Lower Bounds on Stabilizer Rank

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Abstract

The stabilizer rank of a quantum state ψ is the minimal r such that $|\psi\rangle = \sum_{j=1}^r c_j |\varphi_j\rangle$ for $c_j \in \mathbb{C}$ and stabilizer states φ_j . The running time of several classical simulation methods for quantum circuits is determined by the stabilizer rank of the n-th tensor power of single-qubit magic states.

We prove a lower bound of $\Omega(n)$ on the stabilizer rank of such states, improving a previous lower bound of $\Omega(\sqrt{n})$ of Bravyi, Smith and Smolin [5]. Further, we prove that for a sufficiently small constant δ , the stabilizer rank of any state which is δ -close to those states is $\Omega(\sqrt{n}/\log n)$. This is the first non-trivial lower bound for approximate stabilizer rank.

Our techniques rely on the representation of stabilizer states as quadratic functions over affine subspaces of \mathbb{F}_2^n , and we use tools from analysis of boolean functions and complexity theory. The proof of the first result involves a careful analysis of directional derivatives of quadratic polynomials, whereas the proof of the second result uses Razborov-Smolensky low degree polynomial approximations and correlation bounds against the majority function.

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1 Extended Abstract

Algorithms for simulating quantum circuits based on *stabilizer rank decompositions* is a recent powerful paradigm for classical simulation of quantum computation [5, 3, 2]. The computational cost of these algorithms is dominated by a certain natural algebraic and complexity-theoretic rank measure for quantum states, called the *stabilizer rank*, which we now describe.

Clifford circuits are quantum circuits which only apply Clifford gates. This is an important class of quantum circuits which, by the Gottesman-Knill theorem [7, 1], can be efficiently simulated by a classical algorithm. This highly non-obvious theorem follows from the fact that such circuits can only maintain certain states known as *stabilizer states*.

Adding T gates on top of the Clifford gates results in a universal quantum gate set, that is, a set which can approximate every unitary operation. It is then possible, using a simple gadget-based transformation, to "push the T gates to the inputs" and obtain an equivalent circuit, of roughly the same size, which only uses Clifford operations, and is given, as additional auxiliary inputs, sufficiently many copies of qubits in a so-called magic state $|H\rangle$. [4, 5]. This transformation only increases the circuit size by a polynomial factor. This suggests the possibility of simulating a general quantum circuit by decomposing $|H^{\otimes n}\rangle$ as a linear combination of stabilizer states.

More formally, $|\varphi\rangle$ is a stabilizer state if $|\varphi\rangle = U|0^n\rangle$ where U is an n-bit Clifford unitary. The stabilizer rank of a state $|\psi\rangle$, denoted $\chi(\psi)$, is the minimal integer r such that

$$|\psi\rangle = \sum_{j=1}^{r} c_j |\varphi_j\rangle,$$

where for every $1 \leq j \leq r$, $|\varphi_j\rangle$ is a stabilizer state and $c_j \in \mathbb{C}$.

For any n-qubit state, the stabilizer rank is at most 2^n , but much smaller upper bounds were shown for the stabilizer rank of $|H^{\otimes n}\rangle$: it is known that $\chi(H^{\otimes n}) \leq 2^{\alpha n}$ for $\alpha \le \log 3/4$. [5, 10, 12].

Similarly, the δ -approximate stabilizer rank of $|\psi\rangle$, denoted $\chi_{\delta}(\psi)$, is defined as the minimum of $\chi(\varphi)$ over all states $|\varphi\rangle$ such that $||\psi - \varphi||_2 \le \delta$ [2].

A natural question is then what is the limit of such simulation methods. As the running time of the simulation scales with the stabilizer rank, an upper bound which is polynomial (in n) on $\chi(H^{\otimes n})$ will imply that BPP = BQP and even (by simulating quantum circuits with postselection) P = NP [2], and thus seems highly improbable. Much stronger hardness assumptions than $P \neq NP$, such as the exponential time hypothesis, imply that $\chi(H^{\otimes n}) =$ $2^{\Omega(n)}$ [11, 8].

However, in order to understand the power of quantum computation it is important to obtain unconditional impossibility results, and thus we are interested in lower bounds on $\chi(H^{\otimes n})$ and $\chi_{\delta}(H^{\otimes n})$. This seems to be a challenging open problem. As we remark in the full version of this paper, proving super-linear lower bound on $\chi(H^{\otimes n})$ will solve a notable open problem in complexity theory.

Bravyi, Smith and Smolin proved that $\chi(H^{\otimes n}) = \Omega(\sqrt{n})$. In this paper, we improve this lower bound, and also prove the first non-trivial lower bounds for approximate stabilizer rank.

Our results: Improved Lower Bounds on Stabilizer Rank and Approximate Stabilizer Rank

Our first result is an improved lower bound on $\chi(H^{\otimes n})$.

▶ Theorem 1. $\chi(H^{\otimes n}) = \Omega(n)$.

Our second result proves the first non-trivial lower bound for approximate stabilizer rank.

▶ Theorem 2. There exists an absolute constant $\delta > 0$ such that $\chi_{\delta}(H^{\otimes n}) = \Omega(\sqrt{n}/\log n)$.

These theorem together imply an inherent slowdown for any classical algorithm simulating quantum circuits using the stabilizer rank method. They also lay the ground for future work on improving those lower bounds, towards the goal of proving super-polynomial or even exponential lower bounds, as is widely conjectured.

1.2 Technique: Stabilizer States as Quadratic Polynomials

Our main proof idea is using the representation of stabilizer states as quadratic polynomials over affine subspaces of \mathbb{F}_2^n [6, 16]. Using the explicit representation of stabilizer states and the magic states in the standard basis, we translate the question of proving lower bounds into natural questions on boolean functions.

Our proof of Theorem 1 applies a carefully chosen directional (discrete) derivate to the quadratic polynomials obtained from the stabilizer state decomposition, in order to "reduce" a quadratic system of equations to a linear system. It also uses a classical theorem of Kleitman theorem [9] which gives an upper bound on the size of sets of the boolean cube with small diameter.

The proof of Theorem 2 follows a different strategy, and uses ideas from circuit complexity theory such as low-degree polynomial approximates for low-depth circuits, and correlation bounds for low-degree polynomials against the majority function [13, 14, 15].

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