The Central Valuations Monad

Xiaodong Jia
School of Mathematics, Hunan University, Changsha, China

Michael Mislove
Department of Computer Science, Tulane University, New Orleans, LA, USA

Vladimir Zamdzhiev
Université de Lorraine, CNRS, Inria, LORIA, F 54000 Nancy, France

Abstract

We give a commutative valuations monad $\mathcal{Z}$ on the category **DCPO** of dcpo's and Scott-continuous functions. Compared to the commutative valuations monads given in [2], our new monad $\mathcal{Z}$ is larger and it contains all push-forward images of valuations on the unit interval $[0,1]$ along lower semi-continuous maps. We believe that this new monad will be useful in giving domain-theoretic denotational semantics for statistical programming languages with continuous probabilistic choice.

2012 ACM Subject Classification Theory of computation → Denotational semantics

Keywords and phrases Valuations, Commutative Monad, DCPO, Probabilistic Choice, Recursion

Digital Object Identifier 10.4230/LIPIcs.CALCO.2021.18

Category Early Ideas

Acknowledgements We thank the anonymous reviewers for their feedback which led to improvements of this paper. Xiaodong Jia acknowledges the support of NSFC (No. 12001181).

1 Introduction

The valuations monad $\mathcal{V}$ on the category **DCPO** of dcpo's and Scott-continuous functions is a staple of the domain-theoretic approach for denotational semantics of programming languages with probabilistic choice and recursion [3, 4]. For a dcpo D, $\mathcal{V}D$ consists of *subprobability valuations* on D, which are the Scott-continuous functions ν from the set σD of Scott open subsets of D to $[0,1]$ satisfying *strictness* ($\nu(\emptyset) = 0$) and *modularity* ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). The set $\mathcal{V}D$ is a dcpo in the *stochastic order*: $\nu_1 \leq \nu_2$ if and only if $\nu_1(U) \leq \nu_2(U)$ for all $U \in \sigma D$. The *unit* of $\mathcal{V}$ at dcpo D is the map $\eta_D \colon D \to \mathcal{V}D :: x \mapsto \delta_x$, where δ_x is the *Dirac valuation* at x, defined by $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ otherwise. For a Scott-continuous map $f \colon D \to \mathcal{V}E$, the *Kleisli extension* $f^\dagger$ of f is defined by $f^\dagger(\nu)(U) = \int_{x\in X} f(x)(U) d\nu$ for $\nu \in \mathcal{V}D$ and $U \in \sigma E$. The integral in this definition is a Choquet type integral: for a general Scott-continuous function $h \colon D \to [0,1]$, the value of $\int_{x\in X} h d\nu$ is defined to be the Riemann integral $\int_0^1 \nu(h^{-1}(t,1])dt$. Following this, the action of $\mathcal{V}$ on a Scott-continuous function $g \colon D \to E$ between dcpo's D and E is $\mathcal{V}(g) \stackrel{\text{def}}{=} (\eta_E \circ g)^\dagger$; concretely, for $\nu \in \mathcal{V}D$ and $U \in \sigma E$, $\mathcal{V}(g)(\nu)(U) = \nu(g^{-1}(U))$. Subprobability valuations on general topological spaces and the corresponding integral of lower semi-continuous functions against subprobability valuations can be defined similarly [3].

While it is well-known that $\mathcal{V}$ can be restricted to a commutative monad on the category **DOM** of domains and Scott-continuous functions, it is unknown whether $\mathcal{V}$ can be restricted to any Cartesian closed full subcategory of **DOM**. This is known as the *Jung-Tix problem* [5].

One may note that the category **DCPO** itself is Cartesian closed and $\mathcal{V}$ is a monad on it. What does one lose if we use the category **DCPO** and monad $\mathcal{V}$ for semantics? A short answer is that compared to **DOM**, $\mathcal{V}$ is not known to be *commutative* over **DCPO**, which is an important property for the denotational semantics of programming languages.

9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021).
Editors: Fabio Gadducci and Alexandra Silva; Article No. 18; pp. 18:1–18:5
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Commutativity of $\mathcal{V}$ over **DCPO** is equivalent to showing the following Fubini-style equation

$$\int_{x\in D}\int_{y\in E} h(x,y)d\mu d\nu = \int_{y\in E}\int_{x\in D} h(x,y)d\nu d\mu \tag{1}$$

holds for all dcpo's D and E, all Scott-continuous functions $h\colon D\times E\to[0,1]$ and all $\nu\in\mathcal{V}D, \mu\in\mathcal{V}E$. As pointed out in [2], the main difficulty in establishing (1) over **DCPO** is that the Scott topology on the product dcpo $D\times E$ may be different from the product topology $\sigma D\times\sigma E$. Actually, we do know that Equation (1) holds for those functions h that are continuous when $D\times E$ is given the product topology $\sigma D\times\sigma E$ ([4, Lemma 2.37]).

Instead of directly proving (1), we showed (together with Lindenhovius) how to construct three submonads of $\mathcal{V}$ that are commutative on **DCPO**, and used each one to give a sound and (strongly) adequate semantics to `PFPC` (Probabilistic FixPoint Calculus) [2]. The simplest of those three monads is the monad $\mathcal{M}$. For each dcpo D, $\mathcal{M}D$ is defined to be the smallest sub-dcpo of $\mathcal{V}D$ that contains $\mathcal{S}D$, the family of *simple valuations* on D, where a simple valuation is a finite convex sum of Dirac valuations. The other two commutative monads are denoted $\mathcal{W}$ and $\mathcal{P}$ and the following inclusions hold for each dcpo $D : \mathcal{S}D\subseteq\mathcal{M}D\subseteq\mathcal{W}D\subseteq\mathcal{P}D\subseteq\mathcal{V}D$.

Each of our three monads is large enough to interpret *discrete* probabilistic choice in `PFPC` [2]. However, it is unclear if any of these monads is large enough to interpret *continuous* probabilistic choice. In this note, we define a new commutative valuations monad $\mathcal{Z}$ on the category **DCPO** which is larger than $\mathcal{M}, \mathcal{W}$ and $\mathcal{P}$ with the hope of addressing this problem.

2 Central Valuations

Our idea for defining $\mathcal{Z}$ is inspired by the notion of centre in group theory (which always forms an abelian subgroup) and the notion of centre of a premonoidal category (which always forms a monoidal subcategory) [6].

▶ **Definition 1.** *A subprobability valuation ν on a dcpo D is called a* central valuation *if for any dcpo E, any valuation μ on E, and any Scott-continuous function $h\colon D\times E\to[0,1]$, we have*

$$\int_{x\in D}\int_{y\in E} h(x,y)d\mu d\nu = \int_{y\in E}\int_{x\in D} h(x,y)d\nu d\mu.$$

We shall write $\mathcal{Z}D$ for the set of all central valuations on a dcpo D.

It is easy to see that simple valuations are central, and that the central valuations are closed under directed suprema under the stochastic order. Thus, for each dcpo D, $\mathcal{Z}D$ is a sub-dcpo of $\mathcal{V}D$ containing $\mathcal{S}D$. Moreover, we have the following theorem, which can be proved using the *disintegration formula* in [1].

▶ **Theorem 2.** *The assignment $\mathcal{Z}(-)$ extends to a commutative monad over the category* **DCPO** *when equipped with the (co)restricted monad operations of $\mathcal{V}$. In other words, $\mathcal{Z}$ is a commutative submonad of $\mathcal{V}$.*

Proof. The unit of $\mathcal{Z}$ at dcpo D sends each $x\in D$ to δ_x which is obviously a central valuation.

Let $f\colon C\to\mathcal{Z}D$ be a Scott-continuous function. Then f can also be viewed as a Scott-continuous map from C to $\mathcal{V}D$, since $\mathcal{Z}D$ is a sub-dcpo of $\mathcal{V}D$. We prove that $f^\dagger\colon\mathcal{V}C\to\mathcal{V}D$ maps central valuations on C to central valuations on D. Towards this end, we pick μ

from $\mathcal{Z}C$, and assume that E is a dcpo, ν is an arbitrary subprobability valuation on E and $h\colon D \times E \to [0,1]$ is a Scott-continuous map. Then by the disintegration formula (see Lemma 3.1(iii) in [1]) we have that

$$\int_{y\in E}\int_{x\in D} h(x,y)d(f^\dagger(\mu))d\nu = \int_{y\in E}\int_{t\in C}\int_{x\in D} h(x,y)df(t)d\mu d\nu,$$

and the right side of the equation is equal to

$$\int_{t\in C}\int_{x\in D}\int_{y\in E} h(x,y)d\nu df(t)d\mu$$

by the fact that $f(t), t \in D$ and μ are central valuations. Again, by the disintegration formula that is just $\int_{x\in D}\int_{y\in E} h(x,y)d\nu df^\dagger(\mu)$. Hence we have proved that $f^\dagger(\mu)$ is indeed a central valuation provided that μ is. Similar arguments show that the monadic strength also (co)restricts as required. The corresponding (co)restrictions of the monadic operations of $\mathcal{V}$ to $\mathcal{Z}$ validate that $\mathcal{Z}$ is a strong monad on **DCPO**. The commutativity of $\mathcal{Z}$, which is equivalent to Equation (1) holding for all dcpo's D and E and central valuations μ and ν on them, is then obvious by definition of $\mathcal{Z}$. ◀

In fact, it is proved in [2] that all *point-continuous valuations* are central and therefore $\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{Z}D \subseteq \mathcal{V}D$ for each dcpo D. Therefore $\mathcal{Z}$ is the largest commutative submonad of $\mathcal{V}$ known so far. Furthermore, observe that $\mathcal{Z} = \mathcal{V}$ iff $\mathcal{V}$ is a commutative monad on **DCPO**. The latter has been an open problem since 1989, and our simple observation leads us to believe $\mathcal{Z}$ is a very large commutative submonad of $\mathcal{V}$.

It is not difficult to see that, in order to model sampling against continuous probability distributions on the interval $[0,1]$, the monad used for the semantics should at least contain the push-forward images of the Lebesgue valuation on $[0,1]$ (equipped with the metric topology) along lower semi-continuous maps. We can demonstrate even more is true of our new monad $\mathcal{Z}$ (see Theorem 4 below). For this, let us first recall that a space X is called *core-compact* if the set $\mathcal{O}X$ of all open subsets of X is a continuous lattice in the inclusion order. Equivalently, X is core-compact if and only if for each open subset U of X and $x \in U$, there exists an open subset V such that $x \in V \ll U$, where $V \ll U$ means that V is *way-below* U in the sense of domain theory. Many important spaces are core-compact. For example, each locally compact space is core-compact, and in particular, the unit interval with the usual topology is compact Hausdorff, hence locally compact hence core-compact.

▶ **Lemma 3.** *Let X be a core-compact topological space. Let D, E be dcpos, and $f\colon X \to D$ a lower semi-continuous map, i.e., f is continuous when D is equipped with the Scott topology. Then the map $f \times \mathrm{id}_E\colon X \times \Sigma E \to D \times E$ is also lower semi-continuous, where ΣE denotes the topological space $(E, \sigma E)$ and $X \times \Sigma E$ is the topological product of X and ΣE.*

Proof. First, we assume that X is core-compact and prove that $f \times \mathrm{id}_E$ is lower semi-continuous. Towards this end, we pick a Scott open subset O of $D \times E$, and assume that $f \times \mathrm{id}_Y((x_0, e_0)) \in O$, that is $(f(x_0), e_0) \in O$. We must find an open neighbourhood U of x_0 in X and a Scott open neighbourhood V of e_0 in E such that $f \times \mathrm{id}_Y(U \times V) \subseteq O$. We let $A = \{x \in X \mid (f(x), e_0) \in O\}$. Then A is just $f^{-1}(O_{e_0})$, where $O_{e_0} = \{d \in D \mid (d, e_0) \in O\}$. Since $f\colon X \to D$ is lower semi-continuous and O_{e_0} is Scott open in D, we know that A is an open neighbourhood of x_0 in X. Now the core-compactness of X enables us to find, in X, an open subset U and a sequence of open subsets $U_i, i \in \mathbb{N}$ such that $x_0 \in U \ll \cdots \ll U_n \cdots \ll U_1 \ll A$. For each $U_n, n \in \mathbb{N}$, we define $V_n = \{e \mid f(x,e) \in O \text{ for all } x \in U_n\}$ and let $V = \bigcup_{n\in\mathbb{N}} V_n$. Since for each $n \in \mathbb{N}$, $U_n \subseteq A$, we have for all $x \in U_n$, $(f(x), e_0) \in O$. Thus we know that $e_0 \in V_n$ for each $n \in \mathbb{N}$, and hence $e_0 \in V$. Moreover, for any $(x, e) \in U \times V$, there exists a natural number n such that $e \in V_n$, then it follows that $f \times \mathrm{id}_E((x, e)) = (f(x), e) \in f(U) \times V_n \subseteq f(U_n) \times V_n \subseteq O$. The last inclusion is due to the construction of V_n. To sum up, it is true that $f \times \mathrm{id}_E(U \times V) \subseteq O$. Since U is an open subset of X which contains x_0 and $e_0 \in V$, we finish the proof by showing that V is Scott open in E. To this end we let $\{e_i\}_{i\in I}$ be a directed subset of E with $\sup_{i\in I} e_i \in V$. For each $i \in I$, set $W_i = \{x \in X \mid (f(x), e_i) \in O\}$. It is easy to see that $\{W_i \mid i \in I\}$ is a directed family of open subsets of X. Since $\sup_{i\in I} e_i \in V = \bigcup_{n\in\mathbb{N}} V_n$, $\sup_{i\in I} e_i$ is in some V_n. This means that for each $x \in U_n$, $(f(x), \sup_{i\in I} e_i) \in O$. Because O is Scott open, for each $x \in U_n$, there exists $i \in I$ such that $(f(x), e_i) \in O$, i.e., $x \in W_i$. Hence we have that $\sup_{i\in I} e_i \in V_n \subseteq \bigcup_{i\in I} W_i$. Remember that $U_{n+1} \ll U_n$, it follows that $U_{n+1} \subseteq W_j$ for some $j \in I$. By definition of W_j, we know that $f(U_{n+1}) \times \{e_j\} \subseteq O$, which means that $e_j \in V_{n+1}$, this time by definition of V_{n+1}. So we find $j \in I$ with $e_j \in V_{n+1} \subseteq V$, and indeed V is Scott open in E. ◀

▶ **Theorem 4.** *Let X be a core-compact space and f be a lower semi-continuous map from X to a dcpo D. If ν is a valuation on X, then $f_*(\nu) \stackrel{\text{def}}{=} \lambda O \in \sigma D.\nu(f^{-1}(O))$, the* push-forward valuation *along f, is a central valuation on D. In particular, for a core-compact dcpo D, all valuations on D are central, i.e., $\mathcal{V}D = \mathcal{Z}D$.*

Proof. By definition, we prove for any dcpo E, continuous valuations μ on E and Scott-continuous map $h\colon D \times E \to [0, 1]$ the equation

$$\int_{x\in D} \int_{y\in E} h(x, y) d\mu df_*(\nu) = \int_{y\in E} \int_{x\in D} h(x, y) df_*(\nu) d\mu$$

holds.

Note that for each $y \in E$, the map $g \stackrel{\text{def}}{=} (x \mapsto \int_{y\in E} h(x, y)d\mu)\colon D \to [0, 1]$ is Scott-continuous, and $f\colon X \to D$ is lower semi-continuous. Hence for the left side of the above equation we have

$$\int_{x\in D} \int_{y\in E} h(x, y) d\mu df_*(\nu) = \int_{x\in X} g(f(x)) d\nu = \int_{x\in X} \int_{y\in E} h(f(x), y) d\mu d\nu.$$

The first equality follows form the so-called *change-of-variable* formula, which can be found in [4]. As a consequence of it, we also have that

$$\int_{y\in E} \int_{x\in D} h(x, y) df_*(\nu) d\mu = \int_{y\in E} \int_{x\in X} h(f(x), y) d\nu d\mu.$$

Since X is core-compact and the function $f\colon X \to D$ is lower semi-continuous, by Lemma 3 we know that $f \times \mathrm{id}_E\colon X \times \Sigma E \to X \times Y$ is lower semi-continuous. This implies that the map $(x, y) \mapsto h(f(x), y)\colon X \times \Sigma E \to [0, 1]$ is lower semi-continuous. Hence by Lemma 2.37 in [4] we know that $\int_{x\in X} \int_{y\in E} h(f(x), y)d\mu d\nu = \int_{y\in E} \int_{x\in X} h(f(x), y)d\nu d\mu$, which finishes the proof.

The second claim is a straightforward consequence of the first one. ◀

▶ **Theorem 5.** *Let $f : [0, 1] \to D$ be a lower semi-continuous map into a dcpo D. If ν is any continuous valuation on $[0, 1]$, then $f_*(\nu)$ is a central valuation on D.*

Proof. Since $[0, 1]$ is core-compact in the usual topology, the result follows from Theorem 4. ◀

We have not been able to establish the above theorem for any of the monads $\mathcal{M}, \mathcal{W}$ or $\mathcal{P}$, so we believe that $\mathcal{Z}$ is a promising candidate for modeling *continuous* probabilistic choice. We plan to address this in future work.

References

1 Jean Goubault-Larrecq, Xiaodong Jia, and Clément Théron. A Domain-Theoretic Approach to Statistical Programming Languages, 2021. Preprint. URL: `https://arxiv.org/abs/2106.16190`.

2 Xiaodong Jia, Bert Lindenhovius, Michael Mislove, and Vladimir Zamdzhiev. Commutative monads for probabilistic programming languages. In *Logic in Computer Science (LICS 2021)*, 2021. `arXiv:2102.00510`.

3 C. Jones and Gordon D. Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science (LICS '89), Pacific Grove, California, USA, June 5-8, 1989*, pages 186–195. IEEE Computer Society, 1989. `doi:10.1109/LICS.1989.39173`.

4 Claire Jones. *Probabilistic Non-determinism.* PhD thesis, University of Edinburgh, UK, 1990. URL: `http://hdl.handle.net/1842/413`.

5 Achim Jung and Regina Tix. The troublesome probabilistic power domain. In *Comprox III, Third Workshop on Computation and Approximation*, volume 13, pages 70–91, 1998.

6 John Power and Edmund Robinson. Premonoidal Categories and Notions of Computation. *Math. Struct. Comput. Sci.*, 7(5):453–468, 1997. `doi:10.1017/S0960129597002375`.