Coderelictions for Free Exponential Modalities

Jean-Simon Pacaud Lemay
Mathematics and Computer Science Department, Mount Allison University, Sackville, Canada

Abstract

In a categorical model of the multiplicative and exponential fragments of intuitionistic linear logic (MELL), the exponential modality is interpreted as a comonad ! such that each cofree !-coalgebra $!A$ comes equipped with a natural cocommutative comonoid structure. An important case is when ! is a free exponential modality so that $!A$ is the cofree cocommutative comonoid over A. A categorical model of MELL with a free exponential modality is called a Lafont category. A categorical model of differential linear logic is called a differential category, where the differential structure can equivalently be described by a deriving transformation $!A \otimes A \xrightarrow{\mathsf{d}_A} !A$ or a codereliction $A \xrightarrow{\eta_A} !A$. Blute, Lucyshyn-Wright, and O'Neill showed that every Lafont category with finite biproducts is a differential category. However, from a differential linear logic perspective, Blute, Lucyshyn-Wright, and O'Neill's approach is not the usual one since the result was stated in the dual setting and the proof is in terms of the deriving transformation d. In differential linear logic, it is often the codereliction η that is preferred and that plays a more prominent role. In this paper, we provide an alternative proof that every Lafont category (with finite biproducts) is a differential category, where we construct the codereliction η using the couniversal property of the cofree cocommtuative comonoid $!A$ and show that η is unique. To achieve this, we introduce the notion of an infinitesimal augmentation $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$, which in particular is a !-coalgebra and a comonoid morphism, and show that infinitesimal augmentations are in bijective correspondence to coderelictions (and deriving transformations). As such, infinitesimal augmentations provide a new equivalent axiomatization for differential categories in terms of more commonly known concepts. For a free exponential modality, its infinitesimal augmentation is easy to construct and allows one to clearly see the differential structure of a Lafont category, regardless of the construction of $!A$.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Linear logic

Keywords and phrases Differential Categories, Coderelictions, Differential Linear Logic, Free Exponential Modalities, Lafont Categories, Infinitesimal Augmentations

Digital Object Identifier 10.4230/LIPIcs.CALCO.2021.19

Funding *Jean-Simon Pacaud Lemay*: The author is financially supported by a NSERC Postdoctoral Fellowship (PDF) - Award #: 456414649

Acknowledgements The author would like to thank the anonymous referees for their comments and suggestions which helped improve this paper, as well as Robert A. G. Seely, Paul-André Melliès, Rory Lucyshyn-Wright for useful discussions, support of this project, and editorial comments.

1 Introduction

In the multiplicative and exponential fragments of intuitionistic linear logic (MELL) [16, 17], the exponential modality !, read as either "of course" or "bang", admitts four structural rules: promotion, dereliction, contraction, and weakening. A categorical model of MELL [2, 25, 26, 29], often called a linear category, is a symmetric monoidal closed category equipped with a **monoidal coalgebra modality** ! [3, 4] which interprets the exponential modality. Briefly, a monoidal coalgebra modality is a symmetric monoidal comonad, capturing the promotion and dereliction rules, such that for each object A, the cofree !-coalgebra $!A$ comes equipped with a natural cocommutative comonoid structure, capturing the contraction and weakening rules.

9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021).
Editors: Fabio Gadducci and Alexandra Silva; Article No. 19; pp. 19:1–19:21
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

As shown by Lafont in his Ph.D. thesis [19], an important source of examples of monoidal coalgebra modalities are those for which $!A$ is also the cofree cocommutative comonoid over A. Monoidal coalgebra modalities with this extra couniversal property on $!A$ are known as **free exponential modalities** [27] and models of linear logic with free exponential modalities are known as **Lafont categories** [26]. While free exponential modalities have been around since the beginning with Girard's free exponential modality for coherence spaces [16], new free exponential modalities are still being constructed and studied [11, 20, 22, 31], which shows the importance of these kinds of models. In fact, Lafont categories are arguably the most common example of categorical models of MELL. The simplest construction of a free exponential modality is the one obtained by taking the infinite product of all the symmetrized tensor powers of an object. Melliès, Tabareau, and Tasson give a more general construction [27] as a sequential limit of the symmetrized tensor powers of cofree copointed objects. However, not every free exponential modality can be constructed in these ways. For example, the free exponential modality on the category of modules over an arbitrary commutative (semi)ring R is given by cofree cocommutative R-coalgebras, which are often not simple to describe, but their existence and constructions have been well-studied [1, 28, 32].

Differential linear logic [13, 14, 12], as introduced by Ehrhard and Regnier, is an extension of linear logic which includes a differentiation inference rule, as well as a cocontraction, coweaking, and codereliction for the exponential modality. Blute, Cockett, and Seely then introduced differential categories [6], which were the appropriate categorical structure for modelling differential linear logic. A differential category is an additive symmetric monoidal category with a coalgebra modality which comes equipped with a natural transformation $!A \otimes A \xrightarrow{\mathsf{d}_A} !A$, called a **deriving transformation**, satisfying certain equations based on the properties of differentiation from calculus, such as the Leibniz rule (also known as the product rule) and the chain rule. It is important to note that the basic structure of a differential category is weaker than that of a model of linear logic: the base symmetric monoidal category is not assumed to be closed nor to have finite products, and one only requires a coalgebra modality, which drops the requirement that the underlying comonad be symmetric monoidal. For a monoidal coalgebra modality, differential structure can alternatively be axiomatized in terms of a natural transformation $A \xrightarrow{\eta_A} !A$ called a **codereliction** [6, 3], which is also equivalent to Fiore's notion of a **creation map** [15]. Thus for a monoidal coalgebra modality, there is a bijective correspondence between coderelictions and deriving transformations.

There are many examples of differential categories whose coalgebra modality is a free exponential modality. Indeed, Blute, Cockett, and Seely's original examples of differential categories found in [6] were the category of sets and relations, where the free exponential modality is induced by finite multisets, and the category of vector spaces over an arbitrary field, where the free exponential modality is induced by free symmetric algebras [23]. In [21], Laird, Manzonetto, and McCusker use the dual of the free symmetric algebra to construct a variety of differential categories related to game theory. In [8], Clift and Murfet study the category of vector spaces over an algebraically closed field of characteristic 0 as a categorical model of differential linear logic and uses the fact that the free exponential modality in this case admits a very elegant construction. This raises the natural question of whether free exponential modalities (in an appropriate setting) always comes equipped with a codereliction/deriving transformation, and if a Lafont category is always a differential category. The answer is yes! In [5], Blute, Lucyshyn-Wright, and O'Neill showed that, in the presence of finite biproduct, every free exponential modality admits a deriving transformation, and thus every Lafont category with finite biproducts is a differential category. However, from a differential linear logic perspective, Blute, Lucyshyn-Wright, and O'Neill's approach is not the usual one since: (a) the result was stated in the dual setting, and (b) the proof and construction involve the deriving transformation rather than the codereliction. The latter reason is important since in differential linear logic, it is often the codereliction η that is preferred and plays a more central role instead of the deriving transformation d. Therefore, the goal of this paper is to provide an alternative proof that every Lafont category with finite biproducts is a differential category by showing that every free exponential modality comes equipped with a unique codereliction η, which we will construct using the couniversal property of $!A$.

It is always of mathematical interest to have different proofs of the same result, especially when said proofs take different approaches. In this case, the alternative proof presented here has a more differential linear logic "flavour" to it, and should be of use to those who work more with the codereliction rather than the deriving transformation. This will also help clearly unpack the differential structure of an arbitrary Lafont category with finite biproducts, in particular by showing that the differential structure is independent of the construction of the free exponential modality, but depends solely on the couniversal property of the free exponential modality. To prove the desired result, we use the fact that coderelictions are closely linked to !-coalgebras. In [5], Blute, Lucyshyn-Wright, and O'Neill showed that it was possible to construct !-coalgebras using the deriving transformation d. Therefore, it is also possible to construct !-coalgebras using the codereliction η. Readers familiar with the concept of !-coalgebras may think that the codereliction $A \xrightarrow{\eta_A} !A$ is a !-coalgebra structure since it is of the appropriate type. Unfortunately, this is not the case. The reason for this is because, for an arbitrary coalgebra modality, every !-coalgebra is also a cocommutative comonoid. If η_A was a !-coalgebra structure, then A would be a cocommutative comonoid whose comultiplication is given by zero. However, such a comultiplication does not have a counit! To fix this problem, we borrow a trick from Melliès, Tabareau, and Tasson in [27], by considering the free pointed object over A, which in this case is $k \oplus A$, where k is the monoidal unit and $\oplus$ is the biproduct. Then by using the same construction as in [5], we use the codereliction $A \xrightarrow{\eta_A} !A$ to build a !-coalgebra on $k \oplus A$, $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$. A new observation of this paper is that it turns out that the converse is also true!

The main new notion of study in this paper is that of an **infinitesimal augmentation**, which is a natural transformation $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ such that H_A is a !-coalgebra and a comonoid morphism. One of the main results of this paper is that there is a bijective correspondence between infinitesimal augmentations and coderelictions (and deriving transformations). A possible advantage of infinitesimal augmentations compared to deriving transformations and coderelictions, is that the notions of !-coalgebras and comonoid morphisms are well-known, even to those who are not familiar with differential categories, and provide yet another way of understanding differentiation via these commonly understood concepts. In fact, it turns out that infinitesimal augmentations are closely linked to the notion of tangent categories [9, 10]. Furthermore, for a free exponential modality, its infinitesimal augmentation is easily constructed, unique, and satisfies the necessary axioms almost automatically simply by construction. We hope that this paper will help open the door to revisiting other examples of Lafont categories and studying them from a differential category point of view, such as, for example, the Lafont categories with infinite biproducts studied by Laird in [20].

Conventions. In this paper, we will use diagrammatic order for composition: this means that the composite map fg is the map which first does f then g. All commutative diagrams drawn in this paper are assumed to commute.

2 Coalgebra Modalities

In this background section, we review the notions of comonads and their coalgebras, comonoids, (monoidal) coalgebra modalities and their coalgebras, and the Seely isomorphisms. We take the time to provide these definitions for readers less familiar with category theory, to introduce notation, and in trying to keep this paper as self-contained as possible. For a more in-depth introduction, we refer the reader to the following introductory sources [26, 29].

▶ **Definition 1** ([26, Section 6.8]). *A **comonad** on a category $\mathbb{X}$ is a triple $(!, \delta, \varepsilon)$ consisting of a functor $\mathbb{X} \xrightarrow{!} \mathbb{X}$ and two natural transformations $!A \xrightarrow{\delta_A} !!A$ and $!A \xrightarrow{\varepsilon_A} A$ such that:*

$$\begin{array}{ccc} !A & \xrightarrow{\delta_A} & !!A \\ {\scriptstyle \delta_A}\downarrow & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_{!A}} \\ !!A & \xrightarrow[!(\varepsilon_A)]{} & !A \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{\delta_A} & !!A \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_{!A}} \\ !!A & \xrightarrow[!(\delta_A)]{} & !!!A \end{array} \tag{1}$$

*A **!-coalgebra** is a pair (A, ω) consisting of an object A and a map $A \xrightarrow{\omega} !A$ such that:*

$$\begin{array}{ccc} A & \xrightarrow{\omega} & !A \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\omega} & !A \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ !A & \xrightarrow[!(\omega)]{} & !!A \end{array} \tag{2}$$

*For each object A, the **cofree !-coalgebra** over A is the !-coalgebra $(!A, \delta_A)$. A **!-coalgebra morphism** $(A, \omega) \xrightarrow{f} (B, \omega')$ is a map $A \xrightarrow{f} B$ such that:*

$$\begin{array}{ccc} A & \xrightarrow{\omega} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle !(f)} \\ B & \xrightarrow[\omega']{} & !(B) \end{array} \tag{3}$$

*The category of !-coalgebras and !-coalgebra morphisms is denoted $\mathbb{X}^!$ and is also known as the **Eilenberg-Moore category of coalgebras** of the comonad $(!, \delta, \varepsilon)$. There is a forgetful functor $\mathbb{X}^! \xrightarrow{\mathsf{U}^!} \mathbb{X}$, which is defined on objects as $\mathsf{U}^!(A, \omega) = A$ and on maps as $\mathsf{U}^!(f) = f$.*

Coalgebra modalities are comonads on symmetric monoidal categories such that each cofree coalgebra comes equipped with a natural cocommutative comonoid structure. For simplicity, we will work in a symmetric *strict* monoidal category, that is, we will consider the associativity and unit isomorphisms of the monoidal product as strict equalities. For a symmetric monoidal category $\mathbb{X}$, we denote the monoidal product as $\otimes$, the monoidal unit as k, and the natural symmetry isomorphism as $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$. Therefore, $A \otimes k = A = k \otimes A$ and $(A \otimes B) \otimes C = A \otimes B \otimes C = A \otimes (B \otimes C)$.

▶ **Definition 2** ([26, Section 6.3]). *In a symmetric monoidal category $\mathbb{X}$, a **cocommutative comonoid** is a triple (C, Δ, e) consisting of an object C, a map $C \xrightarrow{\Delta} C \otimes C$ called the **comultiplication**, and a map $C \xrightarrow{\mathsf{e}} k$ called the **counit** such that:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes 1_C} \\ C \otimes C & \xrightarrow[1 \otimes \Delta]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} & & C & & \\ & \swarrow^{=} & \downarrow{\scriptstyle \Delta} & \searrow^{=} & \\ C & \xleftarrow[\mathsf{e} \otimes 1_C]{} & C \otimes C & \xrightarrow[1_C \otimes \mathsf{e}]{} & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ & {\scriptstyle \Delta}\searrow & \downarrow{\scriptstyle \sigma} \\ & & C \otimes C \end{array} \tag{4}$$

*A **comonoid morphism** $(C, \Delta, \mathsf{e}) \xrightarrow{f} (D, \Delta', \mathsf{e}')$ is a map $C \xrightarrow{f} D$ which preserves the comultiplication and counit, that is, the following diagrams commute:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes f} \\ D & \xrightarrow[\Delta']{} & D \otimes D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle \mathsf{e}}\searrow & \downarrow{\scriptstyle \mathsf{e}'} \\ & & k \end{array} \tag{5}$$

The category of cocommutative comonoids and comonoid morphisms is denoted $\mathsf{CCom}[\mathbb{X}]$.

For a symmetric monoidal category $\mathbb{X}$, $\mathsf{CCom}[\mathbb{X}]$ is a symmetric monoidal category where the tensor product of cocommutative comonoids is defined as: $(C, \Delta, \mathsf{e}) \otimes (D, \Delta', \mathsf{e}') := \left(C \otimes D, C \otimes D \xrightarrow{\Delta \otimes \Delta'} C \otimes C \otimes D \otimes D \xrightarrow{1_C \otimes \sigma_{C,D} \otimes 1_D} C \otimes D \otimes C \otimes D, C \otimes D \xrightarrow{\mathsf{e} \otimes \mathsf{e}'} k\right)$, and the monoidal unit k admits an obvious canonical monoidal structure $(k, 1_k, 1_k)$. In fact, this symmetric monoidal structure on $\mathsf{CCom}[\mathbb{X}]$ is a finite product structure.

▶ **Definition 3** ([3, Definition 1]). *A **coalgebra modality** on a symmetric monoidal category is a quintuple $(!, \delta, \varepsilon, \Delta, e)$ consisting of a comonad $(!, \delta, \varepsilon)$, a natural transformation $!A \xrightarrow{\Delta_A} !A \otimes !A$, and a natural transformation $!A \xrightarrow{\mathsf{e}_A} K$ such that for each object A, the triple $(!A, \Delta_A, \mathsf{e}_A)$ is a cocommutative comonoid and $(!A, \Delta_A, \mathsf{e}_A) \xrightarrow{\delta_A} (!!A, \Delta_{!A}, \mathsf{e}_{!A})$ is a comonoid morphism.*

Note that naturality of Δ and e is equivalent to asking that for every map $A \xrightarrow{f} B$, $(!A, \Delta_A, \mathsf{e}_A) \xrightarrow{!(f)} (!B, \Delta_B, \mathsf{e}_B)$ is a comonoid morphism. Furthermore, every !-coalgebra of a coalgebra modality comes equipped with a cocommutative comonoid structure [24, Section 4.1]. Indeed, if (A, ω) is an !-coalgebra, then $(A, \Delta^\omega, \mathsf{e}^\omega)$ is a cocommutative comonoid where the comultiplication and counit are defined as follows:

$$\Delta^\omega := \ A \xrightarrow{\omega} !A \xrightarrow{\Delta_A} !A \otimes !A \xrightarrow{\varepsilon_A \otimes \varepsilon_A} A \otimes A \qquad \mathsf{e}^\omega := \ A \xrightarrow{\omega} !A \xrightarrow{\mathsf{e}_A} k \tag{6}$$

It is important to point out that $(A, \Delta^\omega, \mathsf{e}^\omega)$ is in general only a cocommutative comonoid in the base category $\mathbb{X}$ and not in the coEilenberg-Moore category $\mathbb{X}^!$, since the latter does not necessarily have a monoidal product. Furthermore, since δ_A is a comonoid morphism, when applying this construction to a cofree !-coalgebra $(!A, \delta_A)$ we re-obtain Δ_A and e_A, that is, $\Delta^{\delta_A}_A = \Delta_A$ and $\mathsf{e}^{\delta_A}_A = \mathsf{e}_A$. On top of this, every !-coalgebra morphism becomes a comonoid morphism on the induced comonoid structures, that is, if $(A, \omega) \xrightarrow{f} (B, \omega')$ is a !-coalgebra morphism, then $(A, \Delta^\omega, \mathsf{e}^\omega) \xrightarrow{f} (B, \Delta^{\omega'}, \mathsf{e}^{\omega'})$ is a comonoid morphism. Therefore, this induces a functor from the coEilenberg-Moore category to the category of cocommutative comonoids, $\mathbb{X}^! \xrightarrow{\mathcal{I}^!} \mathsf{CCom}[\mathbb{X}]$. In general, however, $\mathcal{I}^!$ is not equivalence.

We now turn our attention to coalgebra modalities with Seely isomorphisms [2, 4, 30], which requires the symmetric monoidal category to have finite products. For a category with finite products, we denote the binary product of objects by $A \times B$ with projection maps $A \times B \xrightarrow{\pi_0} A$ and $A \times B \xrightarrow{\pi_1} B$, pairing operation $\langle -, - \rangle$, and we denote the chosen terminal object as $\top$, with the unique maps to terminal object as $A \xrightarrow{\mathsf{t}_A} \top$.

▶ **Definition 4** ([3, Definition 10]). *In a symmetric monoidal category $\mathbb{X}$ with finite products $\times$ and terminal object $\top$, a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ has **Seely isomorphisms** if the natural transformation $\chi_{A,B} := !(A \times B) \xrightarrow{\Delta_{A \times B}} !(A \times B) \otimes !(A \times B) \xrightarrow{!(\pi_0) \otimes !(\pi_1)} !A \otimes !B$ and the map $\chi_\top : !\top \xrightarrow{\mathsf{e}_\top} K$ are isomorphisms, so $!\top \cong k$ and $!(A \times B) \cong !A \otimes !B$. A coalgebra* *modality with Seely isomorphisms is called a* ***storage modality****. A* ***monoidal storage category*** *(also sometimes known as a (new) Seely category) is a symmetric monoidal category with finite products and a coalgebra modality which has Seely isomorphisms.*

Storage modalities can equivalently be defined as **monoidal coalgebra modalities** [3, Definition 2], which are coalgebra modalities equipped with a natural transformation $!A \otimes !B \xrightarrow{\mathsf{m}_{A,B}} !(A \otimes B)$ and a map $k \xrightarrow{\mathsf{m}_k} !k$ such that the underlying comonad is ! a symmetric monoidal comonad [24, Definition 3.8], and that both Δ and e are both monoidal transformations and !-coalgebra morphisms (which imply that $\mathsf{m}_{A,B}$ and m_k are comonoid morphisms). As explained in [2, 4], every storage modality is a monoidal coalgebra modality, where $\mathsf{m}_{A,B} := !A \otimes !B \xrightarrow{\chi^{-1}_{A,B}} !(A \times B) \xrightarrow{\delta_{A\times B}} !!(A \times B) \xrightarrow{!(\chi_{A,B})} !(!A \otimes !B) \xrightarrow{!(\varepsilon_A \otimes \varepsilon_B)} !(A \otimes B)$ and $\mathsf{m}_k := k \xrightarrow{\chi^{-1}_{\top}} !\top \xrightarrow{\delta_\top} !!\top \xrightarrow{!(\chi_\top)} !k$. It is worth mentioning that there are multiple equivalent ways of defining a monoidal coalgebra modality. One characterization, which is of particular important to this paper, is that the monoidal coalgebra modality coherences are precisely what is required so that the tensor product of the base category becomes a product in the coEilenberg-Moore category. This is explained in detail in [29]. Explicitly, the terminal object is the !-coalgebra (k, m_k), while the product, which we denote $\otimes^!$, is the !-coalgebra defined as $(A, \omega) \otimes^! (B, \omega') := \Big(A \otimes B, A \otimes B \xrightarrow{\omega \otimes \omega'} !(A) \otimes !(B) \xrightarrow{\mathsf{m}_\otimes} !(A \otimes B)\Big)$. As such, the forgetful functor $\mathbb{X}^! \xrightarrow{\mathsf{U}^!} \mathbb{X}$ preserves the symmetric monoidal structure strictly but not the product structure. On the other hand, $\mathbb{X}^! \xrightarrow{\mathcal{I}^!} \mathsf{CCom}[\mathbb{X}]$ preserves the finite product structure strictly.

There is no shortage of examples of (monoidal) coalgebra modalities since every categorical model of MELL admits a monoidal coalgebra modality/storage modality. For example, Hyland and Schalk provide a nice list of such examples in [18, Section 2.4].

3 Coderelictions

In this section, we review the notion of coderelictions, as well as briefly reviewing differential categories and additive bialgebra modalities. In particular, we highlight the bijective correspondence between coderelictions and deriving transformations. For more details on differential categories, we refer the reader to [6, 3].

The underlying categorical structure of a differential category is not only a symmetric monoidal category but that of an *additive* symmetric monoidal category. Indeed, two of the basic properties of the derivative from classical differential calculus require addition: the Leibniz rule and the constant rule. Therefore we must first discuss the additive structure, and so we begin this section by recalling additive structure by starting with the notion of an additive category. Here we mean “additive” in the Blute, Cockett, and Seely sense of the term [6], that is, enriched over commutative monoids. In particular, we do not assume negatives nor do we assume biproducts which differs from other definitions of an additive category found in the literature.

▶ **Definition 5** ([3, Definition 3])**.** *An* ***additive symmetric monoidal category*** *is a symmetric monoidal category $\mathbb{X}$ such that each hom-set $\mathbb{X}(A, B)$ is a commutative monoid with addition $\mathbb{X}(A, B) \times \mathbb{X}(A, B) \xrightarrow{+} \mathbb{X}(A, B)$, $(f, g) \mapsto f + g$, and zero map $0 \in \mathbb{X}(A, B)$, such that composition and the tensor product preserve the additive structure, that is, the following equalities hold: $k(f + g)h = kfh + kgh$ and $k0h = 0$, and $k \otimes (f + g) \otimes h = k \otimes f \otimes h + k \otimes g \otimes h$ and $k \otimes 0 \otimes h = 0$. An* ***additive storage category*** *is a monoidal storage category which is also an additive symmetric monoidal category.*

We first note that if an additive symmetric monoidal category has finite products, then said finite product structure is in fact a finite biproduct structure that is distributive. We denote the zero object as 0, and the biproduct as $\oplus$ with injection maps $A \xrightarrow{\iota_0} A \oplus B$ and $B \xrightarrow{\iota_1} A \oplus B$, which satisfy the biproduct coherence identities with the projection maps, that is, $\pi_0\iota_0 + \pi_1\iota_1 = 1_{A\oplus B}$, $\iota_0\pi_0 = 1_A$, $\iota_1\pi_1 = 1_B$, $\iota_0\pi_1 = 0$, and $\iota_1\pi_0 = 0$. The additive symmetric monoidal structure guarantees that we also have the distributivity laws between the monoidal and biproduct structures: $(A \oplus B) \otimes (C \oplus D) \cong (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D)$ and $A \otimes 0 \cong 0 \cong 0 \otimes A$. It is worth mentioning that every symmetric monoidal category with distributive finite biproducts is an additive symmetric monoidal category, and that conversely, every additive symmetric monoidal category has a finite biproduct completion. With all that said, biproducts are not necessary for the axiomatization of a differential category.

Differential categories were introduced by Blute, Cockett, and Seely in [6] to provide a categorical axiomatization of the basic properties of the differentiation, as well as provide categorical models of Ehrhard and Regnier's differential linear logic [13, 14].

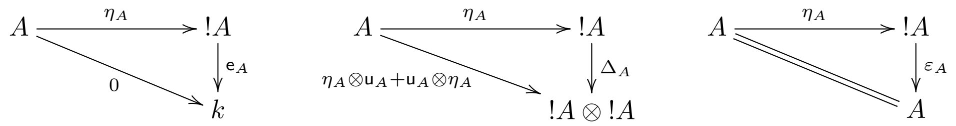
▶ **Definition 6** ([6, Definition 2.4])**.** *A **differential category** is an additive symmetric monoidal category with a coalgebra modality* $(!, \delta, \varepsilon, \Delta, \varepsilon)$ *which comes equipped with a **deriving transformation**, that is, a natural transformation* $!A \otimes A \xrightarrow{\mathsf{d}_A} !A$ *which satisfies the axioms as found in [3, Definition 7]. A **differential storage category** is a differential category with finite products such that its coalgebra modality has Seely isomorphisms.*

The axioms of a deriving transformation include analogues of the product rule, chain rule, that the derivative of a constant map is zero, and that the derivative of a linear map is a constant map. The coKleisli maps of a differential category, that is, the maps of type $!A \xrightarrow{f} B$ are thought of as smooth maps since, in a certain sense, they are differentiable. Indeed, the derivative of a coKleisli map $!A \xrightarrow{f} B$ is the composite $\mathsf{D}[f] := !A \otimes A \xrightarrow{\mathsf{d}_A} !A \xrightarrow{f} B$. This idea is made precise by the fact that the coKleisli category of a differential is a Cartesian differential category [7]. On the other hand, it has been recently shown that the coEilenberg-Moore category of a differential category is a tangent category [9, 10], which we discuss briefly at the end of the next section.

We now turn our attention towards codereliction. To do so, we must first briefly discuss additive bialgebra modalities. Indeed, for an additive symmetric monoidal category with finite biproducts, a storage modality can equivalently be described as an **additive bialgebra modality** [3, Definition 5], which is briefly a coalgebra modality equipped with natural transformations $!A \otimes !A \xrightarrow{\nabla_A} !A$ and $k \xrightarrow{\mathsf{u}_A} !A$ such that $(!A, \nabla_A, \Delta_A, \mathsf{u}_A, \mathsf{e}_A)$ is a bicommutative bialgebra, and other simple coherences hold. As explained in [3, Section 7], $\nabla_A := !A \otimes !A \xrightarrow{\chi^{-1}_{A,A}} !(A \oplus A) \xrightarrow{!(\pi_0+\pi_1)} !A \otimes !A$ and $\mathsf{u}_A := k \xrightarrow{\chi^{-1}_0} !0 \xrightarrow{!(0)} !A$. For differential storage categories, the differential structure can equivalently be described in terms of a codereliction.

▶ **Definition 7** ([3, Definition 9])**.** *For an additive storage category with storage coalgebra modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$*, a **codereliction** is a natural transformation* $A \xrightarrow{\eta_A} !A$ *such that:*

[dC.1] *Constant Rule:* **[dC.2]** *Product Rule:* **[dC.3]** *Linear Rule:*



[dC.4′] *Alternative Chain Rule:*

$$\begin{array}{ccccc} A & & \xrightarrow{\eta_A} & & !A \\ {\scriptstyle \mathsf{u}_A \otimes \eta_A}\downarrow & & & & \downarrow{\scriptstyle \delta_A} \\ !A \otimes !A & \xrightarrow[\delta_A \otimes \eta_{!A}]{} & !!A \otimes !!A & \xrightarrow[\nabla_{!A}]{} & !!A \end{array}$$

[dC.m] *Monoidal Rule:*

$$\begin{array}{ccc} !A \otimes B & \xrightarrow{1_{!A} \otimes \eta_B} & !A \otimes !B \\ {\scriptstyle \varepsilon_A \otimes 1_B}\downarrow & & \downarrow{\scriptstyle \mathsf{m}_{A,B}} \\ A \otimes B & \xrightarrow[\eta_{A \otimes B}]{} & !(A \otimes B) \end{array}$$

We note that the definition of a codereliction provide here is not precisely that found in [3, Definition 9], and is rather defined in terms of the axioms of Fiore's **creation maps** [15]. However, it was shown in [3, Corollary 5] that this axiomatization of a codereliction is equivalent to the original one provided in [6]. As mentioned above, for a storage modality/monoidal coalgebra modality/additive bialgebra modality, coderelictions are in bijective correspondence with deriving transformations.

▶ **Theorem 8** ([3, Theorem 4]). *For an additive storage category with storage modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$*, every codereliction induces a deriving transformation, and every deriving transformation induces a codereliction. Explicitly, if* $A \xrightarrow{\eta_A} !A$ *is a codereliction, then* $\mathsf{d}_A := !A \otimes A \xrightarrow{1_{!A} \otimes \eta_A} !A \otimes !A \xrightarrow{\nabla_A} !A$ *is a deriving transformation. Conversely, if* $!A \otimes A \xrightarrow{\mathsf{d}_A} !A$ *is a deriving transformation, then* $\eta_A := A \xrightarrow{\mathsf{u}_A \otimes 1_A} !A \otimes A \xrightarrow{\mathsf{d}_A} !A$ *is a codereliction. Furthermore, these constructions are inverses of each other, and therefore, there is a bijective correspondence between deriving transformations and coderelictions.*

4 Infinitesimal Augmentations

In this section, we introduce the notion of an infinitesimal augmentation, which is the main novel concept of this paper. We will show that infinitesimal augmentations are equivalent to coderelictions, and therefore provide yet another alternative axiomatization for differential categories. At the end of this section, we discuss the terminology behind the name "infinitesimal augmentation" and the relationship to tangent category structure. The majority of proofs for this section can be found in the appendix.

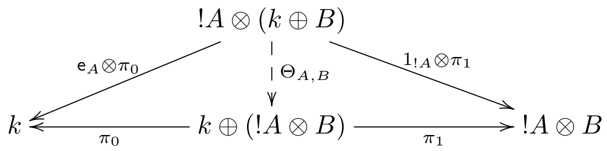
As discussed in the introduction, the basic intuition is that a codereliction $A \xrightarrow{\eta_A} !A$ is not a !-coalgebra structure on A, since A is a missing a comonoid counit. Therefore, we will instead equip $k \oplus A$ with a !-coalgebra. The induced comonoid structure on $k \oplus A$, via the construction of equation (6), should be the canonical one which is conilpotent on the A component. Using element notation, the comultiplication $k \oplus A \xrightarrow{\Lambda_A} (k \oplus A) \otimes (k \oplus A)$ is given by $\Lambda_A(r, a) = (r, 0) \otimes (1, 0) + (0, a) \otimes (1, 0) + (1, 0) \otimes (0, a)$, while the counit is the projection $\pi_0(r, a) = r$. In terms of biproduct distributivity, there are $k \otimes k$, $A \otimes k$ and $k \otimes A$ parts, but no $A \otimes A$ part. So the comultiplication Λ_A is indeed conilpotent on the A component, while the k component is necessary to obtain a counital comonoid.

▶ **Lemma 9.** *In an additive symmetric monoidal category, define the natural transformation* $k \oplus A \xrightarrow{\Lambda_A} (k \oplus A) \otimes (k \oplus A)$ *as* $\Lambda_A := \pi_0(\iota_0 \otimes \iota_0) + \pi_1(\iota_0 \otimes \iota_1) + \pi_1(\iota_1 \otimes \iota_0)$*. Then for every object* A*, the triple* $(k \oplus A, \Lambda_A, \pi_0)$ *is a cocommutative comonoid, and for every map* $A \xrightarrow{f} B$*,* $(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{1_k \oplus f} (k \oplus B, \Lambda_B, \pi_0)$ *is a comonoid morphism.*

Proof. This is straightforward to check and we leave it as an exercise for the reader. ◀

In order to define an infinitesimal augmentation, we will require first defining one extra natural transformation. For an additive storage category with storage coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$, using the universal property of the product, define the natural transformation

$!A \otimes (k \oplus B) \xrightarrow{\Theta_{A,B}} k \oplus (!A \otimes B)$ as the unique map which makes the following diagram commute:



(7)

Or more simply, using the additive structure: $\Theta_{A,B} = (\mathsf{e}_A \otimes \pi_0)\iota_0 + (1_{!A} \otimes \pi_1)\iota_1$.

▶ **Definition 10.** *For an additive storage category with storage modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$, an* ***infinitesimal augmentation*** *is a natural transformation $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ such that:*

[IA.1] *$(k \oplus A, \mathsf{H}_A)$ is an $!$-coalgebra;*

[IA.2] *$(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{\mathsf{H}_A} (!(k \oplus A), \Delta_{k \oplus A}, \mathsf{e}_{k \oplus A})$ is a comonoid morphism;*

[IA.3] *$(!A, \delta_A) \otimes^! (k \oplus B, \mathsf{H}_B) \xrightarrow{\Theta_{A,B}} (k \oplus (!A \otimes B), \mathsf{H}_{!A \otimes B})$ is a $!$-coalgebra morphism.*

The axioms of infinitesimal augmentation are analogous to those of a codereliction. The two diagrams of a !-coalgebra for **[IA.1]** correspond to the Linear Rule **[dC.3]** and the Alternative Chain Rule **[dC.4′]**. The two diagrams of a comonoid morphism for **[IA.2]** correspond to the Constant Rule **[dC.1]** and the Product Rule **[dC.2]**. And lastly, the diagram of a !-coalgebra morphism for **[IA.3]** corresponds to Monoidal Rule **[dC.m]**. It is worth mentioning that it is possible that some of the axioms of an infinitesimal augmentation may be redundant, as was the case for the original definitions of a codereliction and a creation map. However, we've included them here to provide a clear complete story. We now show that the induced comonoid structure from the !-coalgebra structure is the one from Lemma 9.

▶ **Lemma 11.** *If H is an infinitesimal augmentation, then for every object A:*

(i) *$\Delta^{\mathsf{H}_A} = \Lambda_A$ and $\mathsf{e}^{\mathsf{H}_A} = \pi_0$, where Δ^{H_A} and $\mathsf{e}^{\mathsf{H}_A}$ are defined as in equation (6).*

(ii) *$(k, \mathsf{m}_k) \xrightarrow{\iota_0} (k \oplus A, \mathsf{H}_A)$ and $(k \oplus A, \mathsf{H}_A) \xrightarrow{\pi_0} (k, \mathsf{m}_k)$ are $!$-coalgebra morphisms.*

Proof. It follows immediately from **[IA.1]** and **[IA.2]** that (i) holds, which we leave to the reader to check for themselves. For (ii), we use **[IA.1]**, **[IA.2]**, the biproduct identities, naturality of H, and that e is monoidal:

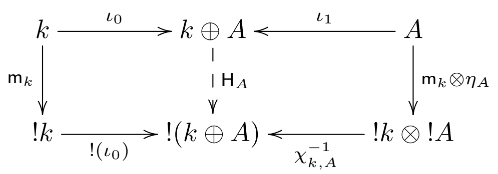
$$\mathsf{H}_A!(\pi_0) = \mathsf{H}_A!(\mathsf{H}_A)!(\mathsf{e}_A) = \mathsf{H}_A\delta_A!(\mathsf{e}_A) = \mathsf{H}_A\mathsf{e}_A\mathsf{m}_k = \pi_0\mathsf{m}_k$$

$$\mathsf{m}_k!(\iota_0) = \iota_0\pi_0\mathsf{m}_k!(\iota_0) = \iota_0\mathsf{H}_A!(\pi_0)!(\iota_0) = \iota_0\mathsf{H}_A!(1_k \oplus 0) = \iota_0(1_k \oplus 0)\mathsf{H}_A = \iota_0\mathsf{H}_A$$

So we conclude that ι_0 and π_0 are !-coalgebra morphisms. ◀

We now show how to construct an infinitesimal augmentation from a codereliction:

▶ **Proposition 12.** *Every codereliction induces an infinitesimal augmentation. Explicitly, for a differential storage category with storage modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ and codereliction $A \xrightarrow{\eta_A} !A$, define the natural transformation $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ as the unique map which makes the following diagram commute (using the couniversal property of the coproduct):*



Or equivalently, using the additive structure: $\mathsf{H}_A := \pi_0\mathsf{m}_k!(\iota_0) + \pi_1(\mathsf{m}_k \otimes \eta_A)\chi^{-1}_{k,A}$. Then H is an infinitesimal augmentation.

Proof. This is actually an application of [5, Theorem 5.1] which we explain in detail in Appendix A. ◀

We now show how to construct a codereliction from an infinitesimal augmentation:

▶ **Proposition 13.** *Every infinitesimal augmentation induces a codereliction. Explicitly, for an additive storage category with coalgebra modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *which has Seely isomorphisms and infinitesimal augmentation* $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$*, define* $\eta_A := A \xrightarrow{\iota_1} k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A) \xrightarrow{!(\pi_1)} !A$*. Then* η *is a codereliction. Therefore, an additive storage category whose storage modality has an infinitesimal augmentation is a differential storage category.*

Proof. See Appendix B. ◀

We now state the first main result of this paper:

▶ **Theorem 14.** *For the storage modality of additive storage category, there is a bijective correspondence between coderelictions and infinitesimal augmentations.*

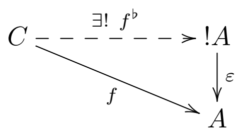
Proof. See Appendix C, where we show that constructions of Proposition 12 and Proposition 13 are inverses of each other. ◀

We conclude this section with a discussion on the terminology behind the name "infinitesimal augmentation". "Augmentation" is a reference to the fact that $k \oplus A$ is always an augmented (co)algebra in the classical sense, in particular since $k \oplus A$ is the (co)free (co)pointed object over A. "Infinitesimal" is related to tangent category terminology. A tangent category [9] is a category equipped with an endofunctor T and various other natural transformations whose axioms generalize the theory of smooth manifolds and their tangent bundles, with the category of smooth manifolds being the canonical example. A representable tangent category is a tangent category with finite products and such that T is a representable functor, that is, $\mathsf{T} \cong (-)^D$ for some exponent object D. The object D is called an **infinitesimal object**. In [10, Section 6] it was shown that, under a mild limit condition, the coEilenberg-Moore category of a differential storage category is a representable tangent category whose infinitesimal object is $(k \oplus k, \mathsf{H}_k)$. Therefore, the !-coalgebra structure of the infinitesimal object is precisely the infinitesimal augmentation. In future work, it would be interesting to further study the connection between infinitesimal augmentations and tangent structure. In particular, infinitesimal augmentations may provide the key in generalizing linear-non-linear adjunctions [2, 25, 26] for differential categories (where one would replace a Cartesian category with a tangent category).

5 Coderelictions for Free Exponential Modalities

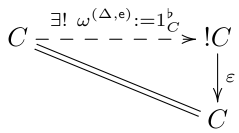
In this section we provide the main objective of this paper, that is, we provide an alternative proof that every additive Lafont category with finite biproducts is a differential storage category. In particular, we will explain how to construct the (necessarily unique) codereliction and induced deriving transformation of the free exponential modality using its couniversal property. In fact, we will first show that every free exponential modality has an infinitesimal augmentation, which is easily constructed using the couniversal property.

▶ **Definition 15** ([27]). *A **free exponential modality** is a coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ such that for each object A, $!A$ is a **cofree cocommutative comonoid** over A, that is, if (C, Δ, e) is a comonoid then for every map $C \xrightarrow{f} A$, there exists a unique comonoid morphism $(C, \Delta, \mathsf{e}) \xrightarrow{f^\flat} (!A, \Delta, \mathsf{e})$ such that the following diagram commutes:*



*A **(additive) Lafont category** is a (additive) symmetric monoidal category with a free exponential modality.*

We should note that here we are using the term "Lafont category" in the sense of Blute, Cockett, and Seely as in [4], which is the same as in [25] but which drops the closed structure requirement. The coEilenberg-Moore category of a free exponential modality is isomorphic to the category of cocommutative comonoids. In other words, for a free exponential modality, every cocommutative comonoid is a !-coalgebra. Explicitly, if (C, Δ, e) is a cocommutative comonoid, then define $(C, \Delta, \mathsf{e}) \xrightarrow{\omega^{(\Delta,\mathsf{e})}} (!C, \Delta_C, \mathsf{e}_C)$ as the unique comonoid morphism such that the following diagram commutes:



Then it follows that $(C, \omega^{(\Delta,\mathsf{e})})$ is a !-coalgebra. Furthermore, if $(C, \Delta, \mathsf{e}) \xrightarrow{f} (D, \Delta', \mathsf{e}')$ is a comonoid morphism, then $(C, \omega^{(\Delta,\mathsf{e})}) \xrightarrow{f} (D, \omega^{(\Delta',\mathsf{e}')})$ is a !-coalgebra morphism. On top of this, it follows that $\Delta^{\omega^{(\Delta,\mathsf{e})}} = \Delta$ and $\mathsf{e}^{\omega^{(\Delta,\mathsf{e})}} = \mathsf{e}$, where $\Delta^{\omega^{(\Delta,\mathsf{e})}}$ and $\mathsf{e}^{\omega^{(\Delta,\mathsf{e})}}$ are defined as in (6). Therefore, this induces a functor $\mathcal{J}^! : \mathsf{CCom}[\mathbb{X}] \to \mathbb{X}^!$ which is inverse to $\mathcal{I}^! : \mathbb{X}^! \to \mathsf{CCom}[\mathbb{X}]$. Furthermore, it is a well-known fact that free exponential modalities are always monoidal coalgebra modalities [2, 25]. Explicitly, $(k, 1_k, 1_k) \xrightarrow{\mathsf{m}_k} (!k, \Delta_k, \mathsf{e}_k)$ and $(!A, \Delta_A, \mathsf{e}_A) \otimes (!B, \Delta_B, \mathsf{e}_B) \xrightarrow{\mathsf{m}_{A,B}} (!(A \otimes B), \Delta_{A\otimes B}, \mathsf{e}_{A\otimes B})$ are the unique comonoid morphisms defined respectively as $\mathsf{m}_k := 1_k^\flat$ and $\mathsf{m}_{A,B} := (\varepsilon_A \otimes \varepsilon_B)^\flat$. As such, in the presence of finite products, every free exponential modality has Seely isomorphisms and is therefore a storage modality. Explicitly, $(!A, \Delta_A, \mathsf{e}_A) \otimes (!B, \Delta_B, \mathsf{e}_B) \xrightarrow{\chi^{-1}_{A,B}} (!(A \times B), \Delta_{A\times B}, \mathsf{e}_{A\times B})$ and $(k, 1_k, 1_k) \xrightarrow{\chi^{-1}_{\top}} (!\top, \Delta_\top, \mathsf{e}_\top)$ are the unique comonoid morphisms defined respectively as $\chi^{-1}_{A,B} := \langle \varepsilon_A \otimes \mathsf{e}_B, \mathsf{e}_A \otimes \varepsilon_B \rangle^\flat$ and $\mathsf{t}_k^\flat$. For an additive Lafont category, it follows that every free exponential modality is thus also an additive bialgebra modality. Explicitly, $(!A, \Delta_A, \mathsf{e}_A) \otimes (!A, \Delta_A, \mathsf{e}_A) \xrightarrow{\nabla_A} (!A, \Delta_A, \mathsf{e}_A)$ and $(k, 1_k, 1_k) \xrightarrow{\mathsf{u}_A} (!A, \Delta_A, \mathsf{e}_A)$ are the unique comonoid morphisms defiend respectively as $\nabla_A := (\varepsilon_A \otimes \mathsf{e}_A + \mathsf{e}_A \otimes \varepsilon_A)^\flat$ and $\mathsf{u}_A := 0^\flat$. So in particular, in the presence of additive structure, for a free exponential modality, $!A$ is also a bicommutative bialgebra.

We now turn our attention to constructing the infinitesimal augmentation for the free exponential modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ of an additive Lafont category with finite biproducts. As shown in Lemma 9, $(k \oplus A, \Lambda_A, \pi_0)$ is a cocommutative comonoid, and therefore admits a canonical !-coalgebra structure. Define the natural transformation $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ as $\mathsf{H}_A := \omega^{(\Lambda,\pi_0)}$, that is, $(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{\mathsf{H}_A} (!(k \oplus A), \Delta_{k\oplus A}, \mathsf{e}_{k\oplus A})$ is the unique comonoid morphism such that the following diagram commutes:

$$\begin{array}{ccc} k \oplus A & \xrightarrow{\exists!\ \mathsf{H}^\flat_A := 1^\flat_{k \oplus A}} & !(k \oplus A) \\ & \searrow^{=} & \downarrow \varepsilon_{k \oplus A} \\ & & k \oplus A \end{array} \tag{8}$$

We now carefully show in steps that H is indeed an infinitesimal augmentation. Starting with the an important, but often overlooked step, of showing that H is natural.

▶ **Lemma 16.** H *is a natural transformation.*

Proof. Consider a map $A \xrightarrow{f} B$. By Lemma 9, $(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{1_k \oplus f} (k \oplus B, \Lambda_B, \pi_0)$ is a comonoid morphism. Therefore, $(1_k \oplus f)\mathsf{H}_B$ and $\mathsf{H}_A!(1_k \oplus f)$ are comonoid morphisms of the same type $(k \oplus A, \Lambda_A, \pi_0) \to (!(k \oplus A), \Delta_{k \oplus A}, \mathsf{e}_{k \oplus A})$. However, we easily compute that:

$$\mathsf{H}_A!(1_k \oplus f)\varepsilon_{k \oplus B} = \mathsf{H}_A\varepsilon_A(1_k \oplus f) = (1_k \oplus f) = (1_k \oplus f)\mathsf{H}_B\varepsilon_{k \oplus B}$$

Since $\mathsf{H}_A!(1_k \oplus f)\varepsilon_{k \oplus B} = (1_k \oplus f)\mathsf{H}_B\varepsilon_{k \oplus B}$, it follows from the couniversal property of $!(k \oplus B)$ that $\mathsf{H}_A!(1_k \oplus f) = (1_k \oplus f)\mathsf{H}_B$. So we conclude that H is a natural transformation. ◀

Next, it follows that **[IA.1]** and **[IA.2]** are automatic by construction.

▶ **Lemma 17.** *For every object A,*

(i) $(k \oplus A, \mathsf{H}_A)$ *is an $!$-coalgebra;*

(ii) $(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{\mathsf{H}_A} (!(k \oplus A), \Delta_{k \oplus A}, \mathsf{e}_{k \oplus A})$ *is a comonoid morphism.*

Proof. Both are automatic by construction since $\mathsf{H}_A := \omega^{(\Lambda, \pi_0)}$. ◀

For the free exponential modality, !-coalgebra morphisms correspond to comonoid morphism. Therefore, in order to prove **[IA.3]**, it is sufficient to show that $\Theta_{A,B}$ is a comonoid morphism of the appropriate type. To do so, we will require the following lemma:

▶ **Lemma 18.** *Let (C, Δ, e) be a cocommutative comonoid. Then for every object A, $(C, \Delta, \mathsf{e}) \xrightarrow{F} (k \oplus A, \Lambda_A, \pi_0)$ is a comonoid morphism if and only if $C \xrightarrow{F} k \oplus A$ is of the form $F = \langle \mathsf{e}, f \rangle$ for some map $C \xrightarrow{f} A$ such that $\Delta(f \otimes f) = 0$.*

Proof. Recall that any map $C \xrightarrow{F} k \oplus A$ satisfies $F = \langle F\pi_0, F\pi_1 \rangle$. Then suppose that $(C, \Delta, \mathsf{e}) \xrightarrow{F} (k \oplus A, \Lambda_A, \pi_0)$ is a comonoid morphism. Since F preserves the counit, it follows that $F\pi_0 = \mathsf{e}$. Next, note that by definition $\Lambda_A(\pi_1 \otimes \pi_1) = 0$. Then since F preserves the comultiplication it follows that $\Delta(F \otimes F)(\pi_1 \otimes \pi_1) = F\Lambda_A(\pi_1 \otimes \pi_1) = 0$. Therefore, $F\pi_1$ satisfies the desired equality, and so $F = \langle \mathsf{e}, F\pi_1 \rangle$ is of the desired form. Conversely, suppose that f is a map which satisfies $\Delta(f \otimes f) = 0$. By definition, it is automatic that $\langle \mathsf{e}, f \rangle$ preserves the counit since $\langle \mathsf{e}, f \rangle \pi_0 = \mathsf{e}$. Next we need to show that $\langle \mathsf{e}, f \rangle$ also preserves the comultiplication. To do so, we will show that $\langle \mathsf{e}, f \rangle \Lambda_A(\pi_i \otimes \pi_j) = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)(\pi_i \otimes \pi_j)$ for $i, j \in \{0, 1\}$.

$$\langle \mathsf{e}, f \rangle \Lambda_A(\pi_0 \otimes \pi_0) = \langle \mathsf{e}, f \rangle \pi_0 = \mathsf{e} = \Delta(\mathsf{e} \otimes \mathsf{e}) = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)(\pi_0 \otimes \pi_0)$$

$$\langle \mathsf{e}, f \rangle \Lambda_A(\pi_0 \otimes \pi_1) = \langle \mathsf{e}, f \rangle \pi_1 = f = \Delta(\mathsf{e} \otimes f) = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)(\pi_0 \otimes \pi_1)$$

$$\langle \mathsf{e}, f \rangle \Lambda_A(\pi_1 \otimes \pi_0) = \langle \mathsf{e}, f \rangle \pi_1 = f = \Delta(f \otimes \mathsf{e}) = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)(\pi_1 \otimes \pi_0)$$

$$\langle \mathsf{e}, f \rangle \Lambda_A(\pi_1 \otimes \pi_1) = 0 = \Delta(f \otimes f) = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)(\pi_1 \otimes \pi_1)$$

Then by the distributivity of the biproduct and the universal property of the product, it follows that $\langle \mathsf{e}, f \rangle \Lambda_A = \Delta\left(\langle \mathsf{e}, f \rangle \otimes \langle \mathsf{e}, f \rangle\right)$. Therefore, $\langle \mathsf{e}, f \rangle$ is a comonoid morphism. ◀

▶ **Corollary 19.** *$(!A, \Delta_A, \mathsf{e}_A) \otimes (k \oplus B, \Lambda_B, \pi_0) \xrightarrow{\Theta_{A,B}} (k \oplus (!A \otimes B), \Lambda_{!A\otimes B}, \pi_0)$ is a comonoid morphism where $!A \otimes (k \oplus B) \xrightarrow{\Theta_{A,B}} k \oplus (!A \otimes B)$ is defined as in (7). Therefore, we also have that $(!A, \delta_A) \otimes^! (k \oplus B, \mathsf{H}_B) \xrightarrow{\Theta_{A,B}} (k \oplus (!A \otimes B), \mathsf{H}_{!A\otimes B})$ is a !-coalgebra morphism.*

Proof. By construction $\Theta_{A,B} = \langle \mathsf{e}_A \otimes \pi_0, 1_{!A} \otimes \pi_1 \rangle$. Then by Lemma 18, to show that $\Theta_{A,B}$ is a comonoid morphism, it suffices to show that $1_{!A} \otimes \pi_1$ satisfies the extra identity, since the first component of $\Theta_{A,B}$ is indeed the counit of $!A \otimes (k \oplus B)$. However by naturality of the symmetry isomorphism and that $\Lambda_B(\pi_1 \otimes \pi_1) = 0$, we easily see that (we omit the subscripts for space):

$$(\Delta \otimes \Lambda)(1 \otimes \sigma \otimes 1)(1 \otimes \pi_1 \otimes 1 \otimes \pi_1) = (\Delta \otimes \Lambda)(1 \otimes 1 \otimes \pi_1 \otimes \pi_1)(1 \otimes \sigma \otimes 1) = 0$$

So $\Theta_{A,B}$ is a comonoid morphism. Furthermore, it is straightforward to check that for $(!A, \Delta_A, \mathsf{e}_A) \otimes (k \oplus B, \Lambda_B, \pi_0)$, its associated !-coalgebra is precisely $(!A, \delta_A) \otimes^! (k \oplus B, \mathsf{H}_B)$. Therefore, since every comonoid morphism is also a !-coalgebra morphism between the induced !-coalgebras, it follows that $\Theta_{A,B}$ is a !-coalgebra morphism. ◀

Bringing all of the above lemmas and corollary together, we obtain:

▶ **Proposition 20.** *For an additive Lafont category with free exponential modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ and finite biproducts, H as defined in equation (8) is an infinitesimal augmentation, and furthermore it is the unique infinitesimal augmentation for the free exponential modality.*

Proof. Lemma 16 shows that H is a natural transformation, while Lemma 17 and Corollary 19 show that H satisfies **[IA.1]** and **[IA.2]** respectively. So H is indeed an infinitesimal augmentation. Now suppose that H' was another infinitesimal augmentation. By Lemma 11, we have that $\Delta^{\mathsf{H}_A} = \Lambda_A = \Delta^{\mathsf{H}'_A}$ and $\mathsf{e}^{\mathsf{H}_A} = \pi_0 = \mathsf{e}^{\mathsf{H}'_A}$. Therefore, the !-coalgebras $(k \oplus A, \mathsf{H}_A)$ and $(k \oplus A, \mathsf{H}'_A)$ both induce the same cocommutative comonoid $(k \oplus A, \Lambda_A, \pi_0)$. However, since the coEilenberg-Moore category of the free exponential modality is isomorphic to the category of cocommutative comonoids, it follows that $(k \oplus A, \mathsf{H}_A) = (k \oplus A, \mathsf{H}'_A)$. Therefore, $\mathsf{H}_A = \mathsf{H}'_A$. So we conclude that H is the unique infinitesimal augmentation for the free exponential modality. ◀

Therefore, we obtain an alternative proof of Blute, Lucyshyn-Wright, and O'Neill in terms of coderelictions and differential categories (rather than deriving transformations and codifferential categories) which is the main contribution of this paper.

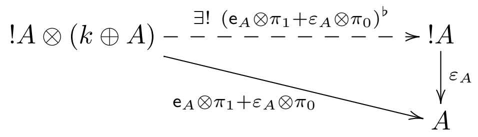
▶ **Theorem 21** ([5, Theorem 4.4])**.** *For an additive Lafont category with free exponential modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ and finite biproducts, the free exponential modality comes equipped with a unique codereliction $A \xrightarrow{\eta_A} !A$ defined as follows, where $(k \oplus A, \Lambda_A, \pi_0) \xrightarrow{\pi^\flat} (!A, \Delta_A, \mathsf{e}_A)$ is the unique comonoid morphism such that the diagram on the right commutes:*

$$\eta_A = \; A \xrightarrow{\iota_1} k \oplus A \xrightarrow{\pi_1^\flat} !A \qquad\qquad \begin{array}{ccc} k \oplus A & \overset{\exists!\ \pi_1^\flat}{\dashrightarrow} & !A \\ & \underset{\pi_1}{\searrow} & \downarrow{\scriptstyle \varepsilon_A} \\ & & A \end{array}$$

Furthermore, this codereliction is precisely the induced codereliction from the infinitesimal augmentation H from Proposition 20 via the construction of Proposition 13. The (necessarily unique) deriving transformation $!A \otimes A \xrightarrow{\mathsf{d}_A} !A$ induced by the construction of Theorem 8 is equal to the following composition,

$$\mathsf{d}_A = \; !A \otimes A \xrightarrow{1_{!A} \otimes \iota_1} !A \otimes (k \oplus A) \xrightarrow{(\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat} !A$$

where $(\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat : (!A, \Delta_A, \mathsf{e}_A) \otimes (k \oplus A, \Lambda_A, \pi_0) \to (!A, \Delta_A, \mathsf{e}_A)$ *is the unique comonoid morphism such that the following diagram commutes:*



Therefore, every additive Lafont category with finite biproducts is a differential category.

Proof. By Proposition 20, H is an infinitesimal augmentation and so, by Proposition 13, H induces a codereliction η defined as $\eta_A = \iota_1 \mathsf{H}_A!(\pi_1)$. Since H is the unique infinitesimal augmentation, by the bijective correspondence of Theorem 14, it follows that η must also be the unique codereliction for the free exponential modality. Next, in order to show the desired equality, it suffices to show that $\pi_1^\flat = \mathsf{H}_A!(\pi_1)$. To do so, first note that by definition, $\pi^\flat$ and $\mathsf{H}_A!(\pi_1)$ are comonoid morphisms of the same type $(k \oplus A, \Lambda_A, \pi_0) \to (!A, \Delta_A, \mathsf{e}_A)$. Also note that we have the following equality:

$$\mathsf{H}_A!(\pi_1)\varepsilon_A = \mathsf{H}_A \varepsilon_A \pi_1 = \pi_1$$

Since $\mathsf{H}_A!(\pi_1)\varepsilon_A = \pi_1 = \pi_1^\flat \varepsilon_A$, by the couniversal property of $!A$, it follows that $\mathsf{H}_A!(\pi_1) = \pi_1^\flat$. Therefore, we have that:

$$\eta_A = \iota_1 \mathsf{H}_A!(\pi_1) = \iota_1 \pi_1^\flat$$

Thus we conclude that $\eta_A = \iota_1 \pi_1^\flat$. By Theorem 8, the codereliction η induces a deriving transformation d defined as $\mathsf{d}_A = (1_{!A} \otimes \eta_A)\nabla_A$. In order to show to the desired equality, it suffices to show that $(1_{!A} \otimes \pi_1^\flat)\nabla_A = (\mathsf{e}_A \otimes \pi_1)^\flat$. Using again the same strategy as before, we note that $(1_{!A} \otimes \pi_1^\flat)\nabla_A$ and $(\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat$ are comonoid morphisms of the same type $(!A, \Delta_A, \mathsf{e}_A) \otimes (k \oplus A, \Lambda_A, \pi_0) \to (!A, \Delta_A, \mathsf{e}_A)$. We also compute the following equality:

$$(1_{!A} \otimes \pi_1^\flat)\nabla_A \varepsilon_A = (1_{!A} \otimes \pi_1^\flat)(\mathsf{e}_A \otimes \varepsilon_A) + (1_{!A} \otimes \pi_1^\flat)(\varepsilon_A \otimes \mathsf{e}_A) = \mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0$$

Since $(1_{!A} \otimes \pi_1^\flat)\nabla_A \varepsilon_A = \mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0 = (\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat \varepsilon_A$, by the couniversal property of $!A$, it follows that $(1_{!A} \otimes \pi_1^\flat)\nabla_A = (\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat$. Therefore, we have that:

$$\mathsf{d}_A = (1_{!A} \otimes \eta_A)\nabla_A = (1_{!A} \otimes \iota_1)(1 \otimes \pi_1^\flat)\nabla_A = (1_{!A} \otimes \iota_1)(\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat$$

Thus we conclude that $\mathsf{d}_A = (1_{!A} \otimes \iota_1)(\mathsf{e}_A \otimes \pi_1 + \varepsilon_A \otimes \pi_0)^\flat$. ◀

6 Examples

In this section, we provide some examples of free exponential modalities and their coderelictions. Other interesting examples of differential categories with free exponential modalities are studied in [3, 5, 6, 8, 12].

▶ **Example 22.** Let REL be the category of sets and relations, where recall that the tensor product is given by the Cartesian product of sets, $X \otimes Y = X \times Y$, the monoidal unit is a chosen singleton, $k = \{*\}$, the biproduct is given by the disjoint union of sets, $X \oplus Y = X \sqcup Y$. REL is also a Lafont category where for a set X, $!X$ is the set of finite multisets of elements of X. We will denote finite multisets as $[\![x_1, \ldots, x_n]\!]$, $x_i \in X$, where recall that we can have multiple copies of the same element in a finite multiset. The codereliction is the relation which associates an element of X to the bag containing only said element, $\eta_X := \{(x, [\![x]\!]) |\ \forall x \in X\}$ $\subseteq X \times !X$. The deriving transformation is the relation which adds an element into the bag, $\mathsf{d}_X := \{(([\![x_1, \dots, x_n]\!], x), [\![x_1, \dots, x_n, x]\!]) \mid \forall x_i, x \in X\}$. The infinitesimal extension relates the element of the singleton to all possible bags of copies of the singleton element, and relates an element of X to bags of copies of the singleton element with said element added in: $\mathsf{H}_X := \{(*, [\![\underbrace{*, \dots, *}_{n\text{-copies}}]\!]) |\ \forall n \in \mathbb{N}\} \cup \{(x, [\![\underbrace{*, \dots, *}_{n\text{-copies}}, x]\!]) |\ \forall x \in X, n \in \mathbb{N}\} \subseteq (\{*\} \sqcup X) \times !(\{*\} \sqcup X)$. For more details on this example, see [6, Section 2.5.1].

▶ **Example 23.** Let k be a field, and let VEC_k be the category of k-vector spaces and k-linear maps between them. Its dual VEC_k^{op} is a Lafont category where for a k-vector space V, $!V = \mathsf{Sym}(V)$, the free symmetric algebra over V. Note that in VEC_k, $\mathsf{Sym}(V)$ is the free commutative k-algebra over V, and therefore $\mathsf{Sym}(V)$ is the cofree cocomutative comonoid over V in VEC_k^{op}. In particular, if X is a basis of V, then $\mathsf{Sym}(V) \cong k[X]$, where the latter is the polynomial ring over X. We will express η, d, and H in terms of polynomials, and if their types look backwards, it is because we expressing them in VEC_k. The codereliction $K[X] \xrightarrow{\eta_V} V$ is defined as picking out the degree 1 terms of the polynomial, that is, its x_i terms. This can be described as follows: $\eta_V(p(\vec{x})) = \sum\limits_{i=1}^{n} \frac{\partial p(\vec{x})}{\partial x_i}(\vec{0}) x_i$. Note that evaluating a polynomial at zero extracts its constant term. The constant term of $\frac{\partial p(\vec{x})}{\partial x_i}$ is precisely the scalar factor of x_i. Therefore, $\frac{\partial p(\vec{x})}{\partial x_i}(0) x_i$ are precisely the degree 1 terms of $p(\vec{x})$. The deriving transformation $k[X] \xrightarrow{\mathsf{d}_V} k[X] \otimes V$ is defined as mapping a polynomial to its sum of its partial derivatives: $\mathsf{d}_V(p(\vec{x})) = \sum\limits_{i=1}^{n} \frac{\partial p(\vec{x})}{\partial x_i} \otimes x_i$. For the infinitesimal extension, note that $\mathsf{Sym}(k \oplus V) \cong k[X, y]$, therefore $k[X, y] \xrightarrow{\mathsf{H}_V} k \oplus V$ is defined as $\mathsf{H}_V(p(\vec{x}, y)) = p(\vec{0}, 1) + \sum\limits_{i=1}^{n} \frac{\partial p(\vec{x}, y)}{\partial x_i}(\vec{0}, 1) x_i$. We note that this example can be generalized to the category of modules over any commutative semiring. For more details on this example, see [6, Section 2.5.3].

▶ **Example 24.** Example 22 and Example 23 are in fact examples of the same general construction of a free exponential modality given by the product of the symmetric tensor powers, that is, $!A = \prod\limits_{n \in \mathbb{N}} \mathsf{S}_n(A)$, where $\prod\limits_{n \in \mathbb{N}}$ is the countable product and $\mathsf{S}_n(A)$ is the equalizer of all permutations $A^{\otimes^n} \xrightarrow{\cong} A^{\otimes^n}$. The codereliction is defined as the "injection" of A into $!A$ since $\mathsf{S}_1(A) = A$, that is, let $!A \xrightarrow{\pi_n} \mathsf{S}_n(A)$ be the projection map of the product (where note that $!A \xrightarrow{\pi_1} A$), then the codereliction $A \xrightarrow{\eta_A} !(A)$ is defined as the unique map (using the universal property of the product) such that $\eta\pi_1 = 1_A$ and $\eta\pi_n = 0$ for $n \neq 1$. We stress that not all free exponential modalities arise in this manner, as explained in [27].

▶ **Example 25.** Let k be a field, then VEC_k is a Lafont category where for a k-vector space V, $!V$ is the cofree cocommutative k-coalgebra over V. When k is algebraically closed and has characteristic zero (such as $\mathbb{C}$), then $!V$ admits a nice expression [28]: if X is a basis for V, then $!V \cong \bigoplus\limits_{v \in V} k[X]$. In this case, the codereliction $V \xrightarrow{\eta_V} \bigoplus\limits_{v \in V} k[X]$ maps basis elements $x \in X$ to the monomial in the $0 \in V$ component: $\eta_V(x) = (x)_0$. This differential category, and its resulting model of differential linear logic, was studied in detail by Clift and Murfet in [8]. We note that this example can be generalized to the category of modules over any commutative semiring, though the cofree cocommutative coalgebra may not have as nice a form. It is also important to observe that this example is not of the form of Example 24.

References

1 M. Barr. Coalgebras over a commutative ring. *Journal of Algebra*, 32(3):600–610, 1974.

2 G.M. Bierman. What is a categorical model of intuitionistic linear logic? In *International Conference on Typed Lambda Calculi and Applications*, pages 78–93. Springer, 1995.

3 R. F. Blute, J. R. B. Cockett, J.-S. P. Lemay, and R. A. G. Seely. Differential categories revisited. *Applied Categorical Structures*, 28:171–235, 2020. doi:10.1007/s10485-019-09572-y.

4 R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Cartesian differential storage categories. *Theory and Applications of Categories*, 30(18):620–686, 2015.

5 R. F Blute, R. B. B. Lucyshyn-Wright, and K O'Neill. Derivations in codifferential categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 57:243–280, 2016.

6 R. F. Blute, Blute, J. R. B. Cockett, and R. A. G. Seely. Differential categories. *Mathematical structures in computer science*, 16(6):1049–1083, 2006.

7 R. F. Blute, Blute, J. R. B. Cockett, and R. A. G. Seely. Cartesian differential categories. *Theory and Applications of Categories*, 22(23):622–672, 2009.

8 J. Clift and D. Murfet. Cofree coalgebras and differential linear logic. *Mathematical Structures in Computer Science*, 30(4):416–457, 2020. doi:10.1017/S0960129520000134.

9 J. R. B. Cockett and G. S. H. Cruttwell. Differential structure, tangent structure, and SDG. *Applied Categorical Structures*, 22(2):331–417, 2014.

10 J.R. B. Cockett, J.-S. P. Lemay, and R. B. B. Lucyshyn-Wright. Tangent Categories from the Coalgebras of Differential Categories. In *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*, volume 152 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 17:1–17:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/LIPIcs.CSL.2020.17.

11 R. Crubillé, T. Ehrhard, M. Pagani, and C. Tasson. The free exponential modality of probabilistic coherence spaces. In *International Conference on Foundations of Software Science and Computation Structures*, pages 20–35. Springer, 2017.

12 T. Ehrhard. An introduction to differential linear logic: proof-nets, models and antiderivatives. *Mathematical Structures in Computer Science*, 28:995–1060, 2018.

13 T. Ehrhard and L. Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309(1):1–41, 2003.

14 T. Ehrhard and L. Regnier. Differential interaction nets. *Theoretical Computer Science*, 364(2):166–195, 2006.

15 M.P. Fiore. Differential structure in models of multiplicative biadditive intuitionistic linear logic. In *International Conference on Typed Lambda Calculi and Applications*, pages 163–177. Springer, 2007.

16 J.-Y. Girard. Linear logic. *Theoretical computer science*, 50(1):1–101, 1987.

17 J.-Y. Girard and Y. Lafont. Linear logic and lazy computation. In *International Joint Conference on Theory and Practice of Software Development*, pages 52–66. Springer, 1987.

18 M. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theoretical computer science*, 294(1-2):183–231, 2003.

19 Y. Lafont. *Logiques, catégories et machines.* PhD thesis, PhD thesis, Université Paris 7, 1988.

20 J. Laird. Weighted models for higher-order computation. *Information and Computation*, 275:104645, 2020.

21 J. Laird, G. Manzonetto, and G. McCusker. Constructing differential categories and deconstructing categories of games. *Information and Computation*, 222:247–264, 2013.

22 J. Laird, G. Manzonetto, G. McCusker, and M. Pagani. Weighted relational models of typed lambda-calculi. In *Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 301–310. IEEE Computer Society, 2013.

23 S. Lang. Algebra, revised 3rd ed. *Graduate Texts in Mathematics*, 211, 2002.

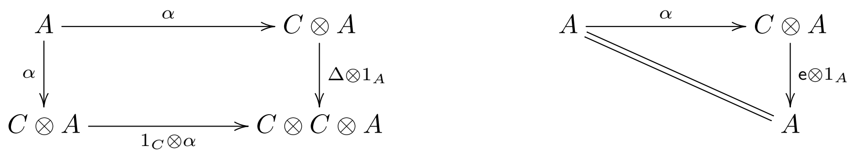
24 J.-S. P. Lemay. Lifting Coalgebra Modalities and MELL Model Structure to Eilenberg-Moore Categories. *Logical Methods in Computer Science*, Volume 15, Issue 4, November 2019. doi:10.23638/LMCS-15(4:8)2019.

25 P.-A. Melliès. Categorical models of linear logic revisited. *HAL preprint, hal-00154229*, 2003.
26 P.-A. Melliès. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.
27 P.-A. Melliès, N. Tabareau, and C. Tasson. An explicit formula for the free exponential modality of linear logic. *Mathematical Structures in Computer Science*, pages 1–34, 2017.
28 D. Murfet. On Sweedler's cofree cocommutative coalgebra. *Journal of Pure and Applied Algebra*, 219(12):5289–5304, 2015.
29 A. Schalk. What is a categorical model of linear logic? *Manuscript, available from http://www.cs.man.ac.uk/ schalk/notes/llmodel.pdf*, 2004.
30 R. A. G. Seely. Linear logic, *-autonomous categories and cofree coalgebras. In *Categories in Computer Science and Logic (1987)*, pages 371–382. American Mathematical Society, 1989.
31 S. Slavnov. On banach spaces of sequences and free linear logic exponential modality. *Mathematical Structures in Computer Science*, 29(2):215–242, 2019. `doi:10.1017/S0960129517000251`.
32 M. E. Sweedler. Hopf algebras. mathematical lecture note series, 1969.

A Proof of Proposition 12

The key to this proof is that we make use of the dual of [5, Theorem 5.1]. To do so, we must first recall the definition of comodules of a cocommutative comonoid.

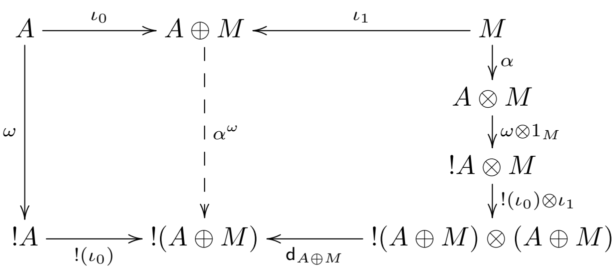
▶ **Definition 26.** *In a symmetric monoidal category, for a cocommutative comonoid* (C, Δ, e)*, a* (C, Δ, e)*-**comodule** is a pair* (M, α) *consisting of an object* M *and a map* $M \xrightarrow{\alpha} C \otimes M$ *called the **coaction**, such that:*



For a coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *and a* $!$*-coalgebra* (A, ω)*, a* (A, ω)*-**comodule** is a* $(A, \Delta^\omega, \mathsf{e}^\omega)$*-comodule, where* Δ^ω *and* e^ω *are defined as in (6).*

As explained in [5], one can use the deriving transformation d to construct new $!$-coalgebras using $!$-coalgebras and their comodules.

▶ **Theorem 27** ([5, Theorem 5.1, Proposition 5.4])**.** *In a differential category with coalgebra modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$*, deriving transformation* d*, and finite biproducts* $\oplus$*, if* (A, ω) *is a* $!$*-coalgebra and* (M, α) *a* (A, ω)*-comodule, define the map* $A \oplus M \xrightarrow{\alpha^\omega} !(A \oplus M)$ *as the unique map which makes the following diagram commute (using the couniversal property of the coproduct):*



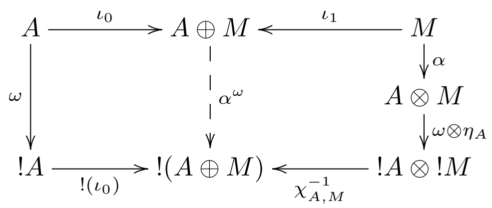
Alternatively using the additive structure, $\alpha^\omega := \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes 1_M)(!(\iota_0) \otimes \iota_1)\mathsf{d}_{A \oplus M}$*. Then* $(A \oplus M, \alpha^\omega)$ *is an* $!$*-coalgebra. Furthermore, the following equalities hold:*

$$\Delta^{\alpha^\omega} := \pi_0 \Delta^\omega(\iota_0 \otimes \iota_0) + \pi_1 \alpha(\iota_0 \otimes \iota_1) + \pi_1 \sigma_{A,M}(\iota_1 \otimes \iota_0) \qquad \mathsf{e}^{\alpha^\omega} = \pi_0 \mathsf{e}^\omega$$

where $A \oplus M \xrightarrow{\Delta^{\alpha^\omega}} (A \oplus M) \otimes (A \oplus M)$ *and* $A \oplus M \xrightarrow{\mathsf{e}^{\alpha^\omega}} k$ *are defined as in (6).*

For a storage modality, α^ω can alternatively be defined using the Seely isomorphisms and codereliction:

▶ **Lemma 28.** *In a differential storage category with storage modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *and codereliction* η*, if* (A, ω) *is a* $!$*-coalgebra and* (M, α) *a* (A, ω)*-comodule, then* α^ω *from the previous theorem can alternatively be described as the unique map which makes the following diagram commute:*



Alternatively using the additive structure, the following equality holds:

$$\alpha^\omega = \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes \eta_M)\chi^{-1}_{A,M}$$

Proof. Using that $\mathsf{d}_A = (1_{!A} \otimes \eta_A)\nabla_A$ and $\chi^{-1}_{A,M} = (!(\iota_0) \otimes !(\iota_1))\nabla_{A \oplus M}$, we easily see that:

$$\begin{aligned}
\alpha^\omega &= \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes 1_M)(!(\iota_0) \otimes \iota_1)\mathsf{d}_{A \oplus M} \\
&= \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes 1_M)(!(\iota_0) \otimes \iota_1)(1_{!(A \oplus M)} \otimes \eta_{A \oplus M})\nabla_{A \oplus M} \\
&= \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes \eta_M)(!(\iota_0) \otimes !(\iota_1))\nabla_{A \oplus M} \\
&= \pi_0 \omega !(\iota_0) + \pi_1 \alpha(\omega \otimes \eta_M)\chi^{-1}_{A,M}
\end{aligned}$$

So we conclude that the desired equality holds. ◀

We will now explain how $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ as constructed in Proposition 12 is of the form α^ω for a specific $!$-coalgebra and comodule. First observe that every object is a comodule of the monoidal unit, that is, for every object A, $(A, 1_A)$ is a $(k, 1_k, 1_k)$-comodule. Also note that it is easy to see that for the $!$-coalgebra (k, m_k), its associated comonoid is precisely $(k, 1_k, 1_k)$, that is, $\Delta^{\mathsf{m}_k} = 1_k$ and $\mathsf{e}^{\mathsf{m}_k} = 1_k$. Therefore, for every object A, $(A, 1_A)$ is a (k, m_k)-comodule.

▶ **Lemma 29.** *In a differential storage category with storage modality* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *and codereliction* η*, for* $k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A)$ *as constructed in Proposition 12, the following equality holds:* $\mathsf{H}_A = 1_A^{\mathsf{m}_k}$*, where* $1_A^{\mathsf{m}_k}$ *is defined as in Theorem 27.*

Proof. Recall that $\mathsf{H}_A := \pi_0 \mathsf{m}_k !(\iota_0) + \pi_1(\mathsf{m}_k \otimes \eta_A)\chi^{-1}_{k,A}$. By Lemma 28, since $\alpha = 1_A$ and $\omega = \mathsf{m}_k$, we clearly see that $\mathsf{H}_A = 1_A^{\mathsf{m}_k}$. ◀

▶ **Corollary 30.** H *satisfies* **[IA.1]***, that is, for every object* A*,* $(k \oplus A, \mathsf{H}_A)$ *is a* $!$*-coalgebra.*

Proof. This follows immediately from Theorem 27 and Lemma 29. ◀

It remains to prove **[IA.3]**, which we compute directly.

▶ **Lemma 31.** H *satisfies* **[IA.3]***, that is,* $(!A, \delta_A) \otimes^! (k \oplus B, \mathsf{H}_B) \xrightarrow{\Theta_{A,B}} (k \oplus (!A \otimes B), \mathsf{H}_{!A \otimes B})$ *is a* $!$*-coalgebra morphism.*

Proof. Recall that by construction $\Theta_{A,B} = (\mathsf{e}_A \otimes \pi_0)\iota_0 + (1_{!A} \otimes \pi_1)\iota_1$. We must show that $(\delta_A \otimes \mathsf{H}_B)\mathsf{m}_{!A,k\oplus B}!(\Theta_{A,B}) = \Theta_{A,B}\mathsf{H}_{!A\otimes B}$. By brute force computation, we show that:

$$
\begin{aligned}
&(\delta_A \otimes \mathsf{H}_B)\mathsf{m}_{!A,k\oplus B}!(\Theta_{A,B}) = \\
&= (\delta_A \otimes \pi_0)(1_{!!A} \otimes \mathsf{m}_k)(1_{!!A} \otimes !(\iota_0)\mathsf{m}_{!A,k\oplus B}!(\Theta_{A,B}) \\
&+ (\delta_A \otimes \pi_1)(1_{!!A} \otimes \mathsf{m}_k \otimes \eta_B)(1_{!!A} \otimes \chi^{-1}_{k,B})\mathsf{m}_{!A,k\oplus B}!(\Theta_{A,B}) \\
&= (\delta_A \otimes \pi_0)(1_{!A} \otimes \mathsf{m}_k)\mathsf{m}_{!A,k}!(1_{!A} \otimes \iota_0)!(\Theta_{A,B}) \\
&+ (\delta_A \otimes \pi_1)(1_{!!A} \otimes \mathsf{m}_k \otimes \eta_B)(\Delta_{!A} \otimes 1_{!k} \otimes 1_{!B})(1_{!!A} \otimes \sigma_{!!A,!k} \otimes 1_{!B}) \\
&(\mathsf{m}_{!A,!k} \otimes \mathsf{m}_{!A,!B})\left(!(1_{!A} \otimes \iota_0) \otimes !(1_{!A} \otimes \iota_1)\right)\nabla_{!A\otimes(k\oplus B)}!(\Theta_{A,B}) \\
&= (\delta_A \otimes \pi_0)!(1_{!A} \otimes \iota_0)!(\Theta_{A,B}) \\
&+ (\delta_A \otimes \pi_1)(\Delta_{!A} \otimes \eta_B)(1_{!!A} \otimes \mathsf{m}_{!A,!B})\left(!(1_{!A} \otimes \iota_0) \otimes !(1_{!A} \otimes \iota_1)\right)\nabla_{!A\otimes(k\oplus B)}!(\Theta_{A,B}) \\
&= (\delta_A \otimes \pi_0)!(\mathsf{e}_A)!(\iota_0) \\
&+ (\delta_A \otimes \pi_1)(\Delta_{!A} \otimes 1_B)(1_{!!A} \otimes \varepsilon_{!A} \otimes 1_B)(1_{!!A} \otimes \eta_{!A\otimes B}) \\
&\left(!(1_{!A} \otimes \iota_0) \otimes !(1_{!A} \otimes \iota_1)\right)\left(!(\Theta_{A,B}) \otimes !(\Theta_{A,B})\right)\nabla_{k\oplus(!A\otimes B)} \\
&= (\mathsf{e}_A \otimes \pi_0)\mathsf{m}_k!(\iota_0) \\
&+ (\delta_A \otimes \pi_1)(\Delta_{!A} \otimes 1_B)(1_{!!A} \otimes \varepsilon_{!A} \otimes 1_B)(1_{!!A} \otimes \eta_{!A\otimes B})(!(\mathsf{e}_A) \otimes 1_{!(!A\otimes(k\oplus B))}) \\
&(!(\iota_0) \otimes !(\iota_1))\nabla_{k\oplus(!A\otimes B)} \\
&= (\mathsf{e}_A \otimes \pi_0)\mathsf{m}_k!(\iota_0) \\
&+ (\Delta_A \otimes \pi_1)(\delta_A \otimes \delta_A \otimes 1_B)(!(\mathsf{e}_A) \otimes \varepsilon_{!A} \otimes 1_B)(1_{!!A} \otimes \eta_{!A\otimes B})\chi^{-1}_{k,!A\otimes !A\otimes B} \\
&= (\mathsf{e}_A \otimes \pi_0)\mathsf{m}_k!(\iota_0) \\
&+ (\Delta_A \otimes \pi_1)(\mathsf{e}_A \otimes 1_{!A} \otimes 1_B)(\mathsf{m}_k \otimes \eta_{!A\otimes B})\chi^{-1}_{k,!A\otimes !A\otimes B} \\
&= (\mathsf{e}_A \otimes \pi_0)\mathsf{m}_k!(\iota_0) + (1_{!A} \otimes \pi_1)(\mathsf{m}_k \otimes \eta_{!A\otimes B})\chi^{-1}_{k,!A\otimes !A\otimes B} \\
&= \Theta_{A,B}\pi_0\mathsf{m}_k!(\iota_0) + \Theta_{A,B}\pi_1(\mathsf{m}_k \otimes \eta_!A \otimes B)\chi^{-1}_{k,!A\otimes B} \\
&= \Theta_{A,B}\mathsf{H}_{!A\otimes B}
\end{aligned}
$$

So we conclude that $\Theta_{A,B}$ is a !-coalgebra morphism and that H satisfies **[IA.3]**. ◀

So we conclude that H is indeed an infinitesimal extension.

B Proof of Proposition 13

By [3, Corollary 5], to show that a natural transformation $A \xrightarrow{\eta_A} !A$ is a codereliction it in fact suffices to show that η satisfies **[dC.3]**, **[dC.4]**, and **[dC.m]**. So let $k\oplus A \xrightarrow{\mathsf{H}_A} !(k\oplus A)$ be an infinitesimal extension and recall that $\eta_A : A \to !A$ is defined as follows $\eta_A := \iota_1\mathsf{H}_A!(\pi_1)$.

▶ **Lemma 32.** *η satisfies* **[dC.3]**.

Proof. We must show that $\eta_A\varepsilon_A = 1_A$. So we compute that:

$$\eta_A\varepsilon_A = \iota_1\mathsf{H}_A!(\pi_1)\varepsilon_A = \iota_1\mathsf{H}_A\varepsilon_{k\oplus A}\pi_1 = \iota_1\pi_1 = 1_A$$

So we conclude that η satisfies **[dC.3]**. ◀

▶ **Lemma 33.** *η satisfies* **[dC.4]**.

Proof. We must show that $\eta_A \delta_A = (\mathsf{u}_A \otimes \eta_A)(\delta_A \otimes \eta_{!A})\nabla_{!A}$. First note that we have the following equality:

$$\begin{aligned}
\mathsf{H}_A!(\pi_1) &= \pi_0\iota_0\mathsf{H}_A!(\pi_1) + \pi_1\iota_1\mathsf{H}_A!(\pi_1)\\
&= \pi_0\mathsf{m}_k!(\iota_0)!(\pi_1) + \pi_1\iota_1\mathsf{H}_A!(\pi_1)\\
&= \pi_0\mathsf{m}_k!(0) + \pi_1\iota_1\mathsf{H}_A!(\pi_1)\\
&= \pi_0\mathsf{u}_A + \pi_1\iota_1\mathsf{H}_A!(\pi_1)\\
&= \pi_0\mathsf{u}_A + \pi_1\eta_A
\end{aligned}$$

So $\mathsf{H}_A!(\pi_1) = \pi_0\mathsf{u}_A + \pi_1\eta_A$. Therefore, we compute:

$$\begin{aligned}
\eta_A\delta_A &= \iota_1\mathsf{H}_A!(\pi_1)\delta_A\\
&= \iota_1\mathsf{H}_A\delta_{k\oplus A}!!(\pi_1)\\
&= \iota_1\mathsf{H}_A!(\mathsf{H}_A)!!(\pi_1)\\
&= \iota_1\mathsf{H}_A!\left(\mathsf{H}_A!(\pi_1)\right)\\
&= \iota_1\mathsf{H}_A!\left(\pi_0\mathsf{u}_A + \pi_1\eta_A\right)\\
&= \iota_1\mathsf{H}_A\Delta_{k\oplus A}(!(\pi_0)\otimes !(\pi_1))(!(\mathsf{u}_A \otimes !(\eta_A))\nabla_{!A}\\
&= \iota_1\Lambda_A(\mathsf{H}_A\otimes\mathsf{H}_A)(!(\pi_0)\otimes !(\pi_1))(!(\mathsf{u}_A \otimes !(\eta_A))\nabla_{!A}\\
&= \iota_1\Lambda_A(\pi_0\otimes\mathsf{H}_A)(\mathsf{m}_k\otimes !(\pi_1))(!(\mathsf{u}_A \otimes !(\eta_A))\nabla_{!A}\\
&= \iota_1\mathsf{H}_A(\mathsf{u}_A\otimes !(\pi_1))(\delta_A \otimes !(\eta_A))\nabla_{!A}\\
&= (\mathsf{u}_A\otimes\eta_A)(\delta_A \otimes !(\eta_A))\nabla_{!A}\\
&= (\mathsf{u}_A\otimes\eta_A)(\delta_A \otimes \eta_{!A})\nabla_{!A}
\end{aligned}$$

So we conclude that η satisfies **[dC.4]**. ◀

▶ **Lemma 34.** *η satisfies* **[dC.m]**.

Proof. We must show that $(1_{!A}\otimes\eta_B)\mathsf{m}_{A,B} = (\varepsilon_A\otimes 1_B)\eta_{A\otimes B}$.

$$\begin{aligned}
(1_{!A}\otimes\eta_B)\mathsf{m}_{A,B} &= (1_{!A}\otimes\iota_1)(1_{!A}\otimes\mathsf{H}_B)(1_{!A}\otimes !(\pi_1))\mathsf{m}_{A,B}\\
&= (1_{!A}\otimes\iota_1)(\delta_A\otimes\mathsf{H}_B)(!(\varepsilon_A)\otimes !(\pi_1))\mathsf{m}_{A,B}\\
&= (1_{!A}\otimes\iota_1)(\delta_A\otimes\mathsf{H}_B)\mathsf{m}_{!A,k\oplus B}!\left(\varepsilon_A\otimes\pi_1\right)\\
&= (1_{!A}\otimes\iota_1)(\delta_A\otimes\mathsf{H}_B)\mathsf{m}_{!A,k\oplus B}!\left(1_{!A}\otimes\pi_1\right)!\left(\varepsilon_A\otimes 1_B\right)\\
&= (1_{!A}\otimes\iota_1)(\delta_A\otimes\mathsf{H}_B)\mathsf{m}_{!A,k\oplus B}!(\Theta_{A,B})!(\pi_1)!\left(\varepsilon_A\otimes 1_B\right)\\
&= (1_{!A}\otimes\iota_1)\Theta_{A,B}\mathsf{H}_{!A\otimes B}!(\pi_1)!\left(\varepsilon_A\otimes 1_B\right)\\
&= \iota_1\mathsf{H}_{!A\otimes B}!(\pi_1)!\left(\varepsilon_A\otimes 1_B\right)\\
&= \eta_{!A\otimes B}!\left(\varepsilon_A\otimes 1_B\right)\\
&= (\varepsilon_A\otimes 1_B)\eta_{A\otimes B}
\end{aligned}$$

So we conclude that η satisfies **[dC.m]**. ◀

So we conclude that η is a codereliction.

C Proof of Theorem 14

We must show that that constructions of Proposition 12 and Proposition 13 are inverses of each other. So starting with a codereliction η, we compute:

$$\iota_1\mathsf{H}_A!(\pi_1) = (\mathsf{m}_k\otimes\eta_A)\chi^{-1}_{k,A}!(\pi_1) = (\mathsf{m}_k\otimes\eta_A)(\mathsf{e}_k\otimes 1_{!A}) = \eta_A$$

Next starting with an infinitesimal extension H, in Lemma 33 we showed that $\mathsf{H}_A!(\pi_1) = \pi_0 \mathsf{u}_A + \pi_1 \eta_A$. Therefore, we compute that:

$$\begin{aligned}
\pi_0 \mathsf{m}_k!(\iota_0) + \pi_1(\mathsf{m}_k \otimes \eta_A)\chi^{-1}_{k,A} &= \pi_0 \left(\mathsf{m}_k \otimes \mathsf{u}_A\right) \chi^{-1}_{k,A} + \pi_1 \left(\mathsf{m}_k \otimes \eta_A\right) \chi^{-1}_{k,A} \\
&= \pi_0 \mathsf{u}_A \left(\mathsf{m}_k \otimes 1_{!A}\right) \chi^{-1}_{k,A} + \pi_1 \eta_A \left(\mathsf{m}_k \otimes 1_{!A}\right) \chi^{-1}_{k,A} \\
&= \left(\pi_0 \mathsf{u}_A + \pi_1 \eta_A\right) \left(\mathsf{m}_k \otimes 1_{!A}\right) \chi^{-1}_{k,A} \\
&= \mathsf{H}_A!(\pi_1) \left(\mathsf{m}_k \otimes 1_{!A}\right) \chi^{-1}_{k,A} \\
&= \mathsf{H}_A \left(\mathsf{m}_k \otimes !(\pi_1)\right) \chi^{-1}_{k,A} \\
&= \Lambda_A(\pi_0 \otimes \mathsf{H}_A) \left(\mathsf{m}_k \otimes !(\pi_1)\right) \chi^{-1}_{k,A} \\
&= \Lambda_A(\mathsf{H}_A \otimes \mathsf{H}_A) \left(!(\pi_0) \otimes !(\pi_1)\right) \chi^{-1}_{k,A} \\
&= \mathsf{H}_A \Delta_{k \oplus A} \left(!(\pi_0) \otimes !(\pi_1)\right) \chi^{-1}_{k,A} \\
&= \mathsf{H}_A \chi_{k,A} \chi^{-1}_{k,A} \\
&= \mathsf{H}_A
\end{aligned}$$

So we conclude that coderelictions are in bijective correspondence with infinitesimal augmentations.