/\varepsilon)$.

² Given an accuracy parameter $\varepsilon > 0$, we say that \hat{Z} is an ε -approximation to the quantity Z if $e^{-\varepsilon}Z \leq \hat{Z} \leq e^\varepsilon Z$. A fully polynomial randomised approximation scheme (FPRAS) for $Z_{G,q,\beta}$ is a randomised algorithm that, given an accuracy parameter $\varepsilon > 0$ and a graph $G = (V_G, E_G)$ as input, outputs a random variable that is an ε -approximation to $Z_{G,q,\beta}$ with probability at least $3/4$, in time $\text{poly}(|V_G|, 1/\varepsilon)$.

for every $u \in \gamma$ and $u' \in \gamma$ the graph distance in G between u and u' is at least 2. We define $\Omega_G = \{\Gamma \subseteq \mathcal{C}_G \mid \forall \gamma, \gamma' \in \Gamma, \gamma \sim \gamma'\}$ to be the set of all sets of mutually compatible polymers of \mathcal{C}_G ; elements of Ω_G are called polymer configurations. We define the partition function as $Z_G = \sum_{\Gamma \in \Omega_G} \prod_{\gamma \in \Gamma} w_G(\gamma)$, and the Gibbs distribution on $\Gamma \in \Omega_G$ as $\mu_G(\Gamma) = \prod_{\gamma \in \Gamma} w_G(\gamma) / Z_G$. We use (\mathcal{C}_G, w_G) to denote the polymer model.

► **Example 4** (The polymer model $(\mathcal{C}_{G,q}^r, w_{G,\beta})$, [20]). Consider the q -state ferromagnetic Potts model with parameter β , and let $r \in [q]$ be a colour. Let G be a graph and set $g_v = \{r\}$ for every $v \in V_G$. The weight of a polymer $\gamma = (V_\gamma, \sigma_\gamma)$ is defined as $w_{G,\beta}(\gamma) = e^{-\beta B_\gamma}$, where B_γ denotes the number of edges from V_γ to $V_G \setminus V_\gamma$ plus the number of edges of G with both endpoints in V_γ that are bichromatic under σ_γ . We let $\mathcal{P}_{G,q}^r$ denote the set of all polymers and the set of allowed polymers $\mathcal{C}_{G,q}^r$ to be the set of polymers $\gamma \in \mathcal{P}_{G,q}^r$ such that $|V_\gamma| < |V_G|/2$. Note that a polymer configuration Γ consisting of the polymers $\gamma_1, \dots, \gamma_k$ corresponds to a colouring σ , where a vertex v takes the colour $\sigma_{\gamma_i}(v)$ if $v \in V_{\gamma_i}$ for some $i \in [k]$, and the colour r otherwise; moreover, $e^{\beta|E_G|} \prod_{i \in [k]} w_G(\gamma_i) = w_G(\sigma)$.

Polymer models have been used to approximate the partition function of spin systems on bounded-degree host graphs. There are several existing algorithmic frameworks which can be used to sample from these resulting polymer models. One such deterministic algorithm uses the polynomial interpolation method of Barvinok [1] combined with the cluster expansion to approximate the partition function of the polymer model (see [16] for more details). Typical running times for these deterministic algorithms are of the form $n^{O(\log(\Delta))}$, where Δ is the maximum degree of the host graph, though for polymer models these have been improved to give a running time of $n^{1+o_\Delta(1)}$, see [20]. Another approach, described in Section 4 of the full version [11], uses a Markov chain called the polymer dynamics to sample from μ_G (see also [7] for more details). The running times of algorithms obtained using the Markov chain approach are usually faster and of the form $O(n \log n)$. Both of these approaches work for roughly the same range of parameters, and the essential condition required is that the weight of each polymer decays exponentially in the number of vertices it contains. To obtain our results, we give a simple generic way to modify this condition, as detailed below.

For a vertex $v \in V_G$ we write $\deg_G(v)$ to denote the degree of v in G , and for a vertex subset $S \subseteq V_G$ we write $\deg_G(S)$ to denote $\sum_{v \in S} \deg_G(v)$.

► **Definition 5.** Let $q \geq 2$ be an integer, \mathcal{G} be a class of graphs, and $\mathcal{F}_\mathcal{G} = \{(\mathcal{C}_G, w_G) \mid G \in \mathcal{G}\}$ be a family of q -spin polymer models. We say that $\mathcal{F}_\mathcal{G}$ satisfies the polymer sampling condition with constant $\tau \geq 3 \log(8e^3(q-1))$ if $w_G(\gamma) \leq e^{-\tau \deg_G(V_\gamma)}$ for all $G \in \mathcal{G}$ and all $\gamma \in \mathcal{C}_G$.³

Using Definition 5, we will show (Lemma 8, below) that if a “computationally feasible” family of polymer models on a class of graphs \mathcal{G} satisfies this new condition, then there is an efficient algorithm which, given a graph $G \in \mathcal{G}$, approximately samples from the Gibbs distribution of the polymer model corresponding to G .

The new polymer sampling condition in Definition 5 is analogous to the original one in [7] except that the original condition requires the weight of a polymer to decay exponentially in its size, and in particular that the constant τ is sufficiently big relative to the maximum degree of G . The new condition relaxes this, allowing us to choose the constant τ in Definition 5 so that it does not depend on the maximum degree of the host graph, which is how we overcome the limitations of previous work. Technically, the improvement comes from the fact that previous work relies on bounding the number of connected vertex subsets of a given size

³ Unless we specify otherwise, the base of all logarithms in this paper is assumed to be e .

(with bounds that depend on the maximum degree of the graph), but here we are able to instead rely on the following lemma which bounds the number of connected vertex subsets with a given total degree and this enables us to avoid restricting the maximum degree of the graph. The new condition, which replaces the notion of “size” with total degree, fits well with the original abstract polymer model framework of [21], where the notion of the “size” of a polymer is an abstract function.

► **Lemma 6.** *Let $G = (V_G, E_G)$ be a graph, $v \in V_G$, and $\ell \geq 1$ be an integer. The number of connected vertex subsets $S \subseteq V_G$ such that $v \in S$ and $\deg_G(S) = \ell$ is at most $(2e)^{2\ell-1}$.*

In addition to the bound on the number of connected vertex subsets in Lemma 6, we will use the fact that these connected vertex subsets can be enumerated in time exponential in the total degree ℓ (see Lemma 21 of the full version). Although the bound in Lemma 6 is exponential in ℓ , this will be mitigated by the fact that the new polymer sampling condition ensures that the weight of each polymer is exponentially small in its total degree. The new polymer sampling condition therefore allows us to prove that the following condition holds – this condition is analogous to the polymer mixing condition of [7], except that we consider edges instead of vertices. For a polymer $\gamma \in \mathcal{P}_G$, let E_γ denote the set of edges of G with at least one endpoint in V_γ .

► **Definition 7.** *Let $q \geq 2$ be an integer, \mathcal{G} be a class of graphs, and $\mathcal{F}_\mathcal{G} = \{(\mathcal{C}_G, w_G) \mid G \in \mathcal{G}\}$ be a family of q -spin polymer models. We say that $\mathcal{F}_\mathcal{G}$ satisfies the polymer mixing condition with constant $\theta \in (0, 1)$ if $\sum_{\gamma' \sim \gamma} |E_{\gamma'}| \cdot w_G(\gamma') \leq \theta |E_\gamma|$ for all $G \in \mathcal{G}$ and all $\gamma \in \mathcal{C}_G$.*

In contrast to the conditions in [7], the two new conditions consider edges since we modify the polymer dynamics algorithm to sample edges instead of vertices. Subject to these new conditions, the techniques of [7] can be adapted to show that the modified polymer dynamics mixes rapidly, therefore giving the efficient algorithm for sampling from the Gibbs distribution of a polymer model. We give the details of the modified dynamics in Section 4 of the full version [11].

Finally, in order to use the modified polymer dynamics as an efficient algorithm for computing an approximate sample from μ_G , we will need a mild computational condition for polymers. More precisely, we say that a family of polymer models $\{(\mathcal{C}_G, w_G) \mid G \in \mathcal{G}\}$ is *computationally feasible* if for all $G \in \mathcal{G}$ and all $\gamma \in \mathcal{P}_G$, it is possible to decide whether $\gamma \in \mathcal{C}_G$ and to compute $w_G(\gamma)$, if it is, in $O(e^{\deg_G(V_\gamma)})$ time. Computational feasibility serves exactly the same purpose as it did in Definition 3 of [7], which requires that the same operations are able to be carried out in time depending on $|V_\gamma|$ (instead of $\deg_G(V_\gamma)$ that we use here).

In Section 4 of the full version [11], we prove the following lemma which gives an efficient algorithm for sampling⁴ from the Gibbs distribution of a polymer model and for approximating its partition function. In order to prove the lemma, we extend the polymer dynamics algorithm of [7] to the unbounded degree setting. The proof of the lemma uses the fact (see Lemma 18 of the full version) that the polymer sampling condition implies the polymer mixing condition.

⁴ Given an accuracy parameter $\varepsilon > 0$, we say that a random variable X is an ε -sample from the probability distribution μ if the total variation distance between the distribution of X and μ is at most ε .

► **Lemma 8.** *Let $q \geq 2$ be an integer, \mathcal{G} be a class of graphs, and $\mathcal{F}_{\mathcal{G}}$ be a family of computationally feasible q -spin polymer models satisfying the polymer sampling condition.*

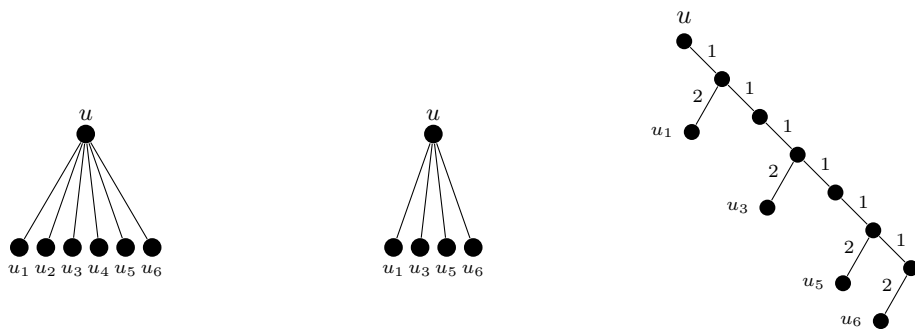
There are randomised algorithms which, given as input a graph $G \in \mathcal{G}$ with m edges and an accuracy parameter $\varepsilon > 0$, output an ε -sample from μ_G in $O(m \log \frac{m}{\varepsilon} \log \frac{1}{\varepsilon})$ time, and an ε -approximation to Z_G , with probability at least $3/4$, in $O(m^2 \log(\frac{m}{\varepsilon})^3)$ time.

We next give the proof of Lemma 6 which is one of the key technical ingredients allowing us to relax the bounded-degree restriction and to remove the dependence of the constant τ in Definition 5 on the maximum degree of the host graph, as noted earlier.

Proof of Lemma 6. Let $N(G, v, \ell)$ be the set of subtrees $T = (V_T, E_T)$ of G such that $v \in V_T$, $\deg_G(V_T) = \ell$. We will show that $|N(G, v, \ell)| \leq (2e)^{2\ell-1}$, which gives us the desired result for the following reason. Let $S \subseteq V_G$ be a connected vertex subset such that $v \in S$ and $\deg_G(S) = \ell$. Since S is connected, it has at least one spanning tree $T = (V_T = S, E_T)$ such that $v \in V_T$ and $\deg_G(V_T) = \ell$. Since S is the unique connected vertex subset that T spans, this gives us an injective map from the set of all connected vertex subsets containing v with total degree ℓ , to $N(G, v, \ell)$.

We now give an injective map from $N(G, v, \ell)$ to $T^*(2\ell, 3)$ – the set of subtrees of size 2ℓ that contain the root, of the infinite rooted 3-regular tree. By a result of Bollobás [3, p. 129], we know that $|T^*(2\ell, 3)|$ is at most $(2e)^{2\ell-1}$. Let $T = (V_T, E_T)$ be a subtree from $N(G, v, \ell)$. We will map T to a rooted subtree $T' = (V_{T'}, E_{T'})$ from $T^*(2\ell, 3)$. For each vertex of V_G , fix an ordering of its neighbours. In the infinite rooted 3-regular tree, label the edges incident to the root with $\{1, 2, 3\}$, and for each other vertex label the edges connecting it to its two children with $\{1, 2\}$. As we construct T' , we will label its edges so that it is clear which subtree from $T^*(2\ell, 3)$ we are constructing, we will also label some of its vertices. We construct T' as follows (see Figure 1 for an example of the following construction).

1. Add the root to $V_{T'}$ and label it v .
2. While there is a labelled vertex of T' (call its label u) such that u has a child w in T but no vertex of T' is labelled w , then we do the following. First, we create a path P of length $\deg_G(u)$ where each edge is labelled 1. We then connect the vertex of T' labelled u to P via an edge labelled 1. Finally, for $1 \leq i \leq \deg_G(u)$, we connect a vertex labelled w to the i^{th} vertex of P via an edge labelled 2, if w is the i^{th} neighbour of u in G and w is a child of u in T .



(a) Neighbourhood of u in G . (b) Neighbourhood of u in T . (c) Neighbourhood of u in T' .

■ **Figure 1** Constructing T' .

Each $T \in N(G, v, \ell)$ maps to a different $T' \in T^*(2\ell, 3)$. When constructing T' , we used edge labels from $\{1, 2, 3\}$, therefore the maximum degree of T' is 3. For each $v \in V_T$, we added at most $2 \deg_G(v)$ vertices to T' , therefore the size of T' is at most $2 \deg_G(V_T) = 2\ell$. ◀

3 Application to unbounded-degree graphs

Let $\alpha > 0$ be a real number. We say that a graph G is an α -total-degree expander if, for all connected vertex subsets $S \subseteq V_G$ with $|S| \leq |V_G|/2$, we have $e_G(S, S^c) \geq \alpha \deg_G(S)$, where $e_G(S, S^c)$ denotes the number of edges with one endpoint in S and the other in $S^c := V_G \setminus S$. Let \mathcal{G}_α denote the set of all α -total-degree expanders. Note, every connected $G \in \mathcal{G}_\alpha$ is also an α -expander (i.e., $e_G(S, S^c) \geq \alpha|S|$).

When β is sufficiently large, the polymer model from Example 4 satisfies the polymer sampling condition (Definition 5) with constant $\tau = \alpha\beta$. To see this, consider $\gamma \in \mathcal{C}_{G,q}^r$ and observe that since $B_\gamma \geq e_G(V_\gamma, V_\gamma^c)$ and $|V_\gamma| < |V_G|/2$, it follows that

$$w_{G,\beta}(\gamma) \leq \exp\{-\alpha\beta \deg_G(V_\gamma)\} = e^{-\tau \deg_G(V_\gamma)}, \tag{1}$$

where $\tau \geq 3 \log(8e^3(q-1))$ if $\beta \geq \frac{3}{\alpha} \log(8e^3(q-1))$.

We may therefore apply Lemma 8 in order to efficiently sample from the ferromagnetic Potts model and to estimate Z_G for $G \in \mathcal{G}_\alpha$, provided that β is sufficiently large. The proof of the following theorem is in Section 5 of the full version [11].

► **Theorem 9.** *Let $\alpha > 0$ be a real number. Let $q \geq 3$ be an integer and $\beta \geq \frac{3}{\alpha} \log(8e^3(q-1))$ be a real. For the Potts model on $G \in \mathcal{G}_\alpha$, there is a poly-time approximate sampling algorithm for $\mu_{G,q,\beta}$ and an FPRAS for $Z_{G,q,\beta}$.*

In fact, for $n = |V_G|$ and $m = |E_G|$, if the desired accuracy ε satisfies $\varepsilon \geq e^{-n}$ then the running time of the sampler is $O(m \log \frac{m}{\varepsilon} \log \frac{1}{\varepsilon})$ and the running time of the FPRAS is $O(m^2 (\log \frac{m}{\varepsilon})^3)$.

3.1 Expansion of random graphs with specified degree sequences

Let d be a real number. In this section, we will show that a graph $G \sim \mathcal{G}(n, \vec{x})$ for a degree sequence $\vec{x} \in \mathcal{D}_{n,d}$ is whp an α -total-degree-expander for some constant $\alpha > 0$, i.e., that $G \in \mathcal{G}_\alpha$.

To work with $G \sim \mathcal{G}(n, \vec{x})$, we consider the standard configuration model, where a random multigraph $H = (V_H, E_H)$ with the given degree sequence \vec{x} is sampled by the following process. For each $i \in [n]$, we attach x_i half-edges to the vertex i . We then sample a uniformly random perfect matching on the half-edges to give E_H . This uniformly random perfect matching can be sampled by performing the following until no half-edges remain: choose any remaining half-edge, choose another remaining half-edge uniformly at random, then pair these two half-edges and remove them from the set of remaining half-edges. We write $H \sim \text{CM}(n, \vec{x})$. Note, for two vertices $i, j \in V_H$ such that $i \neq j$, the probability that a half edge attached to i and a half edge attached to j are paired is

$$p_{\{i,j\}} = \frac{x_i x_j}{2m - 1}, \text{ where } m = \frac{1}{2} \sum_{k=1}^n x_k, \tag{2}$$

and similarly the probability that two half-edges of i are connected is $p_{\{i,i\}} = \frac{x_i(x_i-1)}{2m(2m-1)}$.

We first prove results about $\text{CM}(n, \vec{x})$, since asymptotic properties of $\text{CM}(n, \vec{x})$ can easily be transferred back to $\mathcal{G}(n, \vec{x})$ using the following straightforward consequence of [18, Theorem 1.1]. A proof is included in the full version [11] for completeness.

► **Lemma 10.** *Let d be a positive real number. For every positive integer n , let \mathcal{E}_n be a set of n -vertex multigraphs. If, for some $\vec{x} \in \mathcal{D}_{n,d}$, $G \sim \mathcal{G}(n, \vec{x})$ and $H \sim \text{CM}(n, \vec{x})$ then the following is true. If $H \in \mathcal{E}_n$ with high probability, then $G \in \mathcal{E}_n$ with high probability.*

For a (multi)graph $H = (V_H, E_H)$ we define the tree-excess to be $t_H = |E_H| - (|V_H| - 1)$; that is, the number of edges more than a tree that H has. First, we show that multigraphs drawn from the configuration model have locally bounded tree excess.

► **Lemma 11.** *Let d be a positive real number. The following is true with high probability when $H = (V_H, E_H)$ is drawn from $\text{CM}(n, \vec{x})$ uniformly over all degree sequences $\vec{x} \in \mathcal{D}_{n,d}$. For all connected vertex sets $S \subseteq V_H$ with $|S| \leq (\log n)^2$ and $\deg_H(S) \geq 36$, we have that $t_{H[S]} \leq \frac{1}{6} \deg_H(S)$.*

Proof. For positive integers k and ℓ , and a non-negative integer t , let the random variable $X_{k,\ell,t}$ denote the number of connected vertex subsets $S \subseteq V_H$ such that $|S| = k$, $\deg_H(S) = \ell$, and $t_{H[S]} = t$. To prove the lemma, we will show that whp

$$\sum_{k \leq \lfloor (\log n)^2 \rfloor} \sum_{\ell \geq 36} \sum_{t \geq \lfloor \ell/6 \rfloor + 1} X_{k,\ell,t} = 0.$$

In fact, we can further restrict the range of summation. From the lower bound in Item 1 of Definition 1, we have that $x_i \geq 3$ for all i , and therefore $\ell \geq 3k$. Item 2 shows that $\sum_i x_i \leq dn$, and therefore $\ell \leq dn$ and $t \leq \ell/2 \leq dn/2$. So, consider any integer ℓ in the range $36 \leq \ell \leq dn$, any integer k in the range $1 \leq k \leq \min\{(\log n)^2, \ell/3\}$, and any integer $t > \ell/6$. There are at most $\binom{n}{k}$ vertex subsets $S \subseteq V_H$ with $|S| = k$ and $\deg_G(S) = \ell$. Let $j = k - 1 + t$ be the number of edges with both endpoints in S . Given such a set S , there are at most $\binom{\ell}{2j}$ possibilities for the set of half-edges in these j edges. On a given set of $2j$ half-edges, there are $(2j - 1)!! = \frac{(2j)!}{2^j j!}$ perfect matchings. Using the upper bound on the degrees from Item 1 of Definition 1, the probability that a set of j edges is present in H is at most

$$\frac{n^{2\rho}}{2m-1} \frac{n^{2\rho}}{2m-3} \cdots \frac{n^{2\rho}}{2m-2j+1} \leq \left(\frac{n^{2\rho}}{2m-2j} \right)^j \leq \left(\frac{n^{2\rho}}{n} \right)^j,$$

where the final inequality follows from the fact that $k \leq (\log n)^2$ and therefore that $2m - 2j \geq \deg_G(S^c) \geq 3|S^c| = 3(n - k) > n$ (as long as n is sufficiently big). We also have that

$$\binom{\ell}{2j} \cdot \frac{(2j)!}{2^j j!} < \frac{\ell!}{(\ell - 2j)! j!} < \frac{\ell^{2j} e^j}{j^j} \leq \left(\frac{e\ell^2}{t} \right)^j.$$

Putting everything together, it follows that

$$\mathbb{E}[X_{k,\ell,t}] \leq \binom{n}{k} \left(\frac{e\ell^2}{t} \right)^{k-1+t} \left(\frac{n^{2\rho}}{n} \right)^{k-1+t} < \left(\frac{e^2 \ell^2}{t} \right)^{k-1+t} \frac{n^{2\rho(k-1+t)}}{n^{t-1}}.$$

Furthermore, since $t > \ell/6$, $k < 2t$, and (by the upper bound in Item 1 of Definition 1) $\ell \leq kn^\rho \leq n^{2\rho}$, we have that

$$\mathbb{E}[X_{k,\ell,t}] < \frac{(6e^2 n^{4\rho})^{3t-1}}{n^{t-1}}. \quad (3)$$

Let

$$X = \sum_{\ell=36}^{dn} \sum_{k=1}^{\lfloor \min\{(\log n)^2, \ell/3\} \rfloor} \sum_{t=\lfloor \ell/6 \rfloor + 1}^{\lfloor dn/2 \rfloor} X_{k,\ell,t}.$$

Since $t > \ell/6 \geq 6$, it follows that $t \geq 7$. For big enough n , (3) shows that $\mathbb{E}[X_{k,\ell,t}] \leq n^{13\rho t}/n^{t-1}$. Since $\rho \leq 2/91$ and $t \geq 7$, $1 - 13\rho \geq 5/7 \geq 5/t$ so $13\rho t \leq t - 5$ and $\mathbb{E}[X_{k,\ell,t}]$ is at most n^{-4} . Taking a union bound over all permissible values for ℓ , k , and t , we find that $\mathbb{E}[X] = o(1)$. Applying Markov's inequality, we have that $\Pr(X > 0) = \Pr(X \geq 1) \leq \mathbb{E}[X] = o(1)$, and the result follows. \blacktriangleleft

To obtain the expansion bounds in Lemmas 13 and 14, we require the following result from [8, Proposition 4.5]. Although this result is stated in [8] in terms of the random graph model, it is first proved for the configuration model, so this is how we state it. Also, Fountoulakis and Reed require that the vector \vec{x} be in $\mathcal{D}_{n,d}$ but this is only important for lifting their result to the random graph model, so it is not relevant for us.

► **Lemma 12.** (Fountoulakis, Reed) *Let $H = (V_H, E_H)$ be drawn from $\text{CM}(n, \vec{x})$ for some length- n degree sequence \vec{x} . For any set $S \subseteq V_H$ we have $\Pr(e_H(S, S^c) = 0) \leq \binom{m}{\deg_H(S)/2}^{-1}$, where $m = \frac{1}{2} \sum_i x_i$.*

Note that Fountoulakis and Reed's lemma was stated for S such that $\deg_H(S)$ is even, but if $\deg_H(S)$ is odd, it is not possible to have $e_H(S, S^c) = 0$. Next, we show that in a multigraph H drawn from the configuration model, small vertex subsets satisfy certain expansion properties.

► **Lemma 13.** *Let d be a positive real number. The following is true with high probability when $H = (V_H, E_H)$ is drawn from $\text{CM}(n, \vec{x})$ uniformly over all degree sequences $\vec{x} \in \mathcal{D}_{n,d}$. For all connected vertex sets $S \subseteq V_H$ with $|S| \leq (\log n)^2$, we have that $e_H(S, S^c) \geq |S|/4$.*

Proof. For positive integers k and ℓ , and a non-negative integer j , let the random variable $X_{k,j,\ell}$ denote the number of connected vertex subsets $S \subseteq V_H$ with $|S| = k$, $e_H(S, S^c) = j$, and $\deg_H(S) = \ell$. By Item 1 of Definition 1, we need only consider ℓ satisfying $3k \leq \ell \leq kn^\rho$. Let

$$X = \sum_{k=1}^{\lfloor (\log n)^2 \rfloor} \sum_{j=0}^{\lfloor k/4 \rfloor} \sum_{\ell=3k}^{\lfloor kn^\rho \rfloor} X_{k,j,\ell}.$$

To prove the lemma we will show that $X = 0$, whp. Consider any integer k in the range $1 \leq k \leq (\log n)^2$, any integer j in the range $0 \leq j < k/4$, and any integer ℓ in the range $3k \leq \ell \leq kn^\rho$. There are at most $\binom{n}{k}$ candidates for vertex sets S with $|S| = k$ and $\deg_H(S) = \ell$. There are then at most $\binom{\ell}{j}$ choices for the j half-edges emanating from vertices of S that will be matched with half-edges emanating from vertices of S^c , once H is drawn. Applying Lemma 12 to the degree sequence derived from \vec{x} by removing the j half-edges (and their partners), the probability that the remaining $\ell - j$ half-edges are matched amongst themselves is at most

$$\left(\frac{m'}{(\ell - j)/2} \right)^{-1} \leq \left(\frac{(\ell - j)}{2m'} \right)^{\frac{(\ell - j)}{2}} \leq \left(\frac{kn^\rho}{n} \right)^{\frac{11k}{8}},$$

where $2m' = (\sum_{i=1}^n x_i) - 2j$ and the last inequality follows (for big enough n) since $11k/4 \leq$

36:10 Fast Mixing via Polymers for Random Graphs with Unbounded Degree

$\ell - j \leq kn^\rho$ and $2m' \geq 3n - 2j > n$. We therefore have that

$$\begin{aligned} \mathbb{E}[X] &\leq \sum_{k=1}^{\lfloor (\log n)^2 \rfloor} \sum_{j=0}^{\lfloor k/4 \rfloor} \sum_{\ell=3k}^{\lfloor kn^\rho \rfloor} \binom{n}{k} \binom{\ell}{j} \left(\frac{kn^\rho}{n} \right)^{\frac{11k}{8}} \\ &\leq \sum_{k=1}^{\lfloor (\log n)^2 \rfloor} \sum_{j=0}^{\lfloor k/4 \rfloor} \sum_{\ell=3k}^{\lfloor kn^\rho \rfloor} \left(\frac{ne}{k} \right)^k \left(\frac{e\ell}{j} \right)^j \left(\frac{kn^\rho}{n} \right)^{\frac{11k}{8}} \\ &\leq \sum_{k=1}^{\lfloor (\log n)^2 \rfloor} \left(\frac{(\log n)^{O(1)} n^{\rho(2+11/8)}}{n^{3/8}} \right)^k. \end{aligned}$$

This is $o(1)$ since $\rho < 1/9$. Applying Markov's inequality, we have that $\Pr(X > 0) = \Pr(X \geq 1) = o(1)$, and the result follows. \blacktriangleleft

The next lemma handles the expansion of sets S with relative big size.

► Lemma 14. *Let d be a positive real number. There is a positive real number α (depending on d) such that the following is true with high probability when $H = (V_H, E_H)$ is drawn from $\text{CM}(n, \vec{x})$ uniformly over all degree sequences $\vec{x} \in \mathcal{D}_{n,d}$. For all connected vertex sets $S \subseteq V_H$ with $(\log n)^2 \leq |S| \leq n/2$, we have that $e_H(S, S^c) \geq \alpha \deg_H(S)$.*

Proof. We will give the proof for vertex sets $S \subseteq V_H$ with $(\log n)^2 \leq |S| \leq n/2$ and $\deg_H(S) > \max\{100d|S|, n/2\}$, the cases where $\deg_H(S) \leq \max\{100d|S|, n/2\}$ follow by arguments that are close to those in [8], and are given in the full version [11].

Let $C = 10^4 d$. By the Cauchy-Schwarz inequality, we have that $|S| \sum_{i \in S} x_i^2 \geq (\deg_H(S))^2 \geq 10^4 d^2 |S|^2$, so using Item 2 of Definition 1 which ensures that $\sum_{i \in S} x_i^2 \leq dn$, we find that $|S| \leq n/C$.

Let $f = (eC)^{1/C}$. The number of sets S satisfying $|S| \leq n/C$ is at most $n \binom{n}{n/C} \leq n f^n$ since there are at most n possibilities for $|S|$ to consider, and for each of them $\binom{n}{|S|} \leq \binom{n}{n/C}$.

Fix any set $S \subseteq V_H$ with $|S| \leq n/C$ and consider the random construction of H , starting from half-edges in S^c (and choosing their mates in the pairing). Let

$$j = \left\lfloor \frac{\deg_H(S^c)}{2} \right\rfloor \geq \left\lfloor \frac{3|S^c|}{2} \right\rfloor \geq \left\lfloor \frac{3n(1 - \frac{1}{C})}{2} \right\rfloor \geq \frac{3n(1 - \frac{2}{C})}{2},$$

where the first inequality uses the fact that each x_i is at least 3 (from Item 1 of Definition 1) and the final inequality uses the fact that n is sufficiently large.

Note that the process initiates a pairing from at least j half-edges in S^c . For each $i \in [j]$, let Y_i be the indicator random variable for the event that the i 'th half-edge from which pairing is initiated connects to an endpoint in S (conditioned on the pairings of the first $i - 1$ half-edges initiated from S^c).

Let $\varepsilon = 3(1 - 2/C)/(8d) \leq 1/2$. Recall from Item 2 in Definition 1 that $\sum_{i=1}^n x_i \leq dn$. For any $t \in [j]$ satisfying $\sum_{i=1}^{t-1} Y_i < \varepsilon n/2$ we have

$$\Pr(Y_t = 1) \geq \frac{\deg_H(S) - \varepsilon n/2}{dn} > \frac{1 - \varepsilon}{2d} \geq \frac{1}{4d}.$$

Now let X_1, \dots, X_j be i.i.d. Bernoulli random variables which are 1 with probability $1/(4d)$. We can couple the evolution of these variables so that, for any $t \in [j]$ satisfying $\sum_{i=1}^{t-1} Y_i < \varepsilon n/2$, we have $\sum_{i=1}^t Y_i \geq \sum_{i=1}^t X_i$. We conclude that $\Pr(\sum_{i=1}^j Y_i < \varepsilon n/2) \leq \Pr(\sum_{i=1}^j X_i < \varepsilon n/2)$.

To conclude we will show that $nf^n \Pr(\sum_{i=1}^j X_i < \varepsilon n/2) = o(1)$, implying that we can take $\alpha = \varepsilon/(2d)$ since $\varepsilon n/2 = \alpha dn \geq \alpha \deg_H(S)$.

Let $X = \sum_{i=1}^j X_i$ and $\delta = 1/2$. Note that $\mathbb{E}[X] = j/(4d)$ and that

$$\frac{(1-\delta)j}{4d} \geq \frac{(1-\delta)3n(1-\frac{2}{C})}{8d} = \frac{\varepsilon n}{2}.$$

By a Chernoff bound, $\Pr(X \leq \varepsilon n/2) \leq \Pr(X \leq (1-\delta)j/(4d)) \leq \exp(-j\delta^2/(8d))$.

To conclude that $nf^n \exp(-j\delta^2/(8d)) = o(1)$ we observe that $f < \exp(3(1-2/C)\delta^2/(16d))$. Taking $\alpha = \varepsilon/(2d)$, we conclude the proof. ◀

We can now prove the following result, which establishes the desired expansion properties of the multigraphs generated by the configuration model.

► **Lemma 15.** *Let d be a positive real number. There is a positive real number α (depending on d) such that the following is true with high probability when $H = (V_H, E_H)$ is drawn from $\text{CM}(n, \vec{x})$ uniformly over all degree sequences $\vec{x} \in \mathcal{D}_{n,d}$. For all connected vertex sets $S \subseteq V_H$ with $|S| \leq n/2$, we have that $e_H(S, S^c) \geq \alpha \deg_H(S)$.*

Proof. We consider three cases.

Case 1. Consider all connected subsets $S \subseteq V_H$ with $(\log n)^2 \leq |S| \leq n/2$. By Lemma 14 there is a positive real number α' such that, whp, every such subset S has $e_H(S, S^c) \geq \alpha' \deg_H(S)$.

Case 2. Consider all connected subsets $S \subseteq V_H$ with $|S| \leq (\log n)^2$ and $\deg_H(S) \geq 36$.

- Consider first those subsets S with $|S| \leq \frac{1}{6} \deg_H(S)$. We have that

$$e_H(S, S^c) = \deg_H(S) - 2(t_{H[S]} + |S| - 1) \geq \frac{2}{3} \deg_H(S) - 2|S| \geq \frac{1}{3} \deg_H(S),$$

by Lemma 11 and our assumption on the size of S .

- Now consider those subsets S with $|S| > \frac{1}{6} \deg_H(S)$, then by Lemma 13, we have that $e_H(S, S^c) \geq |S|/4 \geq \deg_H(S)/24$.

Case 3. Finally, consider connected subsets $S \subseteq V_H$ with $|S| \leq (\log n)^2$ and $\deg_H(S) < 36$.

By Lemma 13, we have that $e_H(S, S^c) \geq |S|/4 \geq 1/4 = 36/144 > \deg_H(S)/144$.

The result follows from the three cases by taking $\alpha = \min\{1/144, \alpha'\} = \alpha'$. ◀

Using the definition of \mathcal{G}_α and Lemma 10, we have the following corollary of Lemma 15.

► **Corollary 16.** *Let d be a real number. There is a positive real number α (depending on d) such that the following holds. With high probability, when $G \sim \mathcal{G}(n, \vec{x})$ for some $\vec{x} \in \mathcal{D}_{n,d}$, it holds that $G \in \mathcal{G}_\alpha$.*

Combining Corollary 16 with Theorem 9 implies our main theorem.

► **Theorem 2.** *Let d be a real number and $q \geq 3$ be an integer. For the ferromagnetic Potts model, there is β_0 such that for all $\beta \geq \beta_0$ there is a poly-time approximate sampling algorithm for $\mu_{G,q,\beta}$ and an FPRAS for $Z_{G,q,\beta}$ that work with high probability on random graphs $G \sim \mathcal{G}(n, \vec{x})$ for any degree sequence $\vec{x} \in \mathcal{D}_{n,d}$.*

Proof. Let d be a real number and let $q \geq 2$ be an integer. Let α be the positive real number from Corollary 16. Let $\beta_0 = \frac{3}{\alpha} \log(8e^3(q-1))$.

Consider $\vec{x} \in \mathcal{D}_{n,d}$ and let G be drawn from $\mathcal{G}(n, \vec{x})$. By Corollary 16, $G \in \mathcal{G}_\alpha$ whp. The result then follows by using the algorithms from Theorem 9. ◀

► Remark 17. The bounds on β in Remark 3 follow from the choice of β_0 in the proof of Theorem 2 and from the fact that $\alpha = O(\frac{1}{d \log d})$ which follows from the proofs of Lemmas 14 and 15. The running time bounds in Remark 3 come from those in Theorem 9 using the fact that $|E_G| = O(n)$ which follows from Item 2 of Definition 1.

References

- 1 Alexander Barvinok. *Combinatorics and complexity of partition functions*, volume 9. Springer, 2016.
- 2 Alexander Barvinok and Guus Regts. Weighted counting of solutions to sparse systems of equations. *Combinatorics, Probability and Computing*, 28(5):696–719, 2019.
- 3 Béla Bollobás. *The art of mathematics: Coffee time in Memphis*. Cambridge University Press, 2006.
- 4 Christian Borgs, Jennifer Chayes, Tyler Helmuth, Will Perkins, and Prasad Tetali. Efficient sampling and counting algorithms for the Potts model on \mathbb{Z}^d at all temperatures. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 738–751, 2020.
- 5 Sarah Cannon and Will Perkins. Counting independent sets in unbalanced bipartite graphs. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1456–1466, 2020.
- 6 Charles Carlson, Ewan Davies, and Alexandra Kolla. Efficient algorithms for the Potts model on small-set expanders. *arXiv preprint arXiv:2003.01154*, 2020.
- 7 Zongchen Chen, Andreas Galanis, Leslie Ann Goldberg, Will Perkins, James Stewart, and Eric Vigoda. Fast algorithms at low temperatures via Markov chains. *Random Struct. Algorithms*, 58(2):294–321, 2021. Theorems 5 and 6 from arxiv.org/abs/1901.0665.
- 8 Nikolaos Fountoulakis and Bruce A Reed. The evolution of the mixing rate of a simple random walk on the giant component of a random graph. *Random Structures & Algorithms*, 33(1):68–86, 2008.
- 9 Tobias Friedrich, Andreas Göbel, Martin S. Krejca, and Marcus Pappik. A spectral independence view on hard-spheres via block dynamics, 2021. [arXiv:2102.07443](https://arxiv.org/abs/2102.07443).
- 10 Andreas Galanis, Leslie Ann Goldberg, and James Stewart. Fast Algorithms for General Spin Systems on Bipartite Expanders. In *45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020)*, pages 37:1–37:14, 2020.
- 11 Andreas Galanis, Leslie Ann Goldberg, and James Stewart. Fast mixing via polymers for random graphs with unbounded degree. *CoRR*, abs/2105.00524, 2021. [arXiv:2105.00524](https://arxiv.org/abs/2105.00524).
- 12 Andreas Galanis, Daniel Stefankovic, Eric Vigoda, and Linji Yang. Ferromagnetic Potts model: Refined #BIS-hardness and related results. *SIAM Journal on Computing*, 45(6):2004–2065, 2016.
- 13 Leslie Ann Goldberg and Mark Jerrum. Approximating the partition function of the ferromagnetic Potts model. *Journal of the ACM*, 59(5):1–31, 2012.
- 14 Christian Gruber and Hervé Kunz. General properties of polymer systems. *Communications in Mathematical Physics*, 22(2):133–161, 1971.
- 15 Tyler Helmuth, Matthew Jenssen, and Will Perkins. Finite-size scaling, phase coexistence, and algorithms for the random cluster model on random graphs. *arXiv preprint arXiv:2006.11580*, 2020.
- 16 Tyler Helmuth, Will Perkins, and Guus Regts. Algorithmic Pirogov–Sinai theory. *Probability Theory and Related Fields*, 176(3):851–895, 2020.
- 17 Jeroen Huijben, Viresh Patel, and Guus Regts. Sampling from the low temperature Potts model through a Markov chain on flows. *CoRR*, abs/2103.07360, 2021. [arXiv:2103.07360](https://arxiv.org/abs/2103.07360).
- 18 Svante Janson. The probability that a random multigraph is simple. II. *Journal of Applied Probability*, 51(A):123–137, 2014.
- 19 Matthew Jenssen and Peter Keevash. Homomorphisms from the torus, 2020. [arXiv:2009.08315](https://arxiv.org/abs/2009.08315).

- 20 Matthew Jenssen, Peter Keevash, and Will Perkins. Algorithms for #BIS-hard problems on expander graphs. *SIAM Journal on Computing*, 49(4):681–710, 2020.
- 21 Roman Kotecký and David Preiss. Cluster expansion for abstract polymer models. *Communications in Mathematical Physics*, 103(3):491–498, 1986.
- 22 Chao Liao, Jiabao Lin, Pinyan Lu, and Zhenyu Mao. Counting independent sets and colorings on random regular bipartite graphs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019)*, pages 34:1–34:12, 2019.