# Which Categories Are Varieties?

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## Abstract

Categories equivalent to single-sorted varieties of finitary algebras were characterized in the famous dissertation of Lawvere. We present a new proof of a slightly sharpened version: those are precisely the categories with kernel pairs and reflexive coequalizers having an abstractly finite, effective strong generator. A completely analogous result is proved for varieties of many-sorted algebras provided that there are only finitely many sorts. In case of infinitely many sorts a slightly weaker result is presented: instead of being abstractly finite, the generator is required to consist of finitely presentable objects.

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## 1 Introduction

In his dissertation in 1963 Lawvere not only introduced algebraic theories. He also characterized finitary single-sorted varieties as precisely those categories, up to equivalence, which have

**(1)** finite limits and coequalizers,
**(2)** effective equivalence relations, and
**(3)** an abstractly finite, regularly projective regular generator $G$.

Regular projectivity means that the hom-functor of $G$ preserves regular epimorphisms. Abstract finiteness states that every morphism from $G$ to a copower of $G$ factorizes through a finite subcopower; this is much weaker than being finitely generated.

In Condition (1) of the dissertation coequalizers are not included. But they are used in the proof with no explanation, so this is just a small typo. Condition (2) can be avoided if a bit more than regular projectivity is required of $G$, as observed by Pedicchio und Wood [11]. We follow their idea and call an object $G$ *effective* if its hom-functor preserves coequalizers of equivalence relations. Given a regularly projective regular generator $G$ in a category $\mathcal{K}$, then it is effective iff $\mathcal{K}$ has effective equivalence relations (Proposition 25 below).

In software specification one typically works with many-sorted algebras, and the purpose of our paper is to generalize Lawvere's result to that case and improve it slightly: in (1) we need only to assume that kernel pairs and reflexive coequalizers exist. In (3) it is sufficient (in case of single-sorted varietie) to assume that $G$ is an abstractly finite, effective, strong generator. For many-sorted varieties the concept of an *abstractly finite set* of objects was presented [1]. In case of finitely many sorts we obtain a completely analogous result to that above: a category with kernel pairs and reflexive coequalizers is equivalent to a finitary


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$S$-sorted variety iff it has an abstractly finite strong generator formed by finitely many effective objects. The proof is based on the fact that for a monad $T$ on $\mathbf{Set}^S$ to be finitary it is sufficient that it be *finitely bounded* (which, for $S$ finite, has been shown in [3]). This means that every element of $TX$ lies in the image of $Tm$ for some finite $S$-sorted subset $m\colon M \hookrightarrow X$.

For infinitely many sorts an analogous, but slightly weaker result, is proved: in place of abstract finiteness one has to work with finitely presentable objects in the generator. In Example 39 below we demonstrate that the (expected) stronger result does not hold.

The results presented here are not substantially new. For example in [1] single-sorted varieties are characterized as cocomplete categories with a finitely generated, effective regular generator. But the message of our paper is that the beautiful result of Lawvere can be sharpened a little bit and extended to many-sorted varieties by applying the categorical methods developed in the subsequent 58 years. Many-sorted varieties have also been characterized as precisely the strongly locally presentable categories in [5]. We show that this result is an easy corollary of our main theorems.

**Related Work.** The concept of an abstractly finite set of objects was introduced in [1] and [2], where it was claimed that for many-sorted varieties the result of Lawvere completely generalizes. But the proof (based on the Birkhoff Variety Theorem) was not correct: see Example 39 which shows that the assumption of finitely many sorts is essential.

## 2 Abstractly Finite Objects

There are several concepts generalizing finite sets to "finite" objects of a category. Among the most important ones are that an object $A$ of $\mathcal{K}$ is *finitely presentable* or *finitely generated* if its hom-functor $\mathcal{K}(A,-)$ preserves directed colimits (or directed colimits of monomorphisms, resp.). Lawvere [9] used in his characterization of varieties a weaker concept, abstract finiteness. He commented that it had been introduced by Peter Freyd.

We denote by

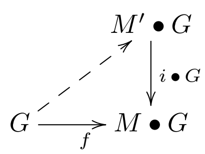
$$M \bullet G = \coprod_M G$$

the copower of an object $G$ indexed by a set $M$. By a *subcopower* is meant the morphism

$$i \bullet M' \colon M' \bullet G \to M \bullet G$$

where $i\colon M' \hookrightarrow M$ is the inclusion map of a subset $M' \subseteq M$.

▶ **Definition 1.** *An object $G$ is called abstractly finite if it has copowers, and every morphism $f\colon G \to M \bullet G$ factorizes through a finite subcopower:*




$(M' \subseteq M$ *finite*$)$

▶ Remark 2.

**(1)** Let $G$ be an object with copowers. If it is finitely generated, then it is abstractly finite – but not vice versa, as we demonstrate in Example 3 below.
In fact, for every set $M \neq \emptyset$ we form a directed diagram of all finite non-empty subcopowers of $M \bullet G$. It has the colimit $M \bullet G$. Its connecting morphisms are

$j \bullet G \colon M' \bullet G \to M'' \bullet G$ for inclusion maps $j \colon M' \hookrightarrow M''$. Since $j$ splits in **Set**, $j \bullet G$ is a split monomorphism. Thus the hom-functor of $G$ preserves this directed colimit. Equivalently, $G$ is abstractly finite.

**(2)** In a single-sorted variety of algebras, a free algebra $G$ on a set $X$ is abstractly finite iff $X$ is finite. This follows easily from the fact that $M \bullet G$ is the free algebra on $M \times X$ for every set $M$.

► **Examples 3.**

**(1)** *In* **Set** *abstractly finite means finite. In the category of vector spaces it means finite-dimensional. So here finitely generated = abstractly finite.*

**(2)** *In the category of posets the abstractly finite objects are precisely the posets with finitely many connected components. Thus, an abstractly finite object can have an arbitrarily large cardinality.*
*The same is true in the category of unary algebras on one operation or in the category of graphs.*

**(3)** *In the category DCPO of dcpo's, i.e. posets with directed joins (and continuous maps) no nonempty object is finitely generated. In contrast, a dcpo is abstractly finite iff it has finitely many connected components.*

► Remark 4. Our focus is on varieties of $S$-sorted algebras, which we now shortly recall from [5, Chapter 14].

**(1)** By an $S$-sorted signature (for a set $S$) is meant a collection $\Sigma$ of (operation) symbols $\sigma$ with prescribed arities in $S^* \times S$. We write $\sigma \colon s_0 \dots s_{n-1} \to s$ if $\sigma$ has arity $(s_1 \dots s_{n-1}, s)$.

**(2)** The category of $S$-sorted sets is denoted by $\mathbf{Set}^S$. Let $X \in \mathbf{Set}^S$ be an $S$-sorted set of a variables. The $S$-sorted set $F_\Sigma X$ of terms is the least one containing $X$ and such that given an operation symbol $\sigma \colon s_0 \dots s_{n-1} \to s$ and terms $p_i$ of sort $s_i$, then $\sigma(p_0, \dots, p_{n-1})$ is a (composite) term of sort $s$.

**(3)** A $\Sigma$-algebra is a sorted set $A$ equipped with operations $\sigma_A \colon A_{s_0} \times \dots \times A_{s_{n-1}} \to A_s$ for every operation symbol $\sigma \colon s_0 \dots s_{n-1} \to s$. Given another $\Sigma$-algebra $B$, a homomorphism is a sorted map $f \colon A \to B$ preserving the operations: for every $\sigma \colon s_0 \dots s_{n-1} \to s$ we have

$$f_s \cdot \sigma_A = \sigma_B \cdot (f_{s_0} \times \dots \times f_{s_{n-1}}) .$$

We denote by $\Sigma$- **Alg** the category of $\Sigma$-algebras and homomorphisms.

Example: $F_\Sigma X$ is a $\Sigma$-algebra w.r.t. composite terms as operations. This is a free $\Sigma$-algebra on $X$ w.r.t. the inclusion map $\eta \colon X \hookrightarrow F_\Sigma X$.

**(4)** An *equation* using variables $x_i$ of sort $s_i$ $(i = 0, \dots, k-1)$ is an expression

$$\forall x_0 \dots \forall x_{k-1} (t = t')$$

where $t$, $t'$ are terms in $F_\Sigma\{x_0, \dots, x_{n-1}\}$ of the same sort. A $\Sigma$-algebra $A$ *satisfies* this equation if for every sorted function $f \colon \{x_0, \dots, x_{n-1}\} \to A$ the free homomorphism $\bar{f} \colon F_\Sigma\{x_i\} \to A$ fulfils $\bar{f}(t) = \bar{f}(t')$.

**(5)** For every set $\mathcal{E}$ of equations we denote by

$$(\Sigma, \mathcal{E})\text{- } \mathbf{Alg}$$

the full subcategory of $\Sigma$-**Alg** of all algebras satisfying all equations in $\mathcal{E}$. It is easy to see that this is a reflective subcategory of $\Sigma$-**Alg**, thus, it has free algebras on all sorted sets. And it is a complete and cocomplete category.

▶ **Definition 5.** *By a variety is meant a category* $(\Sigma, \mathcal{E})$-**Alg** *for some many sorted signature* $\Sigma$ *and some set* $\mathcal{E}$ *of equations.*

▶ Remark 6. In a variety the usual meaning of "finitely generated", that is, having a finite set of generators, is equivalent to the categorical concept. The same is true about "finitely presentable", that is, presentable by finitely many generators and finitely many relations. See [4] 3.11 and 3.12.

For many-sorted varieties we need to generalize the concept of an abstractly finite object to sets of objects:

▶ **Definition 7.** *A set* $\mathcal{G}$ *of objects is abstractly finite if all coproducts of collections of objects from* $\mathcal{G}$ *exist, and given a morphism* $f : G \to \coprod_{i \in I} G_i$ *with* $G$ *and all* $G_i$ *in* $\mathcal{G}$*, then* $f$ *factorizes through a finite subcoproduct of* $\coprod_{i \in I} G_i$.

The factorization of $f$ is not required to be unique (and coproduct injections are not required to be monic).

▶ **Example 8.** In an $S$-sorted variety we form the free algebra $G_s$ on one generator of sort $s$ for each $s \in S$. The set $\{G_s\}_{s \in S}$ is abstractly finite. Indeed, the coproduct $\coprod_{i \in I} G_{s_i}$ is precisely a free algebra on the $S$-sorted set $X$ with $X_s = \{i \in I, s_i = s\}$. Every homomorphism $f : G_s \to \coprod_{i \in I} G_{s_i}$ maps the generator $x \in G_s$ to a term over $X$. If $Y \subseteq X$ is the set of all variables that appear in the term $f(x)$, then $f(x)$ lies in the finite subcoproduct $\coprod_{s_i \in Y} G_{s_i}$. Consequently, $f$ factorizes through this subcoproduct.

▶ Remark 9. Recall that a *strong generator* is a set $\mathcal{G}$ of objects such that coproducts of collections of objects from $\mathcal{G}$ exist, and for every object $X$ there exists an epimorphism $c : \coprod_{i \in I} G_i \to X$ with $G_i \in \mathcal{G}$ for $i \in I$ which is extremal (i.e., $c$ does not factorize through any proper subobject of $X$).

A *regular generator* has the stronger property that the following canonical morphism

$$c_X = [f] : \coprod_{G \in \mathcal{G}} \coprod_{f : G \to X} G \to X$$

is a regular epimorphism.

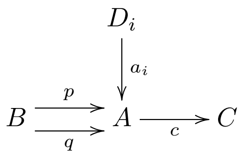
▶ **Example 10.**

**(1)** In a single-sorted variety $\mathcal{K}$ the free algebra $G$ on one generator is an abstractly finite regular generator. Indeed, for every algebra $X$ the coproduct $\coprod_{f : G \to X} G$ is the free algebra of $\mathcal{K}$ generated by $\mathcal{K}(G, X)$, and the canonical morphism $c_X$ is surjective, i.e., a regular epimorphism.

**(2)** In an $S$-sorted variety the set $\{G_s\}_{s \in S}$ from Example 8 is an abstractly finite regular generator. The argument is as above.

▶ Remark 11.

**(1)** In the next theorem we use the standard construction of colimits via coproducts and coequalizers [10] Thm. V.2.2. Let $D : \mathcal{D} \to \mathcal{K}$ be a diagram with objects $D_i$ ($i \in I$). Suppose that the following coproducts exist: $A = \coprod_{i \in I} D_i$ with injections $a_i$, and $B = \coprod_{f : D_i \to D_j} D_i$ with injection $b_f$, where the $f$'s range over all morphisms of $\mathcal{D}$. Then we form the morphisms $p$, $q: B \to A$ with $f$-components $a_i$ and $a_j \cdot f$, resp. for all $f: D_i \to D_j$. Suppose a coequalizer $c$ of $p$ and $q$ exists:




Then the cocone of $c \cdot a_i$ ($i \in I$) is a colimit of $D$.

**(2)** We recall the concept of dense subcategory. Given a full subcategory $\mathcal{G}$ of $\mathcal{K}$, for every object $K$ we form the slice category $\mathcal{G}/K$ of all morphisms $f: G \to K$ with $G \in \mathcal{G}$. The forgetful functor $D_K: \mathcal{G}/K \to \mathcal{K}$ sending $f: G \to K$ to $G$ has the cocone formed by all $f$'s. Then $\mathcal{G}$ is *dense* if this cocone is a colimit of $D_K$ (for every object $K$).

The proof of the following theorem uses ideas of Lawvere's thesis [9]. A shorter proof presented in [1] was not correct.

▶ **Theorem 12.** *Let $G$ be an abstractly finite, regular, singleton generator. Then the full subcategory of finite copowers of $G$ is dense.*

**Proof.** Denote by $\mathcal{G}$ the full subcategory of all $n \bullet G$, $n \in \mathbb{N}$. For every object $K$ we prove that $K = \mathit{colim}\, D_K$. In detail, given an object $L$ and a cocone of $D_K$ denoted by $(-)'$:

$$\frac{n \bullet G \xrightarrow{f} K}{n \bullet G \xrightarrow[f']{} L} \qquad (n \in \mathbb{N})$$

we prove that there exists a unique morphism $h: K \to L$ with $f' = h \cdot f$ for all $f$. The fact that $(-)'$ is a cocone means that

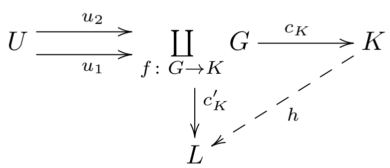
$$f = g \cdot u \quad \text{implies} \quad f' = g' \cdot u \tag{2.1}$$

for all morphisms $u: n \bullet G \to m \bullet G$ and $g: m \bullet G \to K$. In particular, if $f_i: G \to K$ are the components of $f: n \bullet G \to K$ ($i = 1, \dots, n$), then we get the corresponding morphisms $f_i': G \to L$. We then obtain

$$f' = [f_1', \dots, f_n'] \tag{2.2}$$

by applying (2.1) to the coproduct injections on $n \bullet G$.

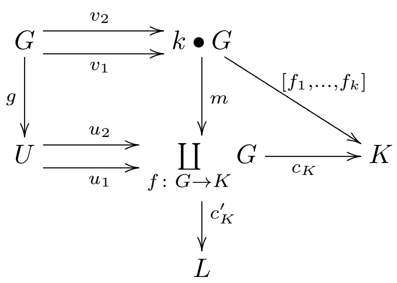
The canonical morphism $c_K: \coprod_{f:\, G \to K} G \to K$ is a coequalizer of a pair $u_1$, $u_2: U \to \coprod_{f:\, G \to K} G$. Denote by $c_K': \coprod_{f:\, G \to K} G \to L$ the morphism with components $f'$ for every $f: G \to K$:




We are going to prove that $c_K' \cdot u_1 = c_K' \cdot u_2$. Then we obtain a factorization $h$ with $c_K' = h \cdot c_K$. This is the desired morphism: for every $f: G \to K$ we then have $f = h \cdot f'$, and due to (2.2) the same holds for every $f\colon n \bullet G \to K$. Uniqueness of $h$ is clear since $G$ is a generator. For proving $c'_K \cdot u_1 = c'_K \cdot u_2$ we just need to verify

$$c'_K \cdot u_1 \cdot g = c'_K \cdot u_2 \cdot g \quad \text{for every} \quad g\colon G \to U\,. \tag{2.3}$$

The morphisms $u_1 g,\ u_2 g\colon G \to \coprod_{f\colon G \to K} G$ both factorize through a finite subcopower, since $G$ is abstractly finite. That is, we have $f_i\colon G \to K$ for $i = 1, \dots, k$ such that for the corresponding coproduct injection $m\colon k \bullet G \to \coprod_{f\colon G\to K} G$ there exist morphisms $v_i$ with $g \cdot u_i = m \cdot v_i$ :




From $c_K \cdot u_1 = c_K \cdot u_2$ we get $c_K \cdot m \cdot v_1 = c_K \cdot m \cdot v_2$, thus, $(c_K \cdot m \cdot v_1)' = (c_K \cdot m \cdot v_2)'$. By applying (2.1) to the morphisms $mv_i$ of $\mathcal{G}$ we get

$$c'_K u_i g = c'_K m v_i = (c_K m v_i)' = (c_K u_i g)'$$

for $i = 1, 2$, which proves (2.3). ◀

▶ Remark 13. The above theorem and proof immediately generalize to non-singleton abstractly finite regular generators $\mathcal{G}$: the closure of $\mathcal{G}$ under finite coproducts is dense.

▶ Remark 14. A pair of morphisms $f_1,\ f_2\colon A \to B$ is called *reflexive* if there exists $d\colon B \to A$ with $f_1 \cdot d = f_2 \cdot d = \mathrm{id}_B$. A category is said to have *reflexive coequalizers* if every reflexive pair has a coequalizer.

▶ **Corollary 15.** *Let $\mathcal{K}$ be a category with reflexive coequalizers. Then it is complete and cocomplete provided that it has an abstractly finite regular generator consisting of regular projectives.*

This follows from Remark 13. Let $\bar{\mathcal{G}}$ be the dense closure of $\mathcal{G}$. Consequently, the functor $E\colon \mathcal{K} \to \left[\bar{\mathcal{G}}^{\mathrm{op}}, \mathbf{Set}\right]$ assigning to $K$ the restriction of $\mathcal{K}(-, K)$ to $\bar{\mathcal{G}}^{\mathrm{op}}$ is full and faithful. Moreover $E$ has a left adjoint: it assigns to $H\colon \bar{\mathcal{G}}^{\mathrm{op}} \to \mathbf{Set}$ the colimit of the category of elements of $H$. The reason why this colimit exists is that in Remark 11(1) the two coproducts exist, since they are formed by object of $\bar{\mathcal{G}}^{\mathrm{op}}$ (and so they are coproducts of objects of $\mathcal{G}$), and the coequalizer $c$ exists because the pair $p,\ q\colon B \to A$ is reflexive. Indeed, the morphism $d\colon A \to B$ whose $i$-component is the coproduct injection corresponding to $\mathrm{id}_{D_i}$ fulfils $p \cdot d = q \cdot d = \mathrm{id}_A$. The details why this yields a left adjoint of $E$ can be found in [1, Corollary 2.12].

We conclude that $\mathcal{K}$ is equivalent to a full reflective subcategory of $[\bar{\mathcal{G}}^{\mathrm{op}}, \mathbf{Set}]$, hence, it is complete and cocomplete.

## 3 Effective Objects

A category is said to have effective equivalence relations if every equivalence relation (see below) is the kernel pair of its coequalizer. We define effective objects as those whose hom-functors preserve coequalizers of equivalence relations. Based on an idea of Pedicchio and Wood [11] we then prove that given a regularly projective regular generator $G$, it is effective iff equivalence relations are effective.

The usual definition of a relation $R$ on an object $A$ is: a subobject of $A \times A$. The restricted projections then form a parallel pair $r_1,\ r_2\colon R \to A$ of morphisms which is collectively monic. Our main theorem makes no assumptions about the existence of products. We therefore introduce relations via parallel pairs:

▶ **Definition 16** ([7] 2.5.2)**.** *Let $A$ be an object of a category $\mathcal{K}$.*

**(1)** *A relation on $A$ is represented by a collectively monic ordered pair of morphisms $r_1$, $r_2\colon R \to A$. Another such pair $r_1'$, $r_2'\colon R' \to A$ represents the same relation iff there is an isomorphism $i\colon R' \to R$ with $r_1' = r_1 i$ and $r_2' = r_2 i$.*
*We speak about "the relation $R$" if $r_1$, $r_2$ are clear.*

**(2)** *A relation $R$ is an equivalence if for every object $X$ of $\mathcal{K}$ the following relation on the hom-set $\mathcal{K}(X, A)$ is an equivalence relation in the usual sense:*

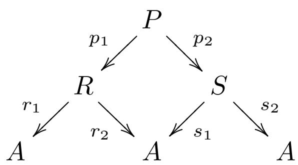
$$\big\{(r_1 f, r_2 f);\, f\colon X \to R\big\}\,.$$

▶ Remark 17. Let $\mathcal{K}$ have finite limits and *regular factorizations* (every morphism factorizes as a regular epimorphism followed by a monomorphism). Then equivalence relations are more intuitive:

**(1)** A relation $R$ on $A$ is precisely a subobject of $A \times A$.
Example: $\Delta_A$ given by $\mathrm{id}_A$, $\mathrm{id}_A$.

**(2)** Every parallel pair of morphisms $f_1$, $f_2\colon B \to A$ represents a relation on $A$ : factorize $\langle f_1, f_2\rangle\colon B \to A \times A$ as a regular epimorphism $e\colon B \to R$ followed by a monomorphism $\langle r_1, r_2\rangle\colon R \to A \times A$. This gives you a relation $R \rightarrowtail A \times A$.

**(3)** A *composite* of relations $r_1$, $r_2\colon R \to A$ and $s_1$, $s_2\colon S \to A$ is the relation $P = R \circ S$ represented by the pair $(r_1 p_1, s_2 p_2)$ for the following pullback $P$ of $r_2$ and $s_1$ :




**(4)** The relation $R^0$ is represented by $r_2$, $r_1\colon R \to A$.

**(5)** A relation $R$ on $A$ is an equivalence relation iff it is reflexive ($\Delta_A \subseteq R$), symmetric ($R^0 \subseteq R$) and transitive ($R \circ R \subseteq R$). This is equivalent to the definition above by [7] Proposition 2.5.1.

**(6)** For every morphism $f\colon A \to B$ the *kernel pair*, which means the pullback of two copies of $f$, is an equivalence relation. By an *effective equivalence* is meant the kernel pair of some morphism.

**(7)** A functor $E\colon \mathcal{K} \to \mathcal{L}$ is said to *reflect isomorphisms* if whenever $Eh$ is invertible in $\mathcal{L}$, then $h$ is invertible in $\mathcal{K}$. Analogously for reflecting regular epimorphisms, limits, etc.

Suppose that $E$ preserves and reflects (a) finite limits and (b) regular factorizations. Then it preserves and reflects relations and relation composition. Since the operation $R \mapsto R^0$ is also preserved and reflected, we conclude that $E$ preserves and reflects equivalence relations.

▶ **Definition 18.** *A category is said to have effective equivalence relations if every equivalence on an object $A$ is effective (the kernel equivalence of some morphism $f\colon A \to B$).*

▶ **Example 19.** Let $\mathcal{K}$ be a variety. A relation on an algebra $A$ is a subalgebra of $A \times A$. And an equivalence is precisely a congruence $R$ on $A$, represented by its projections $r_1$, $r_2\colon R \to A$. Every variety has effective equivalence relations.

▶ **Definition 20.** *An object $A$ is called effective if its hom-functor $\mathcal{K}(A, -)$ preserves coequalizers of equivalence relations.*

▶ **Example 21.** In a variety $\mathcal{K}$ all free algebras are effective. In fact, let us first consider a single-sorted variety. Its forgetful functor $U\colon \mathcal{K} \to \mathbf{Set}$ preserves coequalizers of congruences. (Indeed, if $R$ is a congruence on $A$, then the quotient algebra $A/R$ is formed on the corresponding quotient set. And the canonical map $c\colon A \to A/R$ is precisely the coequalizer of the projections $r_1$, $r_2\colon R \to A$.) Now $U \simeq \mathcal{K}(G, -)$ where $G$ is the free algebra on one generator. Thus, $G$ is effective. And since a free algebra on a set $M$ is precisely $M \bullet G$, its hom-functor is the $M$-copower of $U$, and it also preserves coequalizers of congruences.

The argument for $S$-sorted varieties is analogous, using that the forgetful functor $U\colon \mathcal{K} \to \mathbf{Set}^S$ preserves coequalizers of congruences.

▶ **Definition 22.** *An object $G$ is called regularly projective if its hom-functor preserves regular epimorphisms. More detailed: given a regular epimorphism $e\colon A \to B$, every morphism from $G$ to $B$ factorizes through $e$.*

▶ Remark 23. Let $\mathcal{K}$ be a category with kernel pairs. Recall that kernel pairs are called effective equivalences.

**(1)** Every effective object $G$ is regularly projective.

In fact, given a regular epimorphism $e\colon A \to B$ form its kernel pair $r_1$, $r_2\colon R \to A$. Then $\mathcal{K}(G, e)$ is a coequalizer of $\mathcal{K}(G, r_i)$, thus it is surjective.

**(2)** An object $G$ is regularly projective iff its hom-functor preserves coequalizers of effective equivalences – thus, this is very "near" to being effective.

Indeed, let $r_1$, $r_2\colon R \to A$ be an effective equivalence. Then the coequalizer $e\colon A \to B$ has the kernel equivalence $R$. Since $\mathcal{K}(G, -)$ preserves pullbacks, $\mathcal{K}(G, e)$ has the kernel pair $\mathcal{K}(G, r_i)$, $i = 1, 2$ in $\mathbf{Set}$. We know that $\mathcal{K}(G, e)$ is surjective. This implies that this is the coequalizer of its kernel pair.

▶ Remark 24. Let $\mathcal{K}$ be a category with kernel pairs and their coequalizers and let $\mathcal{G}$ be a regular generator formed by regular projectives.

**(1)** $\mathcal{K}$ has regular factorizations: every morphism $f$ factorizes as a regular epimorphism followed by a monomorphism. Indeed, let $p_1$, $p_2\colon P \to A$ be the kernel pair of $f\colon A \to B$ and $e\colon A \to C$ be a coequalizer of $p_1$, $p_2$. Then we have $m\colon B \to C$ with $f = m \cdot e$. To prove that $m$ is monic, consider $u_1$, $u_2\colon G \to B$ for $G \in \mathcal{G}$: if $mu_1 = mu_2$, we prove $u_1 = u_2$. Since $G$ is regularly projective, there exist morphisms $u_i'\colon G \to A$ with $u_i = e \cdot u_i'$. From $fu_1' = fu_2'$ we conclude that there is $u\colon U \to P$ with $u_i' = p_i u$. Therefore $u_1 = ep_1u = ep_2u = u_2$.

**(2)** The functor $U\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{G}}$ with components $\mathcal{K}(G, -)$ preserves and reflects both regular epimorphisms and isomorphisms. Preservation is clear. Let $f\colon A \to B$ be such that $Uf$ is epic, i.e., sort-wise surjective. Factorize $f = m \cdot e$ where $e\colon A \to C$ is a regular epimorphism and $m\colon C \to B$ is a monomorphism. Then $m$ is invertible because for every $G \in \mathcal{G}$ we see that all morphisms $G \to B$ factorize through $m$. Analogously, $U$ reflects isomorphisms.

**(3)** Consequently, $U$ preserves and reflects equivalences, see Remark 17(7).

▶ **Proposition 25.** *Let $\mathcal{K}$ be a category with kernel pairs and reflexive coequalizers, having a regular generator $\mathcal{G}$ formed by regular projectives. Equivalent are:*
**(1)** *all objects of $\mathcal{G}$ are effective, and*
**(2)** *$\mathcal{K}$ has effective equivalence relations.*

**Proof.** $2 \to 1$. Given an equivalence relations $e_1, e_2 \colon E \to A$ and its coequalizer $c \colon A \to B$, we prove that $\mathcal{K}(G, -)$ preserves it for every $G \in \mathcal{G}$. Let $f \colon \mathcal{K}(G, A) \to X$ be a function with $f \cdot \mathcal{K}(G, e_1) = f \cdot \mathcal{K}(G, e_2)$. We prove that it factorizes through $\mathcal{K}(G, c)$. In other words: for every pair $x_1, x_2 \in \mathcal{K}(G, A)$ merged by $\mathcal{K}(G, c)$ we prove that $f$ also merges it. By assumption, $c \cdot x_1 = c \cdot x_2$. Thus $x_1$, $x_2$ factorize through the kernel pair of $c$, which by (2) is $e_1$, $e_2$. Given $h \colon G \to B$ with $x_i = e_i \cdot h$, we get

$$f(x_i) = f(e_i \cdot h) = \big(f \cdot \mathcal{K}(G, e_i)\big)(h) \qquad (i = 1, 2).$$

This proves $f(x_1) = f(x_2)$.

$1 \to 2$. Let $\mathcal{G}$ consist of effective objects. For every equivalence relation $e_1, e_2 \colon E \to A$ form its coequalizer $c \colon A \to B$ and a kernel pair $e_1', e_2' \colon E' \to A$ of $c$. We have a factorization $h \colon E \to E'$ with $e_k = e_k' \cdot h$ for $k = 1, 2$. Our task is to prove that $h$ is an isomorphism.

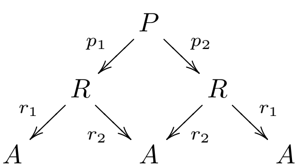
The functor $U = \mathcal{K}(G, -)$ for $G \in \mathcal{G}$ preserves limits and regular epimorphisms since $G$ is a regular projective. Consequently, by Remarks 24 and 17(7) it preserves equivalence relations. Thus $UE$ and $UE'$ are equivalence relations on the set $UA$ and, since $G$ is effective, they have a common coequalizer $Uc$. It follows for the factorization morphism $h$ that $Uh$ is an isomorphism. Consequently, so is $h$. ◀

▶ Remark 26. The above proposition was inspired by the paper [11] in which an object is called an *effective projective* if its hom-functor preserves reflexive coequalizers. Preservation of coequalizers of equivalence relations seems a more suitable condition first because of the above proposition. Second, for varieties of infinitary algebras free algebras are effective but no longer effective projectives.

▶ **Definition 27.** *By a split coequalizer of morphisms $r_1$, $r_2 \colon R \to A$ is meant a morphism $c \colon A \to C$ with $cr_1 = cr_2$ such that there exist morphisms $i \colon C \to A$ and $j \colon A \to R$ with $ci = \mathrm{id}_C$, $jr_1 = \mathrm{id}_A$ and $jr_2 = ic$.*

▶ **Lemma 28.** *In a category with finite limits and regular factorizations for every split coequalizer $c$ of a relation $r_1$, $r_2 \colon R \to A$ the kernel pair of $c$ is the composite relation $R \circ R^0$.*

**Proof.** The relation $R^0$ is represented by $r_2, r_1 \colon R \to A$. Thus $R \circ R^0$ is represented by $r_1 p_1$, $r_1 p_2 \colon P \to A$ where $p_1$, $p_2 \colon P \to R$ is the kernel pair of $r_2$, see Remark 17(3):




The corresponding collectively monic pair $r_1'$, $r_2' \colon R' \to A$ is obtained by factorizing $\langle r_1 p_1, r_1 p_2 \rangle \colon P \to A \times A$ as a regular epimorphism $e \colon P \to R'$ followed by a monomorphism $\langle r_1', r_2' \rangle$. We prove that $r_1'$, $r_2'$ is a kernel pair of $c$.

**(1)** $cr_1' = cr_2'$. Since $e$ is epic, this follows from

$$c \cdot (r_1' e) = c \cdot (r_1 p_1) = cr_2 p_1 = cr_2 p_2 = c \cdot (r_1 p_2) = c \cdot (r_2' e).$$

**(2)** Given $u_1$, $u_2 \colon U \to A$ with $cu_1 = cu_2$, then this pair uniquely factorizes through $r'_1$, $r'_2$. Unicity is clear since $\langle r'_1, r'_2 \rangle$ is monic.

For $i$ and $j$ as in Definition 27 we see that

$$r_2 \cdot (ju_1) = icu_1 = icu_2 = r_2 \cdot (ju_2)\,.$$

Since $p_1$, $p_2$ is the kernel pair of $r_2$, this implies that there exists $h \colon U \to P$ with $ju_1 = p_1 h$ and $ju_2 = p_2 h$. The desired factorization is $eh \colon U \to R'$: we have

$$u_1 = (r_1 j)u_1 = r_1 p_1 h = r'_1(eh)$$

analogously $u_2 = r'_2(eh)$. ◀

▶ Remark 29. Given a reflexive relation $r_1$, $r_2 \colon R \to A$ in a category with finite limits and regular factorizations, then the relations $R$ and $R \circ R^0$ have the same coequalizers. In fact, a morphism $f$ merges the projections of $R$ iff if merges those of $R \circ R^0$. Indeed, let $p_1$, $p_2 \colon P \to A$ be the kernel pair of $f$.

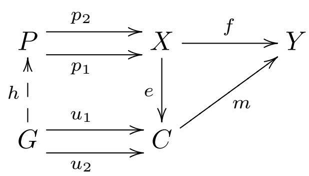
**(1)** If $f$ merges $R$, then $R \subseteq P$ and since $P$ is an equivalence relation, we conclude $R \circ R^0 \subseteq P \circ P^0 = P$.

**(2)** If $f$ merges $R \circ R^0$, then since $R$ is reflexive we have $\Delta_A \subseteq R^0$, hence, $R = R \circ \Delta_A \subseteq R \circ R^0$. Thus $f$ merges $r_1$, $r_2$.

▶ **Proposition 30.** *Let $\mathcal{K}$ be a category with kernel pairs and their coequalizers. Then every strong generator consisting of effective objects is a regular generator.*

**Proof.**

**(1)** $\mathcal{K}$ has regular factorizations. Indeed, given a morphism $f \colon X \to Y$ form its kernel pair $p_1$, $p_2 \colon P \to X$ and a coequalizer $e \colon X \to C$ of $p_1$, $p_2$. We have a unique morphism $m \colon C \to Y$ with $f = m \cdot e$, and our task is to prove that $m$ is monic. Since $\mathcal{G}$ is a strong generator, this is equivalent to proving for every pair $u_1$, $u_2 \colon G \to C$ with $G \in \mathcal{G}$ that $m \cdot u_1 = m \cdot u_2$ implies $u_1 = u_2$.




Since $G$ is regularly projective, $\mathcal{K}(G, -)$ preserves regular epimorphisms. Thus there exist morphisms $v_i \colon G \to X$ with $u_i = e \cdot v_i$ $(i = 1, 2)$. From $m \cdot u_1 = m \cdot v_1$, we derive $f \cdot v_1 = f \cdot v_2$. Therefore, the pair $v_1$, $v_2$ factorizes through the kernel pair via a morphism $h \colon G \to P$ with $v_i = p_i \cdot h$. Thus $u_i = e \cdot p_i \cdot h$ and $e \cdot p_1 = e \cdot p_2$ implies $u_1 = u_2$.

**(2)** Consequently, every extremal epimorphism $e$ is regular: given its regular factorization $e = m \cdot k$, we conclude that $m$ is an isomorphism. Given a strong generator $\mathcal{G}$ consisting of effective objects, we prove that the canonical morphism $c_X$ (Remark 9) is a regular epimorphism for every object $X$. We have an extremal epimorphism $e = [e_i] \colon \coprod_{i \in I} G_i \to X$ for some collection of objects $G_i \in \mathcal{G}$. Define $h \colon \coprod_{i \in I} G_i \to \coprod_{G \in \mathcal{G}} \coprod_{f \colon G \to X}$ to have the $i$-th component equal to the coproduct injection of $e_i \colon G_i \to X$. Then clearly $e = c_X \cdot h$. Since $e$ is an extremal epimorphism, so is $c_X$. Thus $c_X$ is a regular epimorphism. ◀

▶ **Corollary 31.** *Let $\mathcal{K}$ have an abstractly finite strong generator consisting of regular projectives. If $\mathcal{K}$ has reflexive coequalizers and kernel pairs, then it is complete and cocomplete.*

This follows from the above Proposition and Corollary 15.

## 4 A characterization of varieties

▶ **Definition 32** ([3]). *An endofunctor $F$ of $\mathbf{Set}^S$ is called finitely bounded if for every object $X$ and every element $x \in FX$ there exists a finite subobject $m\colon M \hookrightarrow X$ (that is, $\coprod_{s\in S} M_s$ is finite) with $x$ in the image of $Fm$.*

▶ **Proposition 33** ([3]). *If $S$ is a finite set, then an endofunctor of $\mathbf{Set}^S$ is finitely bounded iff it is finitary (i.e., preserves directed colimits).*

▶ **Example 34.** A finitely bounded endofunctor of $\mathbf{Set}^{\mathbb{N}}$ need not be finitary. Consider $F$ assigning to $X$ itself if all but finitely many sorts of $X$ are empty, else $FX$ is the terminal object. $F$ is finitely bounded: given $x \in X$ of sort $n$, let $M \subseteq X$ have all sorts but $n$ empty and $M_n = \{x\}$. Then $x \in FM_n$. But $F$ does not preserves the colimit of the $\omega$-chain of $\mathbb{N}$-sorted sets $X^{(k)} = (X^{(k)})_{n\in\mathbb{N}}$ for $k < \omega$ where $X^{(k)}_n = \emptyset$ if $k > n$, else $X^{(k)}_n = \{0, 1\}$.

The following proof is based on the idea of the proof of Theorem 4.4.5 in [7].

▶ **Theorem 35.** *A category is equivalent to a variety of finitary many-sorted algebras of finitely many sorts iff it has*
**(a)** *kernel pairs and reflexive coequalizers, and*
**(b)** *an abstractly finite, strong generator consisting of finitely many effective objects.*

▶ **Remark.** *If in (b) we require the generator to be regular (rather than just strong), then the assumption that kernel pairs exist can be dropped. See Corollary 15.*

**Proof.** Necessity follows from Remark 4(5) and Examples 21 and 10(2). To prove sufficiency, let $\{G_s\}_{s\in S}$ be a set as in (b) above in $\mathcal{K}$. From Corollary 15 and Remark 23(1) we know that $\mathcal{K}$ is complete and cocomplete. It has regular factorizations by Remark 24.

**(i)** The functor $U\colon \mathcal{K} \to \mathbf{Set}^S$ with components $\mathcal{K}(G_s, -)$ for $s \in S$ has a left adjoint $F\colon \mathbf{Set}^S \to \mathcal{K}$ with

$$FX = \coprod_{s\in S} X_s \bullet G_s \qquad \text{for} \quad X = (X_s)_{s\in S}\,.$$

Denote by $\mathbb{T}$ the corresponding monad on $\mathbf{Set}^S$. It is finitely bounded. Indeed, to give, for an $S$-sorted set $X$, an element of sort $t$ in $TX = UFX$ means to give a morphism $f\colon G_t \to \coprod_{s\in S} X_s \bullet G_s$. Since $\{G_s\}_{s\in S}$ is abstractly finite, there is a finite sorted subset $m\colon M \hookrightarrow X$ such that $f$ factorizes through the subcoproduct $FM \hookrightarrow FX$. That is, $f \in Tm[TM]$.
Since by the above proposition $T$ is finitary, $\mathbf{Set}^{\mathbb{T}}$ is equivalent to a variety, see Theorem A40 in [5]. Thus it is sufficient to prove that $U$ is monadic, then $\mathcal{K}$ is also equivalent to that variety.

**(ii)** To prove that $U$ is monadic we use Beck's Theorem as formulated in [6] Theorem 3.3.10: $U$ is monadic iff (1) $U$ has a left adjoint, (2) $U$ reflects isomorphisms, and (3) for every reflexive pair $r_1, r_2\colon R \to A$ in $\mathcal{K}$ if $Ur_1$, $Ur_2$ have a split coequalizer, then $r_1$, $r_2$ have a coequalizer and $U$ preserves it.

**(1)** The left adjoint is given by $(X_s)_{s\in S} \mapsto \coprod_{s\in S} X_s \bullet G_s$.

**(2)** See Remark 24.

**(3)** Assuming first that $r_1$, $r_2$ are collectively monic, we work with the relation $R$ on $A$. Since $U$ preserves finite limits and reflects isomorphisms, it reflects finite limits. By Remark 24 $\mathcal{K}$ has regular factorizations and $U$ preserves and reflects regular epimorphisms, relation composition and equivalence relations. Indeed, each $G_s$ is regularly projective (Remark 23).

By assumption the relation $\bar{R}$ on $UA$ represented by $Ur_1$, $Ur_2$ has a split coequalizer. Thus $\bar{R} \circ \bar{R}^0$ is an equivalence relation: see Lemma 28. Consequently, $R \circ R^0$ is an equivalence relation on $A$. Let $k\colon A \to K$ be its coequalizer. Then $k$ is, since $R$ is a reflexive relation, the coequalizer of $R$ (Remark 29).

Since each $G_s$ is effective, $U$ preserves coequalizers of equivalence relations. Thus $Uk$ is a coequalizer of $\bar{R} \circ \bar{R}^0$. Arguing as above, we conclude that $Uk$ is also coequalizer of $U\bar{R}$. This proves that $U$ preserves the coequalizer of $r_1$, $r_2$.

Let us next consider $r_1$, $r_2$ arbitrary. For $Ur_1$, $Ur_2$ we have a split coequalizer $c\colon UA \to C$ (Remark 27): there are morphisms $i\colon C \to UA$ and $j\colon UA \to UR$ satisfying $c \cdot i = \mathrm{id}_C$, $(Ur_1) \cdot j = \mathrm{id}_{UA}$ and $(Ur_2) \cdot j = i \cdot c$. Since $\mathcal{K}$ has finite products and regular factorizations, we can factorize $r_1$, $r_2$ as a regular epimorphism $e\colon R \to R'$ followed by a collectively monic pair $r'_1, r'_2\colon R' \to A$. The relation $R'$ is reflexive: given a morphism $d$ with $r_i d = \mathrm{id}_A$, the morphism $ed$ fulfils $r'_i ed = \mathrm{id}_A$. Moreover, the morphisms $Ur'_1$, $Ur'_2$ also have the split coequalizer $c$ : the splitting is given by $i$ and $Ue \cdot j$. Thus $r'_1$, $r'_2$ have a coequalizer preserved by $U$. Since $e$ is epic, the coequalizers of $r_1$, $r_2$ and $r'_1$, $r'_2$ coincide; analogously for $Ue$: this is an epimorphism since each $G_s$ is regularly projective. This concludes the proof. ◀

▶ **Corollary 36.** *A category is equivalent to a single-sorted variety iff it has*
*(a) kernel pairs and reflexive coequalizers, and*
*(b) an effective, abstractly finite singleton strong generator.*

▶ **Example 37.** None of the assumptions on the generator $G$ in the above corollary can be omitted:

**(1)** Abstract finiteness. The one-element space $G$ in the category of compact Hausdorff spaces forms a regular generator and is effective. But not abstractly finite: the copower $\coprod_{\mathbb{N}} G$ is the space $\beta\mathbb{N}$ and almost no morphism from $G$ to $\beta\mathbb{N}$ factorizes through a finite subcopower. Analogously, no nonempty space is abstractly finite. Thus, the above category is not equivalent to a (finitary) variety.

**(2)** Effectivity. In the category DCPO (Example 3) consider the three-element chain 3. This is an abstractly finite strong generator (see [8]). But it is not effective: DCPO does not have effective equivalence relations.

**(3)** Strength. The one-element poset forms an abstractly finite, effective generator of **Pos**. But not a strong one. **Pos** also fails to have effective equivalence relations.

**(4)** Existence of copowers. Let $\mathbf{Set}_0$ be the full subcategory of **Set** on all nonempty sets. Here the terminal object 1 has almost all the required properties: every object is a copower of 1, every morphism from 1 to a coproduct factorizes through a coproduct injection, and 1 is effective. And 1 has “almost all” copowers – but not the empty one! Moreover, $\mathbf{Set}_0$ is clearly not equivalent to a variety: it is not complete.

▶ **Theorem 38.** *A category is equivalent to a variety of finitary, many-sorted algebras iff it has*

**(a)** *kernel pairs and reflexive coequalizers, and*

**(b)** *a strong generator consisting of finitely presentable effective objects.*

The proof is the same as that of the preceding theorem, except that the verification that $T$ is finitary can be left out: since each $G_s$ is finitely presentable, $U$ preserves directed colimits, hence, so does $T = UF$.

▶ **Example 39.** Theorem 35 does not generalize to varieties with infinitely many sorts. We present a category $\mathcal{K}$ whose only finitely presentable object is the initial one (thus, $\mathcal{K}$ is not equivalent to a variety). Yet, $\mathcal{K}$ has coequalizers and an abstractly finite regular generator formed by effective objects.

$\mathcal{K}$ is the full subcategory of $\mathbf{Set}^{\mathbb{N}}$ consisting of the terminal object $\mathbb{1} = (1, 1, 1 \dots)$ and all objects $(X_n)_{n\in\mathbb{N}}$ such that for some $k \in \mathbb{N}$ we have

$$X_n \neq \emptyset \quad \text{iff} \quad n < k\,.$$

$\mathcal{K}$ is closed under coequalizers in $\mathbf{Set}^{\mathbb{N}}$. But not under colimits of $\omega$-chains: consider the chain of inclusions of $A^{(k)} = (A_n^{(k)})\ \ (k < \omega)$ where $A_n^{(k)} = \{0, 1\}$ for $n \leq k$, else $\emptyset$. Then $colim_{k<\omega} A^{(k)} = \mathbb{1}$ in $\mathcal{K}$. And the only object of $\mathcal{K}$ whose hom-functor preserves this colimit is $(\emptyset, \emptyset, \emptyset \dots)$.

However, $\mathcal{K}$ has the abstractly finite regular generator $\{B^{(k)}\}_{k\in\mathbb{N}}$ where $B_n^{(k)} = 1$ for $n \leq k$, else $\emptyset$. Every morphism $f\colon B^{(k)} \to \coprod_{i\in I} B^{(k_i)}$ has the property that $k \leq k_i$ for some $i$, thus $f$ factorizes through the coproduct injection of $B^{(k_i)}$. The verification that each $B^{(k)}$ is effective and that they form a regular generator is easy.

▶ Remark 40. Many-sorted varieties have also been characterized in [5] as precisely the strongly locally finitely presentable categories. We recall this shortly and show how this follows from the above reults.

**(1)** Using [4] Theorem 1.11, a *locally finitely presentable* category can be defined as a cocomplete category with a strong generator formed by finitely presentable objects.

**(2)** Let us recall that a small category is filtered iff colimits in **Set** with that domain commute with finite limits. Analogously one defines a small category to be *sifted* iff colimits in **Set** with that domain commute with finite products. An object $A$ of a category $\mathcal{K}$ is called *perfectly presentable* if its hom-functor preserves sifted colimits (which are colimits of diagrams with sifted domains). If $\mathcal{K}$ has finite coproducts, these are precisely the finitely presentable objects $A$ with $\mathcal{K}(A, -)$ preserving reflexive coequalizers ([5], Thm. 7.7). For example, in every variety all free algebras on finitely many generators are perfectly presentable ([5], Corollary 5.14).

**(3)** Using [5] Theorem 7.7, a *strongly locally finitely presentable* category can be defined as a cocomplete category with a strong generator $\mathcal{G}$ formed by perfectly presentable objects.

▶ **Corollary 41** ([5] Thm. 6.9)**.** *A category is strongly locally finitely presentable iff it is equivalent to a finitary many-sorted variety.*

**Proof.**

**(1)** Let $\mathcal{K}$ be strongly locally finitely presentable. For the strong generator $\mathcal{G}$ in the above remark every member $G \in \mathcal{G}$ is finitely presentable and effective, thus $\mathcal{G}$ is abstractly finite. By Theorem 38 $\mathcal{K}$ is equivalent to a variety.

**(2)** Conversely, every variety is strongly finitely presentable since the regular generator of Example 10(2) consists of perfectly presentable objects. ◀

## Conclusions and Open Problems

Lawvere's characterization of (single-sorted) finitary varieties can be sharpened as follows: these are precisely the categories with reflexive coequalizers, kernel pairs and a strong generator formed by an abstractly finite effective object. We have presented a proof based on the theory of monads. And we have proved that for varieties of many-sorted algebras a completely analogous result holds in case of finitely many sorts: many-sorted varieties are, up to equivalence, precisely the categories with reflexive coequalizers, kernel pairs and an abstractly finite strong generator formed by finitely many effective objects. For infinitely many sorts a somewhat weaker characterization holds: instead of abstract finiteness one requires that the given generator consists of finitely presentable objects.

It is interesting to consider other base categories than **Set** or $\mathbf{Set}^S$. For example **Pos**: can one characterize (finitary) varieties of ordered algebras in a similar way?

Another direction of research are non-finitary varieties: what is an abstract characterization of categories monadic over $\mathbf{Set}^S$, or over **Pos**?

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