The Open Algebraic Path Problem

Jade Master

Department of Mathematics, University of California, Riverside, CA, USA

Abstract

The algebraic path problem provides a general setting for shortest path algorithms in optimization and computer science. We explain the universal property of solutions to the algebraic path problem by constructing a left adjoint functor whose values are given by these solutions. This paper extends the algebraic path problem to networks equipped with input and output boundaries. We show that the algebraic path problem is functorial as a mapping from a double category whose horizontal composition is gluing of open networks. We introduce functional open matrices, for which the functoriality of the algebraic path problem has a more practical expression.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Operational semantics

Keywords and phrases The Algebraic Path Problem, Open Systems, Shortest Paths, Categorical Semantics, Compositionality

Digital Object Identifier 10.4230/LIPIcs.CALCO.2021.20

Related Version *Available at*: https://arxiv.org/abs/2005.06682

Acknowledgements I would like to thank Mike Shulman, John Baez, Christian Williams, Joe Moeller, Rany Tith, Sarah Rovner-Frydman, Zans Mihejez, Oscar Hernandez, Alex Pokorny and Todd Trimble for their helpful comments and contributions. I would also like to thank everyone in my life who supported me during this time. Your work contributed to this paper as well. This paper was written on Tongva land.

1 Introduction

The algebraic path problem is a generalization of the shortest path problem to probability, computing, matrix multiplication, and optimization [18, 9]. Let $([0, \infty], \min, +)$ be the rig of positive real numbers with min as the "additive" monoid and + as the "multiplicative" monoid. A matrix M valued in $[0, \infty]$ is regarded as a distance network and the shortest paths of M between all pairs of vertices may be computed using the geometric series formula: $F(M) = \sum_{n \geq 0} M^n$. The algebraic path problem frames many existing problems as generalizations of the shortest path problem by allowing $[0, \infty]$ to be replaced by a sufficiently nice rig R. Many popular shortest path algorithms can be extended to this more general setting [10] and the algebraic path problem can also be implemented generically using functional programming [5]. In Section 2, we show that finding solutions to the algebraic path problem can be understood as the left adjoint of an adjunction

$$\mathsf{RMat} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{RCat} \quad (F \dashv U)$$

between matrices valued in R and categories enriched in R.

The algebraic path problem deals only with closed systems, i.e. systems which are isolated from their surroundings. On the other hand, open systems are equipped with input and output boundaries, from which they can be composed to form larger and more complicated networks. A research program intiated by Baez, Courser, and Fong aims to provide a theoretical foundation for open systems using cospan formalisms [7, 1]. For a category of

9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021).
Editors: Fabio Gadducci and Alexandra Silva; Article No. 20; pp. 20:1–20:20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

networks C, Baez and Courser defined a symmetric monoidal double category which provides a syntax for composition of open systems in C [1]. In Section 3, we set C equal to RMat, the category of matrices weighted in a quantale R, to obtain a symmetric monoidal double category $\mathsf{Open}(\mathsf{RMat})$. The essence of this double category is gluing. Open R-matrices are represented as cospans with feet given by 0-matrices. Given two such open R-matrices, take their pushout

$$\begin{array}{ccccccc} & & & M +_{0_Y} N & & & \\ & & \nearrow & & \nwarrow & & \\ & M & & & & N & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ 0_X & & & 0_Y & & & 0_Z \end{array}$$

to obtain an open R-matrix whose apex is synthesized from joining M and N along their shared boundary. This, along with the other data and structure of $\mathsf{Open}(\mathsf{RMat})$, provide a syntax for manipulating open R-matrices. The axioms of a symmetric monoidal double category guarantee that this syntax is well-behaved. For example, the word problem for double categories is solvable in quadratic time and double categories are equipped with a string diagram calculus [4, 14].

RCat, the category of R-enriched categories provides a choice of semantics for R-matrices, and can be expressed as R-matrices satisfying some regularity properties. In Section 2, we show how the solution to the algebraic path problem forms the left adjoint F of an adjunction below left

$$\mathsf{RMat} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{RCat} \quad (\bot) \qquad\qquad \bigstar \colon \mathsf{Open}(\mathsf{RMat}) \to \mathsf{Open}(\mathsf{RCat})$$

which provides a mapping from the syntax of R-matrices to the semantics of R-categories. R-categories equipped with input and output boundaries form the horizontal morphisms of a symmetric monoidal double category $\mathsf{Open}(\mathsf{RCat})$. In Section 4, we show how the algebraic path problem functor lifts to a symmetric monoidal double functor providing a coherent semantics for the syntax of *open* R-matrices as shown above right. This symmetric monoidal double functor provides a framework for studying how solutions to the algebraic path problem can be built inductively from gluings of smaller open R-matrices. The axioms of a symmetric monoidal double functor guarantee that this inductive process is well-behaved.

This result is more theoretical than practical. However, there is a subclass of open R-matrices, functional open R-matrices, for which the theory provides useful insight. Functional open R-matrices are roughly open R-matrices where the inputs are all sources and the outputs are all sinks. In Section 5 we show that there is a strict double functor

$$\blacksquare \circ \bigstar_{\text{fxn}} \colon \mathsf{Open}(\mathsf{RMat})_{\text{fxn}} \to \mathsf{Mat}_R$$

where Mat_R is a double category of R-matrices whose horizontal composition is matrix multiplication. This strict double functor gives a series of coherent compositional relationships for the algebraic path problem on functional open R-matrices based on matrix multiplication.

2 The Algebraic Path Problem

The algebraic path problem arises from the observation that various optimization problems can be framed in the same way by varying a sufficiently nice sort of rig. The level of generality for this work will be a commutative quantale, which is sufficient to guarantee existence and uniqueness of solutions to these optimization problems.

▶ **Definition 1.** *A* **quantale** *is a monoidal closed poset with all joins. Explicitly, a quantale is a poset R with an associative, unital, and monotone multiplication $\cdot\colon R \times R \to R$ such that for every index set I*

- *all joins, $\sum_{i \in I} x_i$, exist*
- *$\cdot$ preserves all joins, i.e.*

$$a \cdot \sum_{i \in I} x_i = \sum_{i \in I} a \cdot x_i.$$

A quantale is commutative if its multiplication operation, $\cdot$, is commutative.

A motivating example of such a quantale is the poset $[0, \infty]$ with $+$ as its monoidal product and with join given by infimum. Note that this poset is equipped with the reverse of the usual ordering on $[0, \infty]$. Fong and Spivak show how the shortest path problem on this quantale computes the shortest paths between all pairs of vertices in a given $[0, \infty]$-weighted graph [8, §2.5.3]. Other motivating examples include the rig $([0, 1], \sup, \cdot)$ (whose algebraic path problem corresponds to most likely path in a Markov chain) and the powerset of the language generated by an alphabet (whose algebraic path problem corresponds to the language decided by a nondeterministic finite automata (NFA))[9].

▶ **Definition 2.** *For a commutative quantale R and sets X and Y, an* ***R*-matrix** *$M\colon X \to Y$ is a function $M\colon X \times Y \to R$. For R-matrices $M\colon X \to Y$ and $N\colon Y \to Z$, their matrix product MN is defined by the rule*

$$MN(i, k) = \sum_{j \in Y} M(i, j)N(j, k)$$

where juxtaposition denotes the multiplication of R.

If R is a commutative quantale, R-matrices form a quantale as well.

▶ **Definition 3.** *Let $\mathsf{RMat}(X)$ be the set of X-by-X matrices $M\colon X \times X \to R$. $\mathsf{RMat}(X)$ is equipped with the partial order $\leq$ where $M \leq N$ if and only if $M(i, j) \leq N(i, j)$ for all $i, j \in X$.*

▶ **Proposition 4.** $\mathsf{RMat}(X)$ *is a quantale with*

- *join given by pointwise sum of matrices,*
- *and multiplication given by matrix product.*

The proof of this proposition is left to the reader. All the required properties of $\mathsf{RMat}(X)$ follow from the analogous properties in R.

A square matrix $M\colon X \times X \to R$ represents a complete R-weighted graph whose vertex set is given by X.

▶ **Definition 5.** *Let $M\colon X \times X \to R$ be a square matrix. A* **vertex** *of M is an element $i \in X$. An* **edge** *of M is a tuple of vertices $(a, b) \in X \times X$. A* **path** *in M from a_0 to a_n is a list of adjacent edges $p = ((a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n))$. The* **weight** *of p is defined as the product $l(p) = \Pi_{i=0}^{n-1} M(a_i, a_{i+1})$ in R. For vertices $i, j \in X$, let $P_{ij}^M = \{$ paths in M from i to j $\}$*

Let i and j be vertices of a square matrix $M\colon X \times X \to R$. The algebraic path problem asks to compute the quantity $\sum_{p \in P_{ij}^M} l(p)$ in the quantale R. If R is the quantale $([0,\infty], \inf, +)$ then the weight of an edge M_{ij} represents the distance between vertex i and vertex j and the weight of a path $l(p)$ represents the total distance traversed by p. Summing the weights of all paths between a pair of vertices corresponds to finding the path with the minimum weight.

A more tractable framing of the algebraic path problem can be found by considering matrix powers. The entries of M^2 are given by

$$M^2(i,j) = \sum_{l \in X} M(i,l)M(l,j) = \inf_{l \in X}\{M(i,l) + M(l,j)\}.$$

Because $M(i,l)$ and $M(l,j)$ represent the distance from i to l and from l to j, this infimum computes the cheapest way to travel from i to j while stopping at some l in between. More generally, the entries of M^n for $n \geq 0$ represent the shortest paths between nodes of your graph that occur in exactly n steps. To compute the shortest paths which can occur in any number of steps, we must take the infimum of the matrices M^n over all $n \geq 0$. This pattern replicates for other choices of quantale. Therefore, the **algebraic path problem** seeks to compute

$$F(M) = \sum_{n \geq 0} M^n \tag{1}$$

where M is an R-matrix. The following table summarizes some instances of the algebraic path problem for different choices of R. Fink provides an explanation of the algebraic path problems for $([0,\infty], \leq)$ and $\{T, F\}$ and Foote provides an explanation for the quantales $([0,1], \leq)$ and $(\mathcal{P}(\Sigma), \subseteq)$ [6, 9].

poset	**join**	**multiplication**	**solution of path problem**
$([0,\infty], \geq)$	inf	$+$	shortest paths in a weighted graph
$([0,\infty], \leq)$	sup	inf	maximum capacity in the tunnel problem
$([0,1], \leq)$	sup	$\times$	most likely paths in a Markov process
$\{T, F\}$	OR	AND	transitive closure of a directed graph
$(\mathcal{P}(\Sigma^*), \subseteq)$	$\bigcup$	concatenation	decidable language of a NFA

Note that in this table, $\mathcal{P}(\Sigma^*)$ denotes the power set of the language generated by an alphabet Σ.

Equation (1) is known to category theorists by a different name: the free monoid on M. Framing it in this way gives a categorical proof of existence and uniqueness of $F(M)$. A classic result from [12, §V11] gives a construction of free monoids. MacLane's construction is defined as an adjunction into a category of internal monoids.

▶ **Definition 6.** *Let $(C, \otimes, I)$ be a monoidal category. A* **monoid internal to C** *is an object A of C equipped with morphisms*

$$m\colon A \otimes A \to A \text{ and } i\colon I \to A$$

satisfying the axioms of associativity and unitality expressed as commutative diagrams. A **monoid homomorphism** *from a monoid A to a monoid B is a morphism $f : A \to B$ in C which commutes with the maps m and i of each monoid. Let $\mathsf{Mon}(C)$ be the category where objects are monoids internal to C and morphisms are their homomorphisms.*

▶ **Proposition 7** (MacLane). *Let $(C, \otimes, I)$ be a monoidal category with countable coproducts such that tensoring on both sides preserves these coproducts. Then there is an adjunction below left*

$$C \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Mon}(C) \qquad\qquad F(X) = \sum_{n \geq 0} X^n$$

whose left adjoint is given by the countable coproduct of cartesian powers as shown above right.

The poset $\mathsf{RMat}(X)$ when viewed as a category satisfies the hypotheses of Proposition 7 and therefore admits a free monoid construction.

▶ **Proposition 8.** *There is an adjoint pair*

$$\mathsf{RMat}(X) \underset{U_X}{\overset{F_X}{\rightleftarrows}} \mathsf{Mon}(\mathsf{RMat}(X))$$

where F_X is the monotone map which produces the solution to the algebraic path problem on a matrix and U_X is the natural forgetful map.

Proof. Because $\mathsf{RMat}(X)$ is a quantale, it can be regarded as a monoidal category with all coproducts such that tensoring distributes over these coproducts. The result follows from applying Proposition 7 and noticing that MacLane's construction of free monoids matches Equation 1 in the case when $C = \mathsf{RMat}(X)$. ◀

Monoids internal to $\mathsf{RMat}(X)$ are R-enriched categories.

▶ **Definition 9.** *An R-category C with object set X consists of an element $C(x, y)$ in R for every $x, y \in X$ such that*
- $1 \leq C(x, x)$ *(the identity law), and*
- $C(x, y)C(y, z) \leq C(x, z)$ *(the composition law).*

Let $\mathsf{RCat}(X)$ be the poset whose elements are R-enriched categories with object set X. For R-categories C and D,

$$C \leq D \leftrightarrow C(i, j) \leq D(i, j) \quad \forall i, j \in X$$

▶ **Proposition 10.** $\mathsf{Mon}(\mathsf{RMat}(X))$ *is isomorphic to* $\mathsf{RCat}(X)$*, the poset of categories enriched in R with object set X.*

Proof. The isomorphism in question assigns a matrix $M : X \times X \to R$ to the R-category with $\hom(x, y) = M(x, y)$. The identity law follows from the inequality $1 \leq M$ and the inequality $M^2 \leq M$ implies that for all $y \in X$, $\sum_{y \in X} M(x, y)M(y, z) \leq M(x, z)$. The composition law follows from the fact that any element of R is less than a join which contains it. ◀

Proposition 8 says that each matrix valued in R has a unique, universally characterized solution to the algebraic path problem: namely the free R-category on that matrix. This adjunction can be extended to matrices over an arbitrary set.

▶ **Definition 11.** *Let $f : X \to Y$ be a function and let $M : X \times X \to R$ be an R-matrix. Then the* **pushforward** *of M along f is the matrix $f_*(M) \colon Y \times Y \to R$ defined by*

$$f_*(M)(y, y') = \sum_{(x,x') \in (f \times f)^{-1}(y,y')} M(x, x').$$

▶ **Definition 12.** *Let RMat be the category where objects are square matrices $M \colon X \times X \to R$ on some set X and where a morphism of R-matrices from $M \colon X \times X \to R$ to $N \colon Y \times Y \to R$ is a function $f \colon X \to Y$ satisfying $f_*(M) \leq N$. Let RCat be the full subcategory of RMat consisting of matrices satisfying the axioms of an R-category (see Definition 9).*

The above adjunction may be extended to square matrices over an arbitrary set. We leave the proof of the following proposition to Appendix A.

▶ **Proposition 13.** *The free monoid construction of Proposition 8 extends to an adjunction*

$$\mathsf{RMat} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{RCat}. \quad (\bot)$$

The following proposition will be useful in the next section.

▶ **Proposition 14.** *The above adjunction $F \dashv U$ is idempotent.*

Proof. Every adjunction between posets is idempotent. Therefore the smaller adjunctions $F_X \dashv U_X$ are idempotent. Because F and U are stitched together using these adjunctions, it is idempotent as well. ◀

3 Open R-Matrices

R-matrices are made open by designating some of their vertices to be either inputs or outputs. In this section we show how these open R-matrices are composed by joining the output vertices of one to the input vertices of another and joining the data on the overlap. To define open R-matrices, we need a notion of a discrete R-matrix on a set X i.e. a matrix whose entries are all zero. The map sending a set to its discrete R-matrix is a functor and a left adjoint.

▶ **Proposition 15.** *Let $R \colon \mathsf{RMat} \to \mathsf{Set}$ be the functor which sends an R-matrix to its underlying set of vertices and sends a morphism to its underlying function. Then U has a left adjoint*

$$0 \colon \mathsf{Set} \to \mathsf{RMat} \quad 0_X \colon X \times X \to Y$$

which sends a set X to the R-matrix defined by $0_X(i, j) = 0$ for all i and j in X. F sends a function $f \colon X \to Y$ to the morphism of R-matrices which has f as its underlying function between vertices.

Proof. The natural isomorphism $\phi \colon \mathsf{RMat}(0_X, G) \cong \mathsf{Set}(X, R(G))$ is formed by noting that a morphism $0_X \to R(G)$ is uniquely determined by its underlying function on vertices and every such function obeys the inequality in Definition 12. ◀

A weighted graph can be opened up to its environment by equipping it with inputs and outputs.

▶ **Definition 16.** *Let $M : A \times A \to R$ be an R-matrix. An* **open R-matrix** $M\colon X \to Y$ *is a cospan in* RMat *of the form*

$$\begin{array}{ccccc} & & M & & \\ & \nearrow & & \nwarrow & \\ 0_X & & & & 0_Y \end{array}$$

The idea is that the maps of this cospan point to input and output nodes of the matrix M. Let $M\colon X \to Y$ and $N\colon Y \to Z$

$$\begin{array}{ccccccccc} & & M & & & & N & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ 0_X & & & & 0_Y & & & & 0_Z \end{array}$$

be open R-matrices. The underlying sets of M and N form a diagram

$$\begin{array}{ccccccccc} & & R(M) & & & & R(N) & & \\ & \overset{l}{\nearrow} & & \overset{m}{\nwarrow} & & \overset{n}{\nearrow} & & \overset{o}{\nwarrow} & \\ X & & & & Y & & & & Z \end{array}$$

which generate a pushout

$$\begin{array}{ccccc} & & R(M) +_Y R(N) & & \\ & \overset{a}{\nearrow} & & \overset{b}{\nwarrow} & \\ R(M) & & & & R(N) \\ & \underset{m}{\nwarrow} & & \underset{n}{\nearrow} & \\ & & Y & & \end{array}$$

The functions a and b of this pushout allow the matrices M and N to be compared on equal footing: the pushforwards $a_*(M)$ and $b_*(N)$ both have $R(M) +_Y R(N)$ as their underlying set. The matrices $a_*(M)$ and $b_*(N)$ are combined using pointwise sum.

▶ **Definition 17.** *For open R-matrices $M : X \to Y$ and $N\colon Y \to Z$ as defined above, their* **composite** *is defined by*

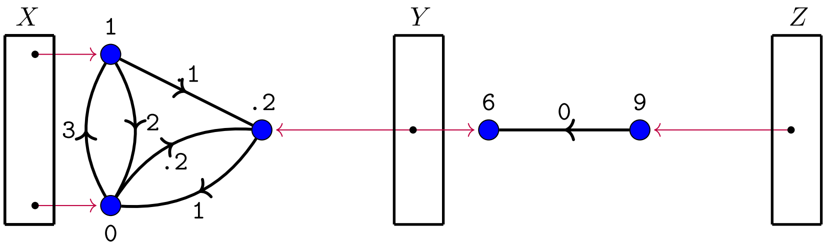
$$N \circ M\colon X \to Z = \begin{array}{ccccc} & & a_*(M) + b_*(N) & & \\ & \overset{\phi^{-1}(a \circ l)}{\nearrow} & & \overset{\phi^{-1}(b \circ r)}{\nwarrow} & \\ 0_X & & & & 0_Z \end{array}$$

where ϕ^{-1} gives the unique morphism out of a discrete R-matrix defined by a function on its underlying set.

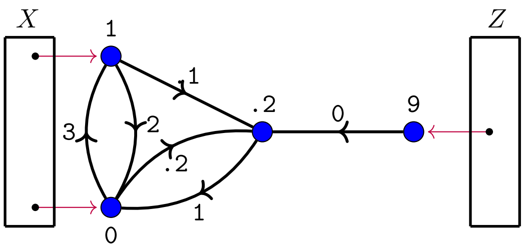
An R-matrix $M\colon X \times X \to R$ can represent a graph with vertex set X weighted in R. Similarly, an open R-matrix, represents an R-weighted graph equipped with inputs and outputs. For example, the $[0, \infty]$-matrices on the sets $\{a, b, c\}$ and $\{d, e\}$

$$\begin{bmatrix} 1 & 2 & .1 \\ 3 & 0 & .2 \\ \infty & 1 & .2 \end{bmatrix} \qquad \begin{bmatrix} 6 & \infty \\ 0 & 9 \end{bmatrix}$$

respectively may be upgraded to open $[0,\infty]$-matrices as follows. The first matrix has input set $\{1,2\}$ and output set $\{3\}$. The mappings of the cospan are given by $1 \mapsto a, 2 \mapsto b$ and $3 \mapsto c$. Similarly, the second matrix has left input set given by $\{3\}$ and right input set given by $\{4\}$. The mappings in the cospan for this open $[0,\infty]$-matrix are given by the assignments $3 \mapsto d$ and $4 \mapsto e$. These two open $[0,\infty]$-matrices are drawn as follows:



In this picture, edges are ommitted when their value is ∞ and a label on a vertex indicates the weight of the edge from that vertex to itself. These two open $[0,\infty]$-matrices are composed by identifying vertices mapped to by a common element of Y.



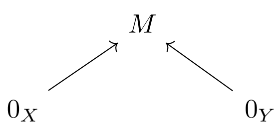
where edges are omitted if their weight is infinite in both directions. The matrix on the apex of this composite is computed by pushing each component matrix forward to the pushout of their underlying sets and adding them together i.e.

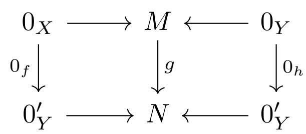
$$\begin{bmatrix} 1 & 2 & .1 & \infty \\ 3 & 0 & .2 & \infty \\ \infty & 1 & .2 & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} + \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & 6 & \infty \\ \infty & \infty & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & .1 & \infty \\ 3 & 0 & .2 & \infty \\ \infty & 1 & .2 & \infty \\ \infty & \infty & 0 & 9 \end{bmatrix}$$

where $+$ denotes the pointwise join of the quantale $[0,\infty]$. The entries of this matrix represent the shortest distances between pairs of vertices in the composite open $[0,\infty]$-matrix. This composition forms the horizontal composition of a symmetric monoidal double category. Note that the double categories considered here are called pseudo–double categories.

▶ **Theorem 18.** *For a quantale R, there is a symmetric monoidal double category* $\mathsf{Open(RMat)}$ *where*

- *objects are sets X,Y,$Z\ldots$*
- *vertical morphisms are functions $f : X \to Y$,*
- *a horizontal morphism $M : X \to Y$ is an open R-matrix below left*





- *vertical 2-morphisms are commutative rectangles shown above right,*
- *vertical composition is ordinary composition of functions,*
- *and horizontal composition is given by the composite operation defined above.*

The symmetric monoidal structure is given by

- *coproducts in* Set *on objects and vertical morphisms,*
- *pointwise coproducts on horizontal morphisms i.e. for open R-matrices,*

$$0_X \rightarrow M \leftarrow 0_Y \qquad 0'_X \rightarrow M' \leftarrow 0'_Y$$

their coproduct is

$$0_{X \sqcup X'} \rightarrow M \sqcup M' \leftarrow 0_{Y \sqcup Y'}$$

and pointise coproduct for two vertical 2-morphisms i.e. for vertical 2-morphisms,

$$\begin{array}{ccccc} 0_X & \longrightarrow & M & \longleftarrow & 0_Y \\ {\scriptstyle 0_f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0_h} \\ 0_Z & \longrightarrow & N & \longleftarrow & 0_Q \end{array} \qquad \begin{array}{ccccc} 0'_X & \longrightarrow & M' & \longleftarrow & 0'_Y \\ {\scriptstyle 0'_f}\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle 0'_h} \\ 0'_Z & \longrightarrow & N' & \longleftarrow & 0'_Q \end{array}$$

their coproduct is

$$\begin{array}{ccccc} 0_{X \sqcup X'} & \longrightarrow & M \sqcup M' & \longleftarrow & 0_{Y \sqcup Y'} \\ {\scriptstyle 0_{f \sqcup f'}}\downarrow & & \downarrow{\scriptstyle g \sqcup g'} & & \downarrow{\scriptstyle 0_{h \sqcup h'}} \\ 0_{Z \sqcup Z'} & \longrightarrow & N \sqcup N' & \longleftarrow & 0_{Q \sqcup Q'} \end{array}$$

Proof. Theorem 3.2.3 of [3] constructs this symmetric monoidal double category as long as

- RMat has coproducts and pushouts, and
- $0\colon$ Set $\to$ RMat preserves pushouts and coproducts.

Because 0 is a left adjoint (Proposition 15) it preserves pushouts and coproducts when they exist so it suffices to prove the following lemma which we do in Appendix A. ◀

▶ **Lemma 19.** RMat *has coproducts and pushouts.*

4 Compositionality of the Algebraic Path Problem

In this section we show how the algebraic path problem functor F: RMat $\to$ RCat extends to a symmetric monoidal double functor

$$\mathsf{Open}(F)\colon \mathsf{Open}(\mathsf{RMat}) \to \mathsf{Open}(\mathsf{RCat}).$$

This double functor describes how the syntax of gluing open R-matrices extends to a series of coherent compositionality laws for the algebraic path problem.

For composable open R-matrices

$$0_X \rightarrow M \leftarrow 0_Y \rightarrow N \leftarrow 0_Z$$

we may apply the algebraic path problem functor F to the entire diagram to get cospans of R-categories

$$1_X \rightarrow F(M) \leftarrow 1_Y \rightarrow F(N) \leftarrow 1_Z$$

where 1_X is the identity matrix on X with respect to matrix multiplication. The pushout in RMat, $F(M) +_{1_Y} F(N)$, is not equal to the solution $F(M +_{0_Y} N)$. The former optimizes only over paths which start in M and end in N. On the other hand, $F(M +_{0_Y} N)$ optimizes over paths which may zig-zag back and forth between M and N, as many times as they like, before arriving at their destination. Therefore, to construct $F(M +_{0_Y} N)$ from its components we turn to the pushout in RCat.

▶ **Proposition 20.** RCat *has pushouts and coproducts.*

Proof. More generally, RCat has all colimits by Corollary 2.14 of [19]. These colimits are constructed via the transfinite construction of free algebras [11]. The idea behind the transfinite construction is that colimits in a category of monoids can be constructed by first taking the colimit of their underlying objects, taking the free monoid on that colimit, and then quotienting out by the equations in your original monoids. Next we provide an explicit description in the case of R-categories. ◀

▶ **Proposition 21.** *For a diagram $D \colon C \to \mathsf{RCat}$, its colimit is given by the formula*

$$\mathrm{colim}_{c \in C} D(c) \cong F(\mathrm{colim}_{c \in C} U(D(c)))$$

Proof. It suffices to show that $F(\mathrm{colim}_{c \in C} U(D(c)))$ satisfies the universal property of $\mathrm{colim}_{c \in C} D(c)$. Let $\alpha \colon \Delta_d \Rightarrow D$ be a cocone from an object $d \in \mathsf{RCat}$ to our diagram D. Because α can be regarded as a cocone in RMat, the universal property of colimits induces a unique map $\mathrm{colim}_{c \in C} U(D(c)) \to U(d)$ of R-matrices. Applying F to this morphism gives a map $F(\mathrm{colim}_{c \in C} U(D(c))) \to FU(d) = d$ where the last equality follows from the adjunction $F \dashv U$ being idempotent as shown in Proposition 14. The above map is a unique morphism satsifying the universal property for $\mathrm{colim}_{c \in C} D(c)$. ◀

▶ **Corollary 22.** *For a diagram below left*

$$M \leftarrow K \rightarrow N \qquad\qquad M +_K N \cong F(U(M) +_{U(K)} U(N))$$

in RCat*, the pushout is given by the isomorphism above right.*

This pushout forms the horizontal composition of a double category of open R-categories.

▶ **Theorem 23.** *There is a symmetric monoidal double category* Open(RCat) *where*

- *objects are sets,*
- *vertical morphisms are functions,*
- *horizontal morphisms are cospans shown below left*

$$\begin{array}{ccc} & M & \\ \nearrow & & \nwarrow \\ 1_X & & 1_Y \end{array} \qquad \begin{array}{ccccc} 1_X & \longrightarrow & M & \longleftarrow & 1_Y \\ {\scriptstyle 1_f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1_h} \\ 1'_X & \longrightarrow & N & \longleftarrow & 1'_Y \end{array}$$

where the apex M satisfies the axioms of an R-category, and

- *vertical 2-morphisms are commuting rectangles shown above right*
- *The horizontal composition is given by pushout of open R-categories i.e. for open R-categories*

$$\begin{array}{ccccccc} & M & & & & N & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ 1_X & & & 1_Y & & & 1_Z \end{array}$$

their pushout is the cospan

$$\begin{array}{ccc} & F(U(M) +_{U(K)} U(N)) & \\ \nearrow & & \nwarrow \\ 1_X & & 1_Y \end{array}$$

The symmetric monoidal structure of Open(RCat) *is given by*

- *coproduct of sets and functions,*
- *pointwise coproduct on horizontal morphisms,*
- *and pointiwise coproduct on vertical 2-morphisms.*

Proof. To construct the desired symmetric monoidal double category, we apply Corollary 2.4 of [1] to the composite left adjoint $\mathsf{Set} \xrightarrow{0} \mathsf{RMat} \xrightarrow{F} \mathsf{RCat}$. ◀

So far we have the commutative diagram of functors

$$\begin{array}{ccc} \mathsf{RMat} & \xrightarrow{F} & \mathsf{RCat} \\ {\scriptstyle 0}\nwarrow & & \nearrow{\scriptstyle 1} \\ & \mathsf{Set} & \end{array}$$

where $1\colon \mathsf{Set} \to \mathsf{RCat}$ is the functor which sends a set X to the identity matrix 1_X. The definition of Open is functorial with respect to this sort of diagram i.e. it induces a symmetric monoidal double functor between the relevant double categories.

▶ **Theorem 24.** *There is a symmetric monoidal double functor*

$$\bigstar\colon \mathsf{Open}(\mathsf{RMat}) \to \mathsf{Open}(\mathsf{RCat})$$

which is

- *the identity on objects and vertical morphisms,*

- *sends an open R-matrix $M\colon X \to Y$ below left*

$$\begin{array}{ccccccc} & & M & & & & FM & & \\ & \nearrow & & \nwarrow & \mapsto & \nearrow & & \nwarrow & \\ 0_X & & & & 0_Y \quad 1_X & & & & 1_Y. \end{array}$$

to the solution of its algebraic path problem $\bigstar(M)\colon X \to Y$ above right, and

- *a vertical 2-morphism of open R-matrices $\alpha\colon M \Rightarrow N$ below left*

$$\begin{array}{ccccc} 0_X & \longrightarrow & M & \longleftarrow & 0_Y \\ {\scriptstyle 0_f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0_h} \\ 0_Z & \longrightarrow & N & \longleftarrow & 0_Q \end{array} \mapsto \begin{array}{ccccc} 1_X & \longrightarrow & FM & \longleftarrow & 1_Y \\ {\scriptstyle 1_f}\downarrow & & \downarrow{\scriptstyle Fg} & & \downarrow{\scriptstyle 1_h} \\ 1'_X & \longrightarrow & FN & \longleftarrow & 1'_Y \end{array}$$

is sent to the 2-morphism $\bigstar(\alpha)\colon M \Rightarrow N$ above right given by pointwise application of F.

Proof. Theorem 4.3 of [1] proves functoriality of the "Open" construction on squares below left

$$\begin{array}{ccc} X & \xrightarrow{F_1} & X' \\ {\scriptstyle L}\uparrow & & \uparrow{\scriptstyle L'} \\ A & \xrightarrow[F_0]{} & A' \end{array} \qquad \begin{array}{ccc} \mathsf{RMat} & \xrightarrow{F} & \mathsf{RCat} \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 1} \\ \mathsf{Set} & = & \mathsf{Set} \end{array}$$

commuting up to natural isomorphism. The result follows from applying this result to the square shown above right. ◀

The definition of symmetric monoidal double functor packages up a lot of information very succinctly. In particular, it contains coherent comparison isomorphism relating the solution of the algebraic path problem on a composite matrix to the solution on its components. For open R-matrices $M : X \to Y$ and $N : Y \to Z$, there is a composition comparison

$$\phi_{MN}\colon \bigstar(M) \circ \bigstar(N) \xrightarrow{\sim} \bigstar(M \circ N) \tag{2}$$

and monoidal comparison

$$\psi_{MM'}\colon \bigstar(M + M') \xrightarrow{\sim} \bigstar(M) + \bigstar(M') \tag{3}$$

giving recipes to break solutions to the algebraic path problem into their components. In other words, the left-hand side of each comparison is computed to determine the right-hand side

Pouly and Kohlas present a similar relationship in the context of valuation algebras. [15, §6.7]. For matrices M and N representing weighted graphs on vertex sets s and t respectively, the solution to the algebraic path problem on the union of their vertex sets is given by

$$F(M) \otimes F(N) = F\left(F(M)^{\uparrow s\cup t} + F(N)^{\uparrow s\cup t}\right)$$

In this formula, $\uparrow s \cup t$ indicates that the matrix is trivially extended to the union of the vertex sets. This formula is less general than comparison (2): it corresponds to the special case when the legs of the open R-matrices are inclusions.

A typical algorithm for the algebraic path problem has spacial complexity $\Theta(n^3)$ where n is the number of vertices in your weighted graph [10]. The comparisons (2) and (3) suggest a strategy for computing the solution to the algebraic path problem which reduces this complexity. First cut your weighted graph into smaller chunks, compute the solution to the algebraic path problem on those chunks, then combine their solutions using (2) and 3). Unfortunately, this strategy will in general take *more* time to compute the solution to the algebraic path problem on a composite because the right hand side of comparison (2) requires three applications of the functor F. However, the situation improves if the open R-matrices are functional.

5 Functional Open Matrices

In this section we define functional open R-matrices, a class of open R-matrices for which the composition comparison $\phi_{MN} \colon \bigstar(M) \circ \bigstar(N) \cong \bigstar(M \circ N)$ can be expressed in terms of matrix multiplication. The one caveat is that this expression requires that the open matrices be restricted to their inputs and outputs. For this we borrow a concept from engineering called "blackboxing" which forgets the internal workings of a system and concentrates only on the relationship it induces between its inputs and outputs.

▶ **Definition 25.** *Let $M \colon X \to Y$ be the open R-category*

$$1_X \xrightarrow{\ i\ } M \xleftarrow{\ o\ } 1_Y.$$

Then the **blackboxing** *of M is the matrix below left*

$$\blacksquare(M) \colon X \times Y \to R \qquad \blacksquare(M)(x, y) = M(i(x), o(y))$$

given by the expression above right.

The $\blacksquare$ operation is extended to all of RCat but the composition is only preserved laxly. The codomain of this extension is the following:

▶ **Definition 26.** *Let Mat_R be the double category where*

- *an object is a set X,Y,Z,. . .*
- *a vertical morphism is a function $f \colon X \to Y$,*
- *a horizontal morphism $M \colon X \to Y$ is a matrix $M \colon X \times Y \to R$,*
- *a vertical 2-morphism from $M \colon X \to Y$ to $N \colon X' \to Y'$ is a square below left*

$$\begin{array}{ccc} X & \xrightarrow{M} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[N]{} & Y' \end{array} \qquad \sum_{x \in f^{-1}(x'),\, y \in g^{-1}(y')} M(x, y) \leq N(x', y')$$

  satisfing the inequality above right for all $x' \in X'$ and $y' \in Y'$.
- *Vertical composition is function composition,*
- *and horizontal composition is given by matrix multiplication.*

In this double category, the composite of matrices M and N is written as the juxtaposition MN. Blackboxing is extended to the double category of open R-categories.

▶ **Proposition 27.** *There is a lax double functor*

$$\blacksquare \colon \mathsf{Open}(\mathsf{RCat}) \to \mathsf{Mat}_R$$

which

- *is the identity on objects,*
- *sends an open R-category $M: X \to Y$ to its blackbox $\blacksquare(M)$,*
- *and sends a vertical 2-cell below left*

$$\begin{array}{ccccc} 1_X & \longrightarrow & M & \longleftarrow & 1_Y \\ {\scriptstyle 1_f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1_h} \\ 1_{X'} & \longrightarrow & N & \longleftarrow & 1_{Y'} \end{array} \quad \mapsto \quad \begin{array}{ccc} X & \xrightarrow{\blacksquare(M)} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\blacksquare(N)]{} & Y' \end{array}$$

to the vertical 2-cell above right.

The blackboxing functor is composed with the algebraic path problem functor to get a lax symmetric monoidal double functor

$$\mathsf{Open}(\mathsf{RMat}) \xrightarrow{\bigstar} \mathsf{Open}(\mathsf{RCat}) \xrightarrow{\blacksquare} \mathsf{Mat}_R$$

This lax symmetric monoidal double functor sends an open R-matrix to the solution of its algebraic path problem only on nodes which start with an input and end with an output. It is natural to ask when this mapping is strictly functorial, as this yields a simple compositional formula for the algebraic path problem:

$$\blacksquare(\bigstar(M \circ N)) = \blacksquare(\bigstar(M))\blacksquare(\bigstar(N)).$$

The double functor $\blacksquare \circ \bigstar$ is strictly functorial on functional open matrices.

▶ **Definition 28.** *Let $M: A \times A \to R$ be an R-matrix. An element $a \in X$ is a* **source** *if for every $b \in X$, $M(b,a) = 0$ and a* **sink** *if $M(a,b) = 0$. A* **functional open *R*-matrix** *is an open R-matrix*

$$\begin{array}{ccc} & M & \\ {\scriptstyle l}\nearrow & & \nwarrow{\scriptstyle r} \\ 0_X & & 0_Y \end{array}$$

such that for every $x \in X$, $l(x)$ is a source and for every $y \in Y$, $r(y)$ is a sink.

Because the pushout of functional open R-matrices is also functional, we can form the following sub-double category.

▶ **Definition 29.** *Let $\mathsf{Open}(\mathsf{RMat})_{fxn}$ be the full sub-symmetric monoidal double category generated by the open R-matrices which are functional.*

▶ **Theorem 30.** *The composite $\blacksquare \circ \bigstar$ restricts to a strict double functor*

$$\blacksquare \circ \bigstar_{fxn} \colon \mathsf{Open}(\mathsf{RMat})_{fxn} \to \mathsf{Mat}_R$$

The proof of this theorem relies on a lemma which resembles the the binomial expansion of $(a+b)^n$ in a ring where $ba = 0$. If a and b represent blackboxings of functional open matrices, then the identity $ba = 0$ indicates that there are no paths which go backwards.

▶ **Lemma 31.** *For functional open R-matrices $M: X \to Y$ and $N: Y \to Z$ we have that*

$$\blacksquare(M +_{1_Y} N)^n = \sum_{i+j=n} \blacksquare(M^i)\blacksquare(N^j)$$

Proof. The entries of the left hand side are expanded as

$$\blacksquare((M +_{1_Y} N)^n)(a_0, a_n) = \sum_{a_1,\cdots,a_{n-1}} (M +_{1_Y} N)(a_0, a_1)(M +_{1_Y} N)(a_1, a_2) \cdots (M +_{1_Y} N)(a_{n-1}, a_n)$$

where the a_i are equivalence classes in $RM +_Y RN$. For a particular term of this sum, let $1 \leq k \leq n$ be the first natural number such that a_k contains an element of RN. Because M and N are functional, for $k \leq i \leq n$ the equivalence classes a_i must also contain an element of RN if our term is nonzero. Therefore for a fixed k the contribution to the above sum is given by

$$\sum M(a_0, a_1) \ldots M(a_{k-1}, a_k) N(a_k, a_{k+1}) \ldots N(a_{n-1}, a_n)$$

which simplifies to

$$\blacksquare(M^k)\blacksquare(N^{n-k})(a_0, a_n).$$

Because k can occur in any entry we have that

$$\blacksquare((M +_{1_Y} N)^n) = \sum_{k \leq n} \blacksquare(M^k)\blacksquare(N^{n-k}) = \sum_{i+j=n} \blacksquare(M^i)\blacksquare(N^j)$$

and this completes the proof. ◀

Proof of Theorem 30: It suffices to prove that for functional open R-matrices $0_X \longrightarrow M \longleftarrow 0_Y$ and $0_Y \longrightarrow N \longleftarrow 0_Z$ the equation

$$\blacksquare(\bigstar(M \circ N)) = \blacksquare(\bigstar(M))\blacksquare(\bigstar(N))$$

holds. Consider the left-hand side:

$$\begin{aligned} \blacksquare\bigstar M \circ N &= \blacksquare \sum_{n\geq 0} (M \circ N)^n \\ &= \sum_{n\geq 0} \blacksquare(M \circ N)^n \\ &= \sum_{n\geq 0} \sum_{i+j=n} \blacksquare(M^i)\blacksquare(N^j) \end{aligned}$$

on the other hand,

$$\begin{aligned} \blacksquare(\bigstar(M))\blacksquare(\bigstar(N)) &= \sum_{i\geq 0} \blacksquare(M^i) \sum_{j\geq 0} \blacksquare(N^j) \\ &= \sum_{i,j\geq 0} \blacksquare(M^i)\blacksquare(M^j) \end{aligned}$$

Both sums contain the term $\blacksquare(M^i)\blacksquare(N^j)$ for every value of i and j, but the left hand side may contain repeated terms. However, because addition is idempotent, repeated terms don't contribute to the sum and the two sides are the same. ◀

6 Conclusion

The functoriality of Theorem 30 might not be surprising. It says that if your open matrices are joined together directionally along bottlenecks, then the computation of the algebraic path problem can be reduced to a computation on components. This strategy has already proven sucessful. In [17], Sairam, Tamassia, and Vitter show how choosing *one way separators* as cuts in a graph, allow for an efficient divide and conquer parallel algorithm for computing shortest paths. In [16] Rathke, Sobocinksi, and Stephens show how the reachability problem on a 1-safe Petri net can be computed more efficiently by cutting it up into more manageable pieces. Theorem 24 provides a framework for compositional formulas of this type. In future work we plan on extending the construction of this theorem to many other sorts of discrete event dynamic systems.

Lemma 31 also holds independent computational interest. The equation given there gives a novel compositional formula for computing the solution to the algebraic path problem. The author has implemented this formula for the special case of Markov processes [13]. We hope that this is the start of a more extensive library, made faster and more reliable by the mathematics developed in this paper.

References

1. John C Baez and Kenny Courser. Structured cospans. *Theory and Applications of Categories*, 35(48):1771–1822, 2020.
2. Francis Borceux. *Handbook of Categorical Algebra: Volume 1, Basic Category Theory*. Cambridge University Press, 1994.
3. Kenny Courser. *Open Systems: a Double Categorical Perspective*. PhD thesis, University of California Riverside, 2020.
4. Antonin Delpeuch. The word problem for double categories. *Theory and Applications of Categories*, 35(1):1–18, 2020.
5. Stephen Dolan. Fun with semirings: a functional pearl on the abuse of linear algebra. In *Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming*, pages 101–110, 2013.
6. Eugene Fink. *A Survey of Sequential and Systolic Algorithms for the Algebraic Path Problem*. University of Waterloo, Department of Mathematics, 1992.
7. Brendan Fong. *The Algebra of Open and Interconnected Systems*. PhD thesis, University of Oxford, 2016.
8. Brendan Fong and David I Spivak. *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press, 2019.
9. Davis Foote. Kleene algebras and algebraic path problems. 2015. Available at edge.edx.org.
10. Peter Höfner and Bernhard Möller. Dijkstra, Floyd and Warshall meet Kleene. *Formal Aspects of Computing*, 24(4-6):459–476, 2012.
11. G Max Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society*, 22(1):1–83, 1980.
12. Saunders Mac Lane. *Categories for the Working Mathematician*. Springer Science & Business Media, 2013.
13. Jade Master. Compositional markov. https://github.com/Jademaster/compositionalmarkov, 2020.
14. David Jaz Myers. String diagrams for double categories and equipments. 2016. Available at https://arxiv.org/abs/1612.02762.
15. Marc Pouly and Jürg Kohlas. *Generic Inference: a Unifying Theory for Automated Reasoning*. John Wiley & Sons, 2012.

16 Julian Rathke, Paweł Sobociński, and Owen Stephens. Compositional reachability in Petri nets. In *International Workshop on Reachability Problems*, pages 230–243. Springer, 2014.

17 Sairam Subramanian, Roberto Tamassia, and Jeffrey Scott Vitter. An efficient parallel algorithm for shortest paths in planar layered digraphs. *Algorithmica*, 14(4):322–339, 1995.

18 Robert Endre Tarjan. A unified approach to path problems. *Journal of the Association for Computing Machinery*, 28(3):577–593, 1981.

19 Harvey Wolff. V-cat and V-graph. *Journal of Pure and Applied Algebra*, 4(2):123–135, 1974.

A Ommitted Proofs

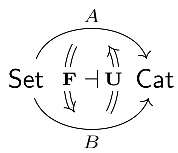
▶ **Proposition 32.** *The free monoid construction of Proposition 8 extends to an adjunction*

$$\mathsf{RMat} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{RCat}. \quad (F \dashv U)$$

Proof. Let $A\colon \mathsf{Set} \to \mathsf{Cat}$ be the functor which sends a set X to the poset $\mathsf{RMat}(X)$ regarded as a category and sends a function $f\colon X \to Y$ to the pushforward functor

$$f_*\colon \mathsf{RMat}(X) \to \mathsf{RMat}(Y).$$

from Definition 11. Analogously, let $B\colon \mathsf{Set} \to \mathsf{Cat}$ be the functor which sends a set X to the poset $\mathsf{RCat}(X)$ and sends a function f to its pushforward functor. The functors F_X for each set X form the components of a natural transformation $\mathbf{F}\colon A \Rightarrow B$. Similarly, the functors U_X form the components of a natural transformation $\mathbf{U}\colon B \Rightarrow A$. Furthermore, these natural transformations form an adjoint pair in the 2-category $[\mathsf{Set}, \mathsf{Cat}]$ of functors $\mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$, natural transformations between them, and modifications. $\mathbf{F}$ and $\mathbf{U}$ are adjoint because an adjoint pair in $[\mathsf{Set}, \mathsf{Cat}]$ is the same as a pair of natural transformations which are adjoint in each component. To summarize, we have a pair of adjoint natural transformations as follows:



A restriction of the Grothendieck construction [2] defines a 2-functor

$$\int\colon [\mathsf{Set}, \mathsf{Cat}] \to \mathsf{CAT}$$

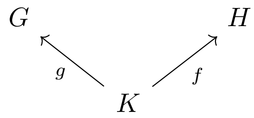
where CAT is the 2-category of large categories. Because every 2-functor preserves adjunctions, the above diagram maps to an adjunction

$$\int A \underset{\int \mathbf{U}}{\overset{\int \mathbf{F}}{\rightleftarrows}} \int B. \quad (\perp)$$

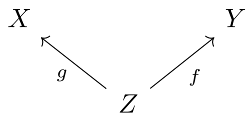
The result follows from the equivalences $\int A \cong \mathsf{RMat}$ and $\int B \cong \mathsf{RCat}$. The desired functors F and U are obtained by composing $\int \mathbf{F}$ and $\int \mathbf{U}$ with these equivalences. ◀

▶ **Lemma 33.** RMat *has coproducts and pushouts.*

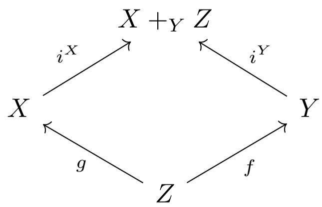
Proof. This is a consequence of Proposition 2.4 of [19] after noting that RMat is the category of R-graphs, the generating data for R-enriched categories. For concreteness and practicality, we offer an explicit construction of pushouts and coproducts here. Let



be a diagram in RMat with



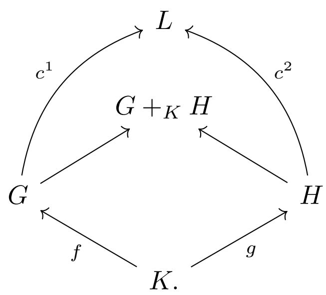
as the underlying diagram of sets. To compute the pushout $G +_K H$ first we take the pushout of sets



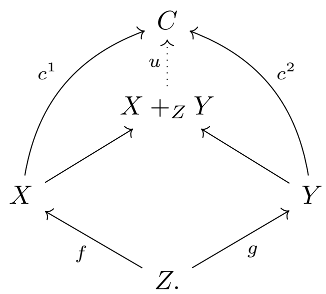
push them forward to get matrices $i^X_*(G)$ and $i^Y_*(H)$ and join them together to get

$$G +_Y H \colon (X +_Y Z) \times (X +_Y Z) \to R = i^X_*(G) + i^Y_*(H)$$

This does indeed define a pushout in RMat. Suppose we have a commutative diagram of R-matrices as follows:



then the underlying diagram of sets induces a unique function u



commuting suitable with c^1 and c^2. The map u is certainly unique, it remains to show that it is well-defined i.e. it satisfies the inequality

$$u_*(G +_K H) \leq L$$

Indeed, for $(x, y) \in C \times C$,

$$\begin{aligned}
u_*(G +_K H)(x, y) &= \sum_{(a,b)\in(u\times u)^{-1}(x,y)} G +_K H(a, b) \\
&= \sum_{(a,b)\in(u\times u)^{-1}(x,y)} i^X_*(G)(a, b) + i^Y_*(H)(a, b) \\
&= \sum_{(a,b)\in(u\times u)^{-1}(x,y)} i^X_*(G)(a, b) + \sum_{(a,b)\in(u\times u)^{-1}(x,y)} i^Y_*(H)(a, b) \\
&= u_*(i^X_*(G))(x, y) + u_*(i^Y_*(H))(x, y)
\end{aligned}$$

However, because

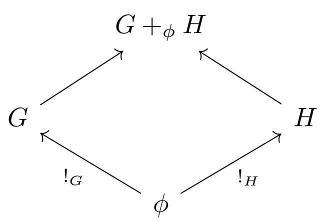
$$u_*(i^X(G)) = c^1_*(G) \text{ and } u_*(i^Y(H)) = c^2_*(G)$$

the above expression is equal to

$$c^1_*(G)(x, y) + c^2_*(H)(x, y)$$

which is less than or equal to $L(x, y)$ because each term is and $+$ is the least upper bound.

For R-matrices $G\colon X \times X \to R$ and $H\colon Y \times Y \to R$, their coproduct is given by the pushout



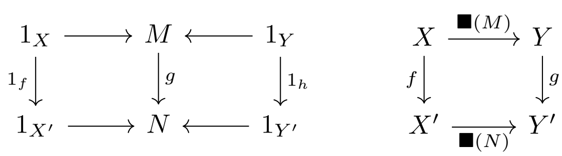
where ϕ is the unique R-matrix on the empty set and $!_G$ and $!_H$ are the unique morphisms into G and H respectively. ◀

▶ **Proposition 34.** *There is a lax double functor*

$$\blacksquare\colon \mathsf{Open}(\mathsf{RCat}) \to \mathsf{Mat}_R$$

which

- *is the identity on objects,*
- *sends an open R-category $M\colon X \to Y$ to its blackbox $\blacksquare(M)$,*
- *and sends a vertical 2-cell below left*



to the vertical 2-cell above right.

Proof. First observe that this lax double functor is well-defined on 2-cells. This amounts to showing that the inequality

$$\sum_{x \in f^{-1}(x'),\, y \in h^{-1}(y')} M(i(x), j(y)) \leq N(i'(x'), j'(y')) \tag{4}$$

holds. Because g is a morphism of R-matrices, we have that

$$\sum_{a \in g^{-1}(i'(x')),\, b \in g^{-1}(j'(y'))} M(a, b) \leq N(i'(x'), j'(y')) \tag{5}$$

Let $M(i(x), j(y))$ be a term on the left hand side of inequality (4). Then by definition, $x' = f(x)$ and $y' = h(y)$ so $a \in g^{-1}(i'(f(x))$ and $b \in g^{-1}(j'(h(y))$. However, because we started with a 2-cell in $\mathsf{Open}(\mathsf{RCat})$, $i' \circ f = g \circ i$ and $j' \circ h = g \circ j$ so we can rewrite inequality (5) as

$$\sum_{a \in g^{-1}(g \circ i(x)),\, b \in g^{-1}(g \circ j(y))} M(a, b) \leq N(i'(x'), j'(y'))$$

The term $M(i(x), j(y))$ of the left hand side of inequality (4) is also a term of the left hand side of inequality (5) so we have that

$$M(i(x), i(y)) \leq \sum_{a \in g^{-1}(g \circ i(x)),\, b \in g^{-1}(g \circ j(y))} M(a, b) \leq N(i'(x'), j'(y'))$$

Because each term on the left hand side of (4) is less than the desired quantity, the join of all the terms will be as well. Therefore the lax double functor is well-defined on 2-cells. Note that Mat_R is locally posetal i.e. for every square below left

$$\begin{array}{ccc} X & \xrightarrow{M} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[N]{} & Y' \end{array} \qquad\qquad \begin{array}{ccc} & 1_X & \\ 1_X & & 1_X \end{array}$$

there is at most one 2-cell filling it. This property makes it so many of the axioms in the definition of lax double functor are satisfied trivially. It suffices to show that the globular composition and identity comparisons exist. The identity morphism in $\mathsf{Open}(\mathsf{RCat})$ on a set X is the cospan above right. The blackbox of this cospan is equal to the identity matrix on X, so the identity comparison is the identity. The composition comparison $\blacksquare(M)\blacksquare(N) \leq \blacksquare(M \circ N)$ follows from the chain of inequalities

$$\begin{aligned} \blacksquare(M)\blacksquare(N) &= \sum_{y \in Y} \blacksquare(M)(x, y)\blacksquare(N)(y, z) \\ &= \sum_{y \in Y} M(i(x), j(y))N(i'(y), j'(z)) \\ &= (M +_{1_Y} N)^2 \\ &\leq \sum_{n \geq 0} (M +_{1_Y} N)^n(i(x), j'(z)) \\ &= \blacksquare(M \circ N)(x, z) \end{aligned}$$

Therefore, $\blacksquare$ is a lax double functor. ◀