Fuzzy Algebraic Theories

Davide Castelnovo
Department of Mathematics, Computer Science and Physics, University of Udine, Italy

Marino Miculan
Department of Mathematics, Computer Science and Physics, University of Udine, Italy

Abstract

In this work we propose a formal system for *fuzzy algebraic reasoning*. The sequent calculus we define is based on two kinds of propositions, capturing equality and existence of terms as members of a fuzzy set. We provide a sound semantics for this calculus and show that there is a notion of free model for any theory in this system, allowing us (with some restrictions) to recover models as Eilenberg-Moore algebras for some monad. We will also prove a completeness result: a formula is derivable from a given theory if and only if it is satisfied by all models of the theory. Finally, leveraging results by Milius and Urbat, we give HSP-like characterizations of subcategories of algebras which are categories of models of particular kinds of theories.

2012 ACM Subject Classification Theory of computation → Logic

Keywords and phrases categorical logic, fuzzy sets, algebraic reasoning, equational axiomatisations, monads, Eilenberg-Moore algebras

Digital Object Identifier 10.4230/LIPIcs.CSL.2022.13

Related Version *Full Version*: https://arxiv.org/abs/2110.10970

Funding *Marino Miculan*: Supported by the Italian MIUR PRIN 2017FTXR7S *IT MATTERS*.

1 Introduction

One of the most fruitful and influential lines of research of Logic in Computer Science is the algebraic study of computation. After Moggi's seminal work [18] showed that notions of computation can be represented as monads, Plotkin and Power [21] approached the problem using operations and equations, i.e., Lawvere theories. Since then, various extensions of the notion of Lawvere theory have been introduced in order to accommodate an ever increasing number of computational notions within this framework; see, e.g., [22, 11, 20], and more recently [3, 4] for *quantitative* algebraic reasoning for probabilistic computations.

Along this line of research, in this work we study algebraic reasoning on *fuzzy sets*. Algebraic structures on fuzzy sets are well known since the seventies (see e.g., [24, 16, 1, 19]). Fuzzy sets are very important in computer science, with applications ranging from pattern recognition to decision making, from system modeling to artificial intelligence. So, one may wonder if it is possible to use an approach similar to above for *fuzzy algebraic reasoning*.

In this paper we answer positively to this question. We propose a sequent calculus based on two kind of propositions, one expressing equality of terms and the other the existence of a term as a member of a fuzzy set. These sequents have a natural interpretation in categories of fuzzy sets endowed with operations. This calculus is sound and complete for such a semantics: a formula is satisfied by all the models of a given theory if and only if it is derivable from it.

It is possible to go further. Both in the classical and in the quantitative settings there is a notion of free model for a theory; we show that is also true for theories in our formal system for fuzzy sets. In general the category of models of a given theory will not be equivalent to the category of Eilenberg-Moore algebras for the induced monad, but we will show that this equivalence holds for theories with sufficiently simple axioms. Finally we will use the techniques developed in [17] to prove two results analogous to Birkhoff's theorem.

30th EACSL Annual Conference on Computer Science Logic (CSL 2022).
Editors: Florin Manea and Alex Simpson; Article No. 13; pp. 13:1–13:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Synopsis. In Section 2 we recall the category $\mathbf{Fuz}(H)$ of fuzzy sets over a frame H. Section 3 introduces the syntax and the rules of fuzzy theories. Then, in Section 4 we introduce the notions of algebras for a signature and of models for a theory; in this section we will also show that the calculus proposed is sound and complete. Section 5 is devoted to free models and it is shown that if a theory is *basic* then its category of models arose as the category of Eilenberg-Moore algebras for a monad on $\mathbf{Fuz}(H)$. In Section 6 we use the results of [17] to prove two HSP-like theorems for our calculus. Finally, Section 7 draws some conclusions and directions for future work. Complete proofs are in the extended version [8].

2 Fuzzy sets

In this section we will recall the definition and some well-known properties of the category of fuzzy sets over a frame H (i.e. a complete Heyting algebra [12]).

▶ **Definition 2.1** ([26, 27]). *Let H be a frame. A* H-fuzzy set *is a pair (A, μ_A) consisting in a set A and a* membership function $\mu_A : A \to H$*. The* support *of μ_A is the set $\mathsf{supp}(A, \mu_A)$ of elements $x \in A$ such that $\mu_A(x) \neq \bot$. An arrow $f : (A, \mu_A) \to (B, \mu_B)$ is a function $f : A \to B$ such that $\mu_A(x) \leq \mu_B(f(x))$ for all $x \in A$.*

We denote by $\mathbf{Fuz}(H)$ the category of H-fuzzy sets and their arrows. We will often drop the explicit reference to the frame H when there is no danger of confusion.

▶ **Proposition 2.2.** *For any frame H, the forgetful functor $\mathcal{V} : \mathbf{Fuz}(H) \to \mathbf{Set}$ has both a left and a right adjoint ∇ and Δ endowing a set X with the function constantly equal to the bottom and the top element of H, respectively.*

Proof. If $\nabla(X)$ and $\Delta(X)$ are, respectively $(X, c_\bot)$ and $(X, c_\top)$, where $c_\bot$ and $c_\top$ are the functions $X \to H$ constant in $\bot$ and $\top$, then for any $X \in \mathbf{Set}$, $\mathrm{id}_X : \mathcal{V}(\Delta(X)) = X \to X = \mathcal{V}(\nabla(X))$ is the counit of $\mathcal{V} \dashv \Delta$ and the unit of $\nabla \dashv \mathcal{V}$. ◀

▶ **Definition 2.3.** *Let $e : A \to B$ and $m : C \to D$ be two arrows in a category $\mathbf{C}$, we say that m has* the left lifting property with respect to e *if for any two arrows $f : A \to C$ and $g : B \to D$ such that $m \circ f = g \circ e$ there exists a unique $k : B \to C$ with $m \circ k = g$.*

A strong monomorphism *is an arrow m which has the left lifting property with respect to all epimorphisms.*

▶ **Proposition 2.4.** *Let $f : (A, \mu_A) \to (B, \mu_B)$ be an arrow of $\mathbf{Fuz}(H)$, then:*
1. *f is a monomorphism iff it is injective; f is an epimorphism iff it is surjective;*
2. *f is a strong monomorphism iff it is injective and $\mu_B(f(x)) = \mu_A(x)$ for all $x \in A$;*
3. *f is a split epimorphism iff for any $b \in B$ there exists $a_b \in f^{-1}(b)$ such that $\mu_B(b) = \mu_A(a_b)$.*

▶ **Definition 2.5** ([13]). *A* proper factorization system *on a category $\mathbf{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ given by two classes of arrows such that:*
- *$\mathcal{E}$ and $\mathcal{M}$ are closed under composition;*
- *every isomorphism belongs to both $\mathcal{E}$ and $\mathcal{M}$;*
- *every $e \in \mathcal{E}$ is an epimorphism and every $m \in \mathcal{M}$ is a monomorphism;*
- *every $m \in \mathcal{M}$ has the left lifting property with respect to every $e \in \mathcal{E}$;*
- *every arrow of $\mathbf{C}$ is equal to $m \circ e$ for some $m \in \mathcal{M}$ and $e \in \mathcal{E}$.*

▶ **Lemma 2.6.** *For any frame H, $\mathbf{Fuz}(H)$ has all products. Moreover the classes of epimorphisms and strong monomorphisms form a proper factorization system on it.*

▶ Remark 2.7. Completeness and the existence of both adjoints to $\mathcal{V}$ can be deduced directly from the fact that $\mathbf{Fuz}(H)$ is topological over $\mathbf{Set}$ [26, Prop. 71.3].

3 Fuzzy Theories

In this section we introduce the syntax and logical rules of fuzzy theories. The first step is to introduce an appropriate notion of signature. Differently from usual signatures, in fuzzy theories constants cannot be seen simply as 0-arity operations, because , as we will see in Section 4, these are interpreted as fuzzy morphisms from the terminal object, and these correspond only to elements whose membership degree is $\top$.

▶ **Definition 3.1.** *A* signature $\Sigma = ()O, \mathsf{ar}, C)$ *is a set* O *of* operations *with an* arity function $\mathsf{ar} : O \to \mathbb{N}_+$ *and a set* C *of* constants*. Signatures form a category* **Sign** *in which an arrow* $\Sigma_1 = (O_1, \mathsf{ar}_1, C_1) \to \Sigma_2 = (O_2, \mathsf{ar}_2, C_2)$ *is a pair* $\mathbf{F} = (F_1, F_2)$ *of functions:* $F_1 : O_1 \to O_2$ *and* $F_2 : C_1 \to C_2$ *with the property that* $\mathsf{ar}_2 \circ F_1 = \mathsf{ar}_1$*.*

An algebraic language $\mathcal{L}$ *is a pair* (Σ, X) *where* Σ *is a signature and* X *a set. The category* **Lng** *of algebraic languages is just* **Sign** $\times$ **Set***.*

▶ **Example 3.2.** The *signature of semigroups* Σ_S in which $O = \{\cdot\}$, $\mathsf{ar}(\cdot) = 2$ and $C = \emptyset$.

▶ **Example 3.3.** The *signature of groups* Σ_G is equal to Σ_S plus an operation $(-)^{-1}$ of arity 1 and a constant e.

Given a language $\mathcal{L}$ we can inductively apply the operations to the set of variables to construct terms, and once terms are built we can introduce equations.

▶ **Definition 3.4.** *Given a language* $\mathcal{L} = (\Sigma, X)$*, the set* $\mathcal{L}$-Terms *is the smallest set s.t.*
- $X \sqcup C \subset \mathcal{L}$-$\mathsf{Terms}$*;*
- *if* $f \in O$ *and* $t_1, \dots, t_{\mathsf{ar}(f)} \in \mathcal{L}$-$\mathsf{Terms}$ *then* $f\big(t_1, \dots, t_{\mathsf{ar}(f)}\big) \in \mathcal{L}$-$\mathsf{Terms}$*.*

▶ **Proposition 3.5.** *There exists a functor* $\mathsf{Terms} : \mathbf{Lng} \to \mathbf{Set}$ *sending* $\mathcal{L}$ *to* $\mathcal{L}$-Terms*.*

▶ **Definition 3.6** (Formulae)**.** *For any language* $\mathcal{L}$ *we define the sets* $\mathsf{Eq}(\mathcal{L})$ *of* equations *as the product* $\mathsf{Eq}(\mathcal{L}) \coloneqq \mathcal{L}\text{-}\mathsf{Terms} \times \mathcal{L}\text{-}\mathsf{Terms}$ *and the set* $\mathsf{M}(\mathcal{L})$ *of* membership propositions *as* $\mathsf{M}(\mathcal{L}) \coloneqq H \times \mathcal{L}\text{-}\mathsf{Terms}$*. We will write* $s \equiv t$ *for* $(s, t) \in \mathsf{Eq}(\mathcal{L})$ *and* $\mathsf{E}(l, t)$ *for* $(l, t) \in \mathsf{M}(\mathcal{L})$*. The set* $\mathsf{Form}(\mathcal{L})$ *of formulae is then defined as* $\mathsf{Eq}(\mathcal{L}) \sqcup \mathsf{M}(\mathcal{L})$*.*

Clearly, a proposition $s \equiv t$ means "s and t are equivalent and hence interchangeable"; on the other hand, $\mathsf{E}(l, t)$ intuitively means "the degree of existence of t is at least l".

▶ **Definition 3.7** (Sequent ant fuzzy theory)**.** *A* sequent $\Gamma \vdash \psi$ *is an element* (Γ, ψ) *of* $\mathsf{Seq}(\mathcal{L}) \coloneqq \mathcal{P}(\mathsf{Form}(\mathcal{L})) \times \mathsf{Form}(\mathcal{L})$*, where* $\mathcal{P}$ *is the (covariant) power set functor. A* fuzzy theory in the language $\mathcal{L}$ *is a subset* $\Lambda \subset \mathsf{Seq}(\mathcal{L})$ *and we will use* $\mathsf{Th}(\mathcal{L})$ *for the set* $\mathcal{P}(\mathsf{Seq}(\mathcal{L}))$*.*

▶ **Notation.** *We will write* $\vdash \phi$ *for* $\emptyset \vdash \phi$*.*

For any function $\sigma : X \to \mathcal{L}$-$\mathsf{Terms}$ *and* $t \in \mathcal{L}$-Terms *we denote* $t[\sigma]$ *the term obtained substituting* $\sigma(x)$ *to any occurence of* x *in* t*. Moreover, for any formula* $\phi \in \mathsf{Form}(\mathcal{L})$ *we define* $\phi[\sigma]$ *as* $t[\sigma] \equiv s[\sigma]$ *if* ϕ *is* $t \equiv s$ *or as* $\mathsf{E}(l, t[\sigma])$ *if* ϕ *is* $\mathsf{E}(l, t)$*. Finally, given* $\Gamma \subset \mathcal{P}(\mathsf{Form}(\mathcal{L}))$ *we put* $\Gamma[\sigma] \coloneqq \{\phi[\sigma] \mid \phi \in \Gamma\}$*.*

▶ **Definition 3.8.** *For any* $\mathcal{L}$*, the* fuzzy sequent calculus *is given by the rules in Figure 1.*

Given a fuzzy theory Λ*, its* deductive closure $\Lambda^{\vdash}$ *is the smallest subset of* $\mathsf{Seq}(\mathcal{L})$ *which contains* Λ *and it is closed under the rules of fuzzy sequent calculus. A sequent is* derivable from Λ *(or simply derivable if* $\Lambda = \emptyset$*) if it belongs to* $\Lambda^{\vdash}$*. We will write* $\vdash_\Lambda \phi$ *if* $\vdash \phi \in \Lambda^{\vdash}$*.*

Finally we say that two theories Λ *and* Θ *are* deductively equivalent *if* $\Lambda^{\vdash} = \Theta^{\vdash}$*.*

$$\frac{\phi \in \Gamma}{\Gamma \vdash \phi}\ \textsc{A} \qquad \frac{\Gamma \vdash \phi}{\Gamma \cup \Delta \vdash \phi}\ \textsc{Weak} \qquad \frac{\{\Gamma \vdash \phi \mid \phi \in \Phi\} \qquad \Phi \vdash \psi}{\Gamma \vdash \psi}\ \textsc{Cut}$$

$$\frac{}{\Gamma \vdash s \equiv s}\ \textsc{Refl} \qquad \frac{\Gamma \vdash s \equiv t}{\Gamma \vdash t \equiv s}\ \textsc{Sym} \qquad \frac{\Gamma \vdash s \equiv t \qquad \Gamma \vdash t \equiv u}{\Gamma \vdash t \equiv u}\ \textsc{Trans}$$

$$\frac{\sigma : X \to \mathcal{L}\text{-Terms} \qquad \Gamma \vdash \psi}{\Gamma[\sigma] \vdash \psi[\sigma]}\ \textsc{Sub} \qquad \frac{f \in O \qquad \{\Gamma \vdash t_i \equiv s_i\}_{i=1}^{\mathsf{ar}(f)}}{\Gamma \vdash f\big(t_1, \dots, t_{\mathsf{ar}(f)}\big) \equiv f\big(s_1, \dots, s_{\mathsf{ar}(f)}\big)}\ \textsc{Cong}$$

$$\frac{}{\Gamma \vdash \mathsf{E}(\bot, t)}\ \textsc{Inf} \qquad \frac{\Gamma \vdash \mathsf{E}(l, t)}{\Gamma \vdash \mathsf{E}(l \wedge l', t)}\ \textsc{Mon} \qquad \frac{\{\Gamma \vdash \mathsf{E}(l_i, t_i)\}_{i=1}^{\mathsf{ar}(f)}}{\Gamma \vdash \mathsf{E}\big(\inf(\{l_i\}_{i=1}^{n}), f\big(t_1, \dots, t_{\mathsf{ar}(f)}\big)\big)}\ \textsc{Exp}$$

$$\frac{S \subset H \qquad \{\Gamma \vdash \mathsf{E}(l, t)\}_{l \in S}}{\Gamma \vdash \mathsf{E}(\sup(S), t)}\ \textsc{Sup} \qquad \frac{\Gamma \vdash t \equiv s \qquad \Gamma \vdash \mathsf{E}(l, t)}{\Gamma \vdash \mathsf{E}(l, s)}\ \textsc{Fun}$$

Figure 1 Derivation rules for the fuzzy sequent calculus.

The next result shows how maps between languages are lifted to theories.

▶ **Proposition 3.9.** *For any $\mathbf{F} : \mathcal{L}_1 \to \mathcal{L}_2$ arrow of* **Lng***:*

1. *there exists a Galois connection $\mathbf{F}_* \dashv \mathbf{F}^*$ between $(\mathsf{Th}(\mathcal{L}_1), \subset)$ and $(\mathsf{Th}(\mathcal{L}_2), \subset)$;*
2. *$\mathbf{F}_*\big(\Lambda_1^{\vdash}\big) \subset (\mathbf{F}_*(\Lambda_1))^{\vdash}$ and $(\mathbf{F}^*(\Lambda_2))^{\vdash} \subset \mathbf{F}^*\big(\Lambda_2^{\vdash}\big)$ for any $\Lambda_1 \in \mathbf{Th}(\mathcal{L}_1)$ and $\Lambda_2 \in \mathbf{Th}(\mathcal{L}_2)$.*

Usually, logics enjoy the so-called “deduction lemma”, which allows us to treat elements of a theory on a par with assumptions in sequents. In fuzzy theories, this holds in a slightly restricted form, as proved next.

▶ **Lemma 3.10** (Partial deduction lemma)**.** *Let Λ be in $\mathsf{Th}(\mathcal{L})$ and $\Gamma \in \mathcal{P}(\mathsf{Form}(\mathcal{L}))$, let also $\Lambda[\Gamma]$ be the theory $\Lambda \cup \{\emptyset \vdash \phi \mid \phi \in \Gamma\}$. Then the following are true:*

1. *$\Gamma \cup \Delta \vdash \psi$ in $\Lambda^{\vdash}$ implies $\Delta \vdash \psi$ in $(\Lambda[\Gamma])^{\vdash}$;*
2. *if $\Delta \vdash \psi$ is derivable from $\Lambda[\Gamma]$ without using rule* Sub *then $\Gamma \cup \Delta \vdash \psi$ is in $\Lambda^{\vdash}$.*

Proof.

1. By hypothesis $\Gamma \cup \Delta \vdash \psi$ is in $\Lambda^{\vdash}$ then, since $\Lambda \subset \Lambda[\Gamma]$, it is also in $(\Lambda[\Gamma])^{\vdash}$. Now, for any $\phi \in \Gamma$ and $\theta \in \Delta$ rules Weak and A give

$$\frac{\vdash \phi}{\Delta \vdash \phi}\ \textsc{Weak} \qquad \frac{}{\Delta \vdash \theta}\ \textsc{A}$$

so $\{\Delta \vdash \phi \mid \phi \in \Gamma \cup \Delta\}$ is contained in $(\Lambda[\Gamma])^{\vdash}$ and we can apply Cut to get the thesis:

$$\frac{\{\Delta \vdash \phi \mid \phi \in \Gamma \cup \Delta\} \qquad \Gamma \cup \Delta \vdash \psi}{\Delta \vdash \psi}\ \textsc{Cut}$$

2. Let us proceed by induction on a derivation of $\Delta \vdash \psi$ from $\Lambda[\Gamma]$.
   - If $\Delta \vdash \psi \in \Lambda[\Gamma]$ then or $\Delta \vdash \psi \in \Lambda$ and we are done, or $\psi \in \Gamma$ and we can conclude since $\Gamma \cup \Delta \vdash \phi_i$ is in $\Lambda^{\vdash}$ by rules A and Weak
   - If $\Delta \vdash \psi$ follows from the application of one of the rules A, Inf or Refl then it belongs to the closure of any theory, by Weak this is true even for $\Gamma \cup \Delta \vdash \psi$ which, in particular, it belongs to $\Lambda^{\vdash}$.

- Suppose that $\Delta \vdash \psi$ comes from an application of WEAK, then there exists Ψ and Φ such that $\Psi \cup \Phi = \Delta$ and $\Psi \vdash \phi$ is in $(\Lambda[\Gamma])^{\vdash}$. By inductive hypothesis we have the following derivation of $\Gamma \cup \Delta \vdash \psi$ from Λ:

$$\frac{\Gamma \cup \Psi \vdash \psi}{\Gamma \cup \Psi \cup \Phi \vdash \psi}\ \textsc{Weak}$$

- If $\Delta \vdash \psi$ is derived with an application of CUT as last rule then there exists a set Θ such that $\{\Delta \vdash \theta \mid \theta \in \Theta\} \cup \{\Theta \vdash \psi\} \subseteq (\Lambda[\Gamma])^{\vdash}$, therefore, by the inductive hypothesis, we have that $\{\Gamma \cup \Delta \vdash \theta \mid \theta \in \Theta\} \cup \{\Gamma \cup \Theta \vdash \psi\}$ is conteined in $\Lambda^{\vdash}$. Now, $\{\Gamma \cup \Delta \vdash \phi \mid \phi \in \Gamma \cup \Theta\} \subset \Lambda^{\vdash}$ by rule A so an application of CUT gives the thesis:

$$\frac{\{\Gamma \cup \Delta \vdash \phi \mid \phi \in \Gamma \cup \Theta\} \qquad \Gamma \cup \Theta \vdash \psi}{\Gamma \cup \Delta \vdash \psi}\ \textsc{Cut}$$

- Any other rule is of the form

$$\frac{\{\Psi \vdash \xi_j\}_{j \in J}}{\Psi \vdash \xi}\ \mathrm{R}$$

therefore, if $\Delta \vdash \psi$ is derived with an application of one of this rules then the set of its premises must be an element of $(\Lambda[\Gamma])^{\vdash}$ of type $\{\Delta \vdash \xi_j\}_{j \in J}$, so by inductive hypothesis $\{\Gamma \cup \Delta \vdash \theta_j\}_{j \in J} \subseteq \Lambda^{\vdash}$ and then the thesis is witnessed by

$$\frac{\{\Gamma \cup \Delta \vdash \xi_j\}_{j \in J}}{\Gamma \cup \Delta \vdash \psi}\ \mathrm{R}$$ ◀

▶ **Example 3.11.** Our first set of running examples is inspired by [19]. Let Σ_S be the signature of semigroups and X a countable set. The theory of *fuzzy semigroups* Λ_S is simply the usual theory of semigroups, i.e given by the sequent (using infix notation)

$$\vdash (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

We get the theory of *left ideal* Λ_{LI} if we add the axioms (one for any $l \in L$):

$$\mathsf{E}(l, y) \vdash \mathsf{E}(l, x \cdot y)$$

Similarly the theory Λ_{RI} of *right ideal* is obtained from the axioms:

$$\mathsf{E}(l, x) \vdash \mathsf{E}(l, x \cdot y)$$

Finally we get the theory of *(bilateral) ideal* Λ_I taking the union of the above theories.

▶ **Example 3.12** ([24, 1, 2])**.** Let Σ_G be the signature of groups and X a countable set. The theory Λ_G of *fuzzy groups* is simply the usual theory of groups, i.e that given by the sequents

$$\vdash x \cdot x^{-1} \equiv e \quad \vdash x^{-1} \cdot x \equiv e \quad \vdash e \cdot x \equiv x \quad \vdash x \cdot x \equiv x \quad \vdash (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

We get the theory Λ_N of *normal fuzzy groups* ([16]) if we add the axioms:

$$\mathsf{E}(l, x) \vdash \mathsf{E}\big(l, y \cdot (x \cdot y^{-1})\big)$$

It can be shown that the sequents $\mathsf{E}(l, x) \vdash \mathsf{E}(l, e)$ and $\mathsf{E}\big(l, y \cdot (x \cdot y^{-1})\big) \vdash \mathsf{E}(l, x)$ are derivable, respectively, from Λ_G and from Λ_N.

4 Fuzzy algebras and semantics

In this section we provide a sound and complete semantics to the syntax and sequents introduced in Section 3. The first step is to define the semantic counterpart of a signature.

▶ **Definition 4.1.** *Given a signature* Σ*, a* Σ-fuzzy algebra $\mathcal{A} := ((A, \mu_A), \Sigma^{\mathcal{A}})$ *is a fuzzy set* (A, μ_A) *and a collection* $\Sigma^{\mathcal{A}} = \{f^{\mathcal{A}} \mid f \in O\} \sqcup \{c^{\mathcal{A}} \mid c \in C\}$ *of arrows:*

$$f^{\mathcal{A}} : (A, \mu_A)^{\mathsf{ar}(f)} \to (A, \mu_A) \qquad c^{\mathcal{A}} : (1, c_\perp) \to (A, \mu_A)$$

where $c_\perp$ *is the constant function in* $\perp$*. A* morphism of Σ-fuzzy algebras $\mathcal{A} \to \mathcal{B}$ *is an arrow* $g : (A, \mu_A) \to (B, \mu_B)$ *such that* $g \circ c^{\mathcal{A}} = c^{\mathcal{B}}$ *and* $f^{\mathcal{B}} \circ g^{\mathsf{ar}(f)} = g \circ f^{\mathcal{A}}$ *for every* $c \in C$ *and* $f \in O$*. We will write* Σ-Alg *for the resulting category of* Σ*-fuzzy algebras.*

▶ Remark 4.2. We will not distinguish between a function from the singleton and its value.

▶ **Definition 4.3.** *Let* $\mathcal{L} = (\Sigma, X)$ *be a language and* $\mathcal{A} = \big((A, \mu_A), \Sigma^{\mathcal{A}}\big)$ *be a* Σ*-algebra.*

An assignment *is simply a function* $\iota : X \to A$*. We define the* evaluation in $\mathcal{A}$ with respect to ι *as the function* $(\text{-})^{\mathcal{A},\iota} : \mathcal{L}\text{-}\mathsf{Terms} \to A$ *by induction:*

- $x^{\mathcal{A},\iota} := \iota(x)$ *if* $x \in X$*;*
- $c^{\mathcal{A},\iota} := c^{\mathcal{A}}$ *if* $c \in C$*;*
- $(f(t_1, \dots, t_{\mathsf{ar}(f)}))^{\mathcal{A},\iota} := f^{\mathcal{A}}(t_1^{\mathcal{A},\iota}, \dots, t_{\mathsf{ar}(f)}^{\mathcal{A},\iota})$ *if* $f \in O$*.*

▶ **Proposition 4.4.** *Let* $\mathcal{A}$ *be a* Σ*-algebra. Given a function* $\sigma : X \to \mathcal{L}\text{-}\mathsf{Terms}$ *and an assignment* $\iota : X \to A$ *define* $\iota_\sigma : X \to A$ *as the assignment sending* x *to* $(\sigma(x))^{\mathcal{A},\iota}$*. Then* $\mathcal{A} \vDash_\iota \phi[\sigma]$ *if and only if* $\mathcal{A} \vDash_{\iota_\sigma} \phi$*.*

Proof. This follows at once noticing that $\big(t[\sigma]\big)^{\mathcal{A},\iota} = t^{\mathcal{A},\iota_\sigma}$ holds for every term t. ◀

▶ **Definition 4.5.** $\mathcal{A}$ satisfies $\phi \in \mathsf{Form}(\mathcal{L})$ with respect to ι*, and we write* $\mathcal{A} \vDash_\iota \phi$*, if* ϕ *is* $\mathsf{E}(l, t)$ *and* $l \leq \mu_A(t^{\mathcal{A},\iota})$ *or if* ϕ *is* $t \equiv s$ *and* $t^{\mathcal{A},\iota} = s^{\mathcal{A},\iota}$*.*

$\mathcal{A}$ satisfies ϕ *if* $\mathcal{A} \vDash_\iota \phi$ *for all* $\iota : X \to A$*, and we write* $\mathcal{A} \vDash \phi$*, similarly, given* $\Gamma \subset \mathsf{Form}(\mathcal{L})$*,* $\mathcal{A} \vDash \Gamma$ *(*$\mathcal{A} \vDash_\iota \Gamma$*) means* $\mathcal{A} \vDash \phi$ *(*$\mathcal{A} \vDash_\iota \phi$*) for any* $\phi \in \Gamma$*.*

Finally, given a sequent $\Gamma \vdash \phi$ *we say that* $\mathcal{A}$ satisfies it with respect to ι *and we will write* $\Gamma \vDash_{\mathcal{A},\iota} \phi$ *if* $\mathcal{A} \vDash_\iota \phi$ *whenever* $\mathcal{A} \vDash_\iota \Gamma$*; if this happens for all assignments* ι *we say that* $\mathcal{A}$ satisfies *the sequent and we will write* $\Gamma \vDash_{\mathcal{A}} \phi$*.*

We say that a Σ*-fuzzy algebra* $\mathcal{A}$ *is a* model *of a fuzzy theory* $\Lambda \in \mathsf{Th}(\mathcal{L})$ *if it satisfies all the sequents in it.* $\mathbf{Mod}(\Lambda)$ *denotes the full subcategory of* Σ-Alg *given by the models of* Λ*.*

Clearly Σ-Alg $= \mathbf{Mod}(\emptyset)$. For any $\Lambda \in \mathsf{Th}(\mathcal{L})$ there exist two forgetful functors $\mathcal{U}_\Lambda : \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(L)$ and $\mathcal{V}_\Lambda : \mathbf{Mod}(\Lambda) \to \mathbf{Set}$. We will write $\mathcal{U}_\Sigma$ and $\mathcal{V}_\Sigma$ for $\mathcal{U}_\emptyset$ and $\mathcal{V}_\emptyset$.

▶ **Proposition 4.6.** *For any signature* Σ*,* $\mathcal{V}_\Sigma$ *has a left adjoint* $\mathcal{F}_\Sigma^{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Mod}(\Lambda)$*.*

Proof. For any set X take the language $\mathcal{L}_X$ and define $\mathcal{F}_\Sigma^{\mathbf{Set}}(X)$ has $(\nabla(\mathcal{L}_X\text{-}\mathbf{Terms}), \Sigma^{\mathcal{F}_\Sigma^{\mathbf{Set}}(X)})$ where $c^{\mathcal{F}_\Sigma^{\mathbf{Set}}(X)} := c$ and for any $f \in O$,

$$f^{\mathcal{F}_\Sigma^{\mathbf{Set}}(X)} : \nabla(\mathcal{L}_X\text{-}\mathbf{Terms})^{\mathsf{ar}(f)} \to \nabla(\mathcal{L}_X\text{-}\mathbf{Terms}) \qquad \big(t_1, \dots, t_{\mathsf{ar}(f)}\big) \mapsto f(t_1, \dots, t_{\mathsf{ar}(f)})$$

It is easy to see that for any $\iota : X \to \mathcal{V}_\Sigma(\mathcal{A})$ the evaluation $(\text{-})^{\mathcal{A},\iota}$ is the unique morphism of Σ-Alg that composed with the inclusion $X \to \mathcal{L}_X\text{-}\mathbf{Terms}$ gives back ι. ◀

We now provide two technical results about interpretations. The first describes how interpretations are moved along morphisms of algebras.

▶ **Proposition 4.7.** *Let $\mathcal{L} = (\Sigma, X)$ be a language, $\Lambda \in \mathsf{Th}(\mathcal{L})$ and $\mathcal{A} = \big((A, \mu_A), \Sigma^{\mathcal{A}}\big)$, $\mathcal{B} = \big((B, \mu_B), \Sigma^{\mathcal{B}}\big)$ be two Σ-algebras. Let also $f : \mathcal{A} \to \mathcal{B}$ be a morphism between them, then:*
1. *$\mathcal{A}$ is a model of Λ if and only if it is a model of $\Lambda^{\vdash}$;*
2. *$f \circ (-)^{\mathcal{A},\iota} = (-)^{\mathcal{B}, f\circ\iota}$ for every assignment $\iota : X \to A$;*
3. *for any assignment $\iota : X \to A$, $\mathcal{A} \vDash_\iota \phi$ entails $\mathcal{B} \vDash_{f\circ\iota} \phi$;*
4. *if $\mathcal{U}_\Sigma(f)$ is a strong monomorphism in $\mathbf{Fuz}(H)$ and $\iota : X \to A$ is an assignment then, for any formula ϕ, $\mathcal{A} \vDash_\iota \phi$ if and only if $\mathcal{B} \vDash_{f\circ\iota} \phi$;*
5. *if $\mathcal{U}_\Sigma(f)$ is a strong monomorphism in $\mathbf{Fuz}(H)$ and $\mathcal{B} \in \mathbf{Mod}(\Lambda)$ then $\mathcal{A} \in \mathbf{Mod}(\Lambda)$.*

We can also move interpretations and theories along morphisms of signatures.

▶ **Definition 4.8.** *For any $\mathbf{F} : \Sigma_1 \to \Sigma_2$ arrow of* **Sign** *and any $\mathcal{A} = \big((A, \mu_A), \Sigma_2^{\mathcal{A}}\big) \in \Sigma_2\text{-}\mathsf{Alg}$, we define $\mathsf{r}_{\mathbf{F}}(\mathcal{A}) = \Big((A, \mu_A), \Sigma_1^{\mathsf{r}_{\mathbf{F}}(\mathcal{A})}\Big) \in \Sigma_1\text{-}\mathsf{Alg}$ putting, for any $f \in O_1$*

$$f^{\mathsf{r}_{\mathbf{F}}(\mathcal{A})} : (A, \mu_A)^{\mathsf{ar}(f)} \to (A, \mu_A) \qquad \big(a_1, \dots, a_{\mathsf{ar}(f)}\big) \mapsto F_2(f)^{\mathcal{A}}\big(a_1, \dots, a_{\mathsf{ar}(f)}\big)$$

and $c^{\mathsf{r}_{\mathbf{F}}(\mathcal{A})} := F_3(c)^{\mathcal{A}}$ for every $c \in C_1$.

▶ **Lemma 4.9.** *Let $\mathcal{L}_1 = (\Sigma_1, X)$ and $\mathcal{L}_2 = (\Sigma_2, Y)$ and $\mathbf{F} = ((F_1, F_2), g) : \mathcal{L}_1 \to \mathcal{L}_2$, then:*
1. *there exists a functor $\mathsf{r}_{\mathbf{F}} : \Sigma_2\text{-}\mathsf{Alg} \to \Sigma_1\text{-}\mathsf{Alg}$ sending $\mathcal{A}$ to $\mathsf{r}_{\mathbf{F}}(\mathcal{A})$;*
2. *$t^{\mathsf{r}_{\mathbf{F}}(\mathcal{A}), \iota\circ g} = (\mathsf{Terms}(\mathbf{F})(t))^{\mathcal{A},\iota}$ for any assignment $\iota : Y \to A$ and $t \in \mathcal{L}_1\text{-}\mathsf{Terms}$;*
3. *for any assignment $\iota : Y \to A$, $\mathsf{r}_{\mathbf{F}}(\mathcal{A}) \vDash_{\iota\circ g} \phi$ if and only if $\mathcal{A} \vDash_\iota \mathsf{Form}(\mathbf{F})(\phi)$;*
4. *If $X = Y$ and $g = \mathrm{id}_X$ then $\mathsf{r}_{\mathbf{F}}$ restricts to a functor $\mathsf{r}_{\mathbf{F},\Lambda} : \mathbf{Mod}(\Lambda) \to \mathbf{Mod}(\mathbf{F}^*(\Lambda))$.*

▶ **Example 4.10.** The models for Λ_S, Λ_{LI}, Λ_{RI} and Λ_I (Example 3.11) are precisely the structures defined in [19], while the models for Λ_G (Example 3.12) are precisely the fuzzy groups as in [24] and those of Λ_N are the structures called *normal fuzzy subgroups* in [2, 1, 16].

Soundness. Now we can proceed proving the soundness of the rules in Figure 1.

▶ **Lemma 4.11.** *Let $\mathcal{L} = (\Sigma, X)$ be a language and $\mathcal{A} = ((A, \mu_A), \Sigma^{\mathcal{A}})$ a Σ-algebra, then:*
1. *for any assignment $\iota : X \to A$ and rule*

$$\frac{\{\Psi_i \vdash \xi_i\}_{i \in I}}{\Psi \vdash \xi}\, R$$

different from Sub*, if $\Psi_i \vDash_{\mathcal{A},\iota} \xi_i$ for all $i \in I$ then $\Psi \vDash_{\mathcal{A},\iota} \xi$ too;*

2. *for any $\sigma : X \to \mathcal{L}\text{-}\mathsf{Terms}$, if $\Gamma \vDash_{\mathcal{A}} \psi$ then $\Gamma[\sigma] \vDash_{\mathcal{A}} \psi[\sigma]$.*

▶ **Corollary 4.12** (Soundness)**.** *If a Σ-algebra satisfies all the premises of a rule of the fuzzy sequent calculus then it satisfies also its conclusion.*

▶ Remark 4.13. Let us see why the deduction lemma (Lemma 3.10) cannot be extended to rule Sub. Take Σ to be the empty set, $X = \{x, y, z\}$ and $H = \{0, 1\}$. Notice that $\Sigma\text{-}\mathsf{Alg} = \mathbf{Fuz}(H)$. We have the derivation

$$\frac{\vdash x \equiv y}{\vdash x \equiv z}\, \textsc{Sub}$$

If the deduction lemma held for Sub, $x \equiv y \vdash x \equiv z$ would be in $\emptyset^{\vdash}$, hence satisfied by any fuzzy set, but (H, id_H) with $\iota : X \to H$ sending x and y to 0 and z to 1 is a counterexample.

▶ Remark 4.14. Let us take $\Sigma = \emptyset$ and $H = \{0, 1\}$ and $X = \{x, y, z\}$ and the derivation as in Remark 4.13. Now, a fuzzy set (A, μ_A) satisfies $\vdash_\iota x \equiv y$ if and only if $\iota(x) = \iota(y)$, thus, if we take (H, id_H) and the assignment ι of the previous example, then $(H, \mathrm{id}_H) \vdash_\iota x \equiv y$ but it does not satisfy $x \equiv z$ with respect to ι.

Completeness. Now we prove that the calculus we have provided in Section 3 is complete. Let us start with the following observation.

▶ Remark 4.15. For any $\Lambda \in \mathsf{Th}(\mathcal{L})$ the relation $\sim_\Lambda$ given by all t and s such that $\vdash_\Lambda t \equiv s$, is an equivalence relation on $\mathcal{L}$-Terms.

Using this equivalence, we can define the model of terms, as done next.

▶ **Definition 4.16.** *Let $\mathcal{L} = (\Sigma, X)$ be a language and $\Lambda \in \mathsf{Th}(\mathcal{L})$, we define $\mathsf{Terms}(\Lambda)$ to be the quotient of $\mathcal{L}$-Terms by $\sim_\Lambda$, moreover, by rule* Fun*, the function*

$$\hat{\mu} : \mathcal{L}\text{-}\mathsf{Terms} \to H \qquad t \mapsto \sup\{l \in H \mid \vdash_\Lambda \mathsf{E}(l, t)\}$$

induces a function $\mu_\Lambda : \mathsf{Terms}(\Lambda) \to H$. For any $f \in O$ and $c \in C$ putting $c^{\mathcal{T}_\Lambda} := [c]$ and

$$f^{\mathcal{T}_\Lambda} : \mathsf{Terms}(\Lambda)^{\mathsf{ar}(f)}, \to \mathsf{Terms}(\Lambda) \qquad \big([t_1], \ldots, [t_{\mathsf{ar}(f)}]\big) \mapsto \big[f\big(t_1, \ldots, t_{\mathsf{ar}(f)}\big)\big]$$

gives us a Σ-algebra $\mathcal{T}_\Lambda = \big((\mathsf{Terms}(\Lambda), \mu_\Lambda), \Sigma^{\mathcal{T}_\Lambda}\big)$, called the Σ-algebra of terms in Λ. *The* canonical assignment *is the function $\iota_{can} : X \to \mathsf{Terms}(\Lambda)$ sending x to its class $[x]$.*

▶ Remark 4.17. Rule Cong assures us that $f^{\mathcal{T}_\Lambda}$ is well defined while Exp implies that it is an arrow of $\mathbf{Fuz}(H)$.

The following Lemma will be needed to prove completeness.

▶ **Lemma 4.18.** *Let $\mathcal{L} = (\Sigma, X)$ be a language and $\Lambda \in \mathsf{Th}(\mathcal{L})$, then:*

1. *for any $\phi \in \mathsf{Form}(\mathcal{L})$ the following are equivalent:*
   - **a.** $\mathcal{T}_\Lambda \vDash \phi$,
   - **b.** $\mathcal{T}_\Lambda \vDash_{\iota_{can}} \phi$,
   - **c.** $\vdash_\Lambda \phi$;
2. *for any assignment $\iota : X \to \mathsf{Terms}(\Lambda)$ and formula ϕ, $\mathcal{T}_\Lambda \vDash_\iota \phi$ if and only if $\vdash_\Lambda \phi[\sigma \circ \iota]$ for one (and thus any) section σ of the quotient $\mathcal{L}$-$\mathsf{Terms} \to \mathsf{Terms}(\Lambda)$;*
3. *$\mathcal{T}_\Lambda = \big((\mathsf{Terms}(\Lambda), \mu_\Lambda), \Sigma^{\mathcal{T}_\Lambda}\big)$ is a model of Λ.*

Let us start with a technical result.

▶ **Proposition 4.19.** *Let $\mathcal{L} = (\Sigma, X)$ be a language, $\Lambda \in \mathsf{Th}(\mathcal{L})$, and $\sigma : \mathsf{Terms}(\Lambda) \to \mathcal{L}$-$\mathsf{Terms}$ a section of the quotient $\mathcal{L}$-$\mathsf{Terms} \to \mathsf{Terms}(\Lambda)$. The equation $t^{\mathcal{T}_\Lambda, \iota} = [t[\sigma \circ \iota]]$ holds for any assignment $\iota : X \to \mathsf{Terms}(\Lambda)$ and $t \in \mathcal{L}$-Terms. In particular $t^{\mathcal{T}_\Lambda, \iota_{can}} = [t]$.*

Now we can proceed with the proof of Lemma 4.18.

Proof of Lemma 4.18.

1. Let us show the three implications. (a)⇒(b) follows from the definition. For the implication (b)⇒(c) we split the cases.
   - ϕ is $t \equiv s$. Then $\mathcal{T}_\Lambda \vDash_{\iota_{can}} \phi$ means

     $$[t] = t^{\mathcal{T}_\Lambda, \iota_{can}} = s^{\mathcal{T}_\Lambda, \iota_{can}} = [s]$$

     thus $t \sim_\Lambda s$ i.e. $\vdash_\Lambda t \equiv s$.
   - ϕ is $\mathsf{E}(l, t)$. Let S be $\{l' \in H \mid \Lambda \vdash \mathsf{E}(l', t)\}$, by hypothesis $\mathcal{T}_\Lambda \vDash_{\iota_{can}} \phi$, so

     $$l \leq \mu_\Lambda\Big(t^{\mathcal{T}_\Lambda, \iota_{can}}\Big) = \mu_\Lambda([t]) = \sup(S)$$

     hence $l = l \wedge \sup(S)$ and, since H is a frame, $l = \sup(\{l \wedge l' \mid l' \in S\})$, by rule Mon we know that that $\vdash_\Lambda \mathsf{E}(l \wedge l', t)$ for all $l' \in S$ and so rule Sup gives us $\vdash_\Lambda \mathsf{E}(l, t)$.

Finally, for (c)⇒(a), let $\iota : X \to \mathsf{Terms}(\Lambda)$ be an assignment and σ a section as in the hypothesis; by rule SUB we get $\vdash_\Lambda \phi[\sigma \circ \iota]$, and by Proposition 4.19 follows the thesis.

2. By Proposition 4.19 we have

$$t^{\mathcal{T}_\Lambda,\iota} = [t[\sigma \circ \iota]] = (t[\sigma \circ \iota])^{\mathcal{T}_\Lambda,\iota_{can}}$$

we can conclude using the previous point.

3. Let $\Gamma \vdash \psi$ be a sequent in Λ with $\Gamma = \{\phi_i\}_{i=1}^n$ and $\iota : X \to \mathsf{Terms}(\Lambda)$ an assignment such that $\mathcal{T}_\Lambda \vDash_\iota \Gamma$. By point 1 above this means that $\vdash_\Lambda \Gamma[\sigma \circ \iota]$ and applying SUB and CUT we can conclude that $\vdash_\Lambda \psi[\sigma \circ \iota]$. By the previous point this is equivalent to $\mathcal{T}_\Lambda \vDash_\iota \psi$. ◀

Since satisfaction of a formula by $\mathcal{T}_\Lambda$ entails its derivability from Λ we can deduce immediately a form of completeness.

▶ **Corollary 4.20** (Completeness for formulae)**.** *For any theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, $\mathcal{A} \vDash \phi$ for all $\mathcal{A} \in \mathbf{Mod}(\Lambda)$ if and only if $\vdash_\Lambda \phi$.*

5 From theories to monads

Given a language $\mathcal{L} = (\Sigma, X)$ and a fuzzy theory $\Lambda \in \mathsf{Th}(\mathcal{L})$ we have a forgetful functor: $\mathcal{U}_\Lambda : \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(L)$. In this section we first show that it has a left adjoint (Section 5.1) and that for a specific class of theories, models correspond to Eilenberg-Moore algebras for the monad induced by this adjunction (Section 5.2).

5.1 The free fuzzy algebra on a fuzzy set

To give the definition of free models (Definition 5.8) we need some preliminary constructions.

▶ **Definition 5.1.** *Let $\mathcal{A}$ be a Σ-algebra and $f : (B, \mu_B) \to \mathcal{U}_\Sigma(\mathcal{A})$ an arrow in $\mathbf{Fuz}(H)$, a Σ-algebra* generated by f in $\mathcal{A}$ *is a morphism $\epsilon : \langle B, \mu_B\rangle_{\mathcal{A},f} \to \mathcal{A}$ such that:*

- *$\mathcal{U}_\Sigma(\epsilon)$ is strong mono;*
- *there exists $\bar{f} : (B, \mu_B) \to \langle B, \mu_B\rangle_{\mathcal{A},f}$ such that $\mathcal{U}_\Sigma(\epsilon) \circ \bar{f} = f$;*
- *if $g : \mathcal{C} \to \mathcal{A}$ is a morphism such that $\mathcal{U}_\Sigma(g)$ is strong monomorphism and $\mathcal{U}_\Sigma(g) \circ h = f$ for some h then there exists a unique $k : \langle B, \mu_B\rangle_{\mathcal{A},f} \to \mathcal{C}$ such that $g \circ k = \epsilon$.*

We can construct $\langle B, \mu_B\rangle_{\mathcal{A},f}$ closing $f(B)$ under the iterated images of the functions $g^{\mathcal{A}}$, when g varies between the operations in O, so we get easily the following.

▶ **Proposition 5.2.** *For any signature Σ, Σ-algebra $\mathcal{A}$ and $f : (B, \mu_B) \to \mathcal{U}_\Sigma(\mathcal{A})$, $\langle B, \mu_B\rangle_{\mathcal{A},f}$ exists and it is unique up to isomorphism.*

▶ Remark 5.3. Proposition 4.7 implies that, given a model $\mathcal{A} = \big((A, \mu_A), \Sigma^{\mathcal{A}}\big)$ of a theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, and a morphism $f : (B, \mu_B) \to (A, \mu_A)$, the Σ-algebra $\langle B, \mu_B\rangle_{\mathcal{A},f}$ is in $\mathbf{Mod}(\Lambda)$.

The next result follows at once noticing that $\langle B, \mu_B\rangle_{\mathcal{A},f}$ is built from $f(B)$ closing it under the interpretation of elements of O.

▶ **Proposition 5.4.** *Let $\mathcal{A}$ be a Σ-algebra and $f : (B, \mu_B) \to \mathcal{U}_\Sigma(\mathcal{A})$, then, for any other Σ-algebra $\mathcal{C}$ and $h : (B, \mu_B) \to \mathcal{U}_\Sigma(\mathcal{C})$ there exists at most one $k : \langle B, \mu_B\rangle_{\mathcal{A},f} \to \mathcal{C}$ such that $k \circ \bar{f} = h$.*

The next definition explains how to extend a theory in a given language with the data of a fuzzy set.

▶ **Definition 5.5.** *Let (M, μ_M) be a fuzzy set, $\mathcal{L} = (\Sigma, X)$ a language with $\Sigma = (O, \mathsf{ar}, C)$. We define $\Sigma[M, \mu_M]$ to be $(O, \mathsf{ar}, C \sqcup M)$ and $\mathcal{L}_{(M,\mu_M)}$ to be $(\Sigma[M, \mu_M], X)$. We have an obvious morphism $\mathbf{I} : \mathcal{L} \to \mathcal{L}_{(M,\mu_M)}$ given by the identities and the inclusion $i_C : C \to C \sqcup M$.*

For any $\Lambda \in \mathsf{Th}(\mathcal{L})$ we define $\Lambda[M, \mu_M] \in \mathcal{L}_{(M,\mu_M)}$ as $\mathbf{I}_*(\Lambda) \cup \overline{(M, \mu_M)}$ where $\overline{(M, \mu_M)} = \{\vdash \mathsf{E}(l, m) \mid m \in M, l \in L$ and $\mu_M(m) \geq l\}$.

▶ Remark 5.6. It is immediate to see that $\mathbf{I}^*(\Lambda[M, \mu_M]) = \Lambda$.

In the next proposition we show that, for any theory Λ, a fuzzy set can be mapped into the term model of the theory Λ extended with it. Hence, the natural candidate to be the free model is the algebra generated by such map.

▶ **Proposition 5.7.** *For any fuzzy set (M, μ_M) and any theory $\Lambda \in \mathsf{Th}(\mathcal{L})$:*

1. *the function $\bar{\eta}_{(M,\mu_M)}$ sending m to the class $[m]$ of the corresponding constant defines an arrow of fuzzy sets $\bar{\eta}_{(M,\mu_M)} : (M, \mu_M) \to \mathsf{Terms}(\Lambda[M, \mu_M])$;*
2. *any element in $\langle M, \mu_M \rangle_{\mathcal{T}_{\Lambda[M,\mu_M]}, \bar{\eta}_{(M,\mu_M)}}$ has a representative without variables;*
3. *$\langle M, \mu_M \rangle_{\mathcal{T}_{\Lambda[M,\mu_M]}, \bar{\eta}_{(M,\mu_M)}}$ is the initial object of $\mathbf{Mod}(\Lambda[M, \mu_M])$.*

▶ **Definition 5.8.** *For any language $\mathcal{L}$, $\Lambda \in \mathsf{Th}(\mathcal{L})$ and $(M, \mu_M) \in \mathbf{Fuz}(H)$ we define the* free model $\mathcal{F}_\Lambda(M, \mu_M)$ of Λ generated by (M, μ_M) *to be* $\mathsf{r}_{\mathbf{I}, \Lambda[M,\mu_M]}\left(\langle M, \mu_M \rangle_{\mathcal{T}_{\Lambda[M,\mu_M]}, \bar{\eta}_{(M,\mu_M)}}\right)$. *We set $\mathsf{T}_\Lambda(M, \mu_M)$ to be $\mathcal{U}_\Lambda(\mathcal{F}_\Lambda(M, \mu_M))$.*

Now it is enough to check that the free model just defined actually provides the left adjoint.

▶ **Theorem 5.9.** *For any language $\mathcal{L}$ and $\Lambda \in \mathsf{Th}(\mathcal{L})$ the functor $\mathcal{U}_\Lambda : \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(L)$ has a left adjoint $\mathcal{F}_\Lambda$.*

Proof. By construction $\bar{\eta}_{(M,\mu_M)}$ factors through $\eta_{(M,\mu_M)} : (M, \mu_M) \to \mathsf{T}_\Lambda(M, \mu_M)$ which sends m to $[m]$. Let now $g : (M, \mu_M) \to \mathcal{U}_\Lambda(\mathcal{B})$ be another arrow in $\mathbf{Fuz}(H)$, we can turn $\mathcal{B}$ into a $\Sigma[M, \mu_M]$-algebra $\mathcal{B}^g$ setting $m^{\mathcal{B}^g}$ to be $g(m)$ for any $m \in M$.

Let us show that $\mathcal{B}^g$ is a model of $\Lambda[M, \mu_M]$. Surely it is a model of Λ since $\mathcal{B}$ is, let $\vdash \mathsf{E}(l, m)$ be a sequent in $\overline{(M, \mu_M)}$, then for any assignment $\iota : V \to B$:

$$\mathcal{B}^g \vDash_\iota \mathsf{E}(l, m) \iff l \leq \mu_B(m^{\mathcal{B}^g, \iota}) l \leq \mu_\Lambda\left(t^{\mathcal{F}_\Lambda(M,\mu_M), \eta_{(M,\mu_M)} \circ \iota}\right) \iff l \leq \mu_B(g(m))$$

but g is an arrow of $\mathbf{Fuz}(H)$ so $\mu_B(g(m)) \geq \mu_M(m)$ and we are done.

Now, since $\mathcal{B}^g$ is a model of $\Lambda[M, \mu_M]$, we can take $\bar{g}$ to be the image through $\mathsf{r}_{\mathbf{I}, \Lambda[M,\mu_M]}$ of the unique arrow $\langle M, \mu_M \rangle_{\mathcal{T}_{\Lambda[(M,\mu_M)]}, \bar{\eta}_{(M,\mu_M)}} \to \mathcal{B}^g$, by construction

$$\bar{g}(\eta_{(M,\mu_M)}(m)) = \bar{g}([m]) = m^{\mathcal{B}^g} = g(m)$$

so $\mathcal{U}_\Lambda(\bar{g}) \circ \eta_{(M,\mu_M)} = g$. Uniqueness follows from Proposition 5.7. ◀

▶ **Definition 5.10.** *Given a theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, its* associated monad $\mathsf{T}_\Lambda : \mathbf{Fuz}(H) \to \mathbf{Fuz}(H)$ *is the composite $\mathcal{U}_\Lambda \circ \mathcal{F}_\Lambda$.*

▶ Remark 5.11. If Λ is the empty theory (in any language), then, by Proposition 4.6, the composittion $\mathcal{F}_\emptyset \circ \nabla$ gives us a functor isomorphic to $\mathcal{F}_\Sigma^{\mathbf{Set}}$.

▶ **Notation.** *We will denote by $\mathcal{F}_\emptyset$ with $\mathcal{F}_\Sigma$ and with T_Σ the monad $\mathsf{T}_\emptyset = \mathcal{U}_\Sigma \circ \mathcal{F}_\Sigma$.*

In this setting we can provide a result similar to Lemma 4.18.

▶ **Lemma 5.12.** *For any language $\mathcal{L} = (\Sigma, X)$ we define the* natural assignment ι_{nat} *as the adjoint to the unit $\nabla(X) \to \mathsf{T}_\Lambda(\nabla(X))$. Then $\mathcal{F}_\Lambda(\nabla(X)) \vDash_{\iota_{nat}} \phi$ if and only if $\vdash_\Lambda \phi$.*

Proof. The implication from the right to the left follows immediately since $\mathcal{F}_\Lambda(\nabla(X))$ is a model for Λ. By adjointness he canonical assignment ι_{can} induces an arrow $\nabla(X) \to \mathcal{U}_{\Lambda[\nabla(X)]}\big(\mathcal{T}_{\Lambda[\nabla(X)]}\big)$, which, in turn, induces a morphism $e : \mathcal{F}_\Lambda(\nabla(X)) \to \mathcal{T}_{\Lambda[\nabla(X)]}$ of Σ-algebras such that, as function between sets, $e \circ \iota_{nat} = \iota_{can}$. Recalling that $\mathbf{I}$ is the arrow $(\Sigma, X) \to (\Sigma[\nabla(X)], X)$ and using Proposition 4.7, Lemma 4.9 and Lemma 4.18:

$$\mathcal{F}_\Lambda(\nabla X) \vDash_{\iota_{nat}} \phi \iff \mathsf{r}_{\mathbf{I},\Lambda[\nabla(X)]}\big(\langle \nabla(X)\rangle_{\mathcal{T}_{\Lambda[\nabla(X)]},\bar{\eta}_{\nabla(X)}}\big) \vDash_{\iota_{nat}} \phi$$
$$\iff \mathsf{r}_{\mathbf{I}}\big(\langle \nabla(X)\rangle_{\mathcal{T}_{\Lambda[\nabla(X)]},\bar{\eta}_{\nabla(X)}}\big) \vDash_{\iota_{nat}} \phi \iff \langle \nabla(X)\rangle_{\mathcal{T}_{\Lambda[\nabla(X)]},\bar{\eta}_{\nabla(X)}} \vDash_{\iota_{nat}} \phi$$
$$\implies \mathcal{T}_{\Lambda[\nabla(X)]} \vDash_{e\circ\iota_{nat}} \phi \iff \mathcal{T}_{\Lambda[\nabla(X)]} \vDash_{\iota_{can}} \phi \iff \vdash_{\Lambda[\nabla(X)]} \phi$$

Now, by definition $\overline{\nabla(X)}$ is equal to $\{\vdash \mathsf{E}(\bot, x) \mid x \in X\}$, therefore $(\Lambda[\nabla(X)])^{\vdash} = \Lambda^{\vdash}$ and we get the thesis. ◀

5.2 Eilenberg-Moore algebras and models

In this section we will compare the category $\mathbf{Mod}(\Lambda)$ of models of some $\Lambda \in \mathsf{Th}(\mathcal{L})$ and $\mathbf{Alg}(\mathsf{T}_\Lambda)$ of Eilenberg-Moore algebras for the corresponding monad T_Λ. First of all we recall the following classic lemma ([7, Prop. 4.2.1] and [14, Theorem VI.3.1]).

▶ **Lemma 5.13.** *Let $\mathcal{L} : \mathbf{C} \to \mathbf{D}$ be a functor with right adjoint $\mathcal{R}$ and let $\mathsf{T} = \mathcal{R} \circ \mathcal{L}$ be its associated monad, then there exists a* comparison functor *$\mathcal{K} : \mathbf{D} \to \mathbf{Alg}(\mathsf{T})$ such that $\mathcal{U}_\mathsf{T} \circ \mathcal{K} = \mathcal{R}$, where $\mathcal{U}_\mathsf{T} : \mathbf{Alg}(\mathsf{T}) \to \mathbf{C}$ is the forgetful functor. $\mathcal{K}$ sends D in $(\mathcal{R}(D), \mathcal{R}(\epsilon_D))$, where ϵ is the counit of the adjunction.*

As a consequence, for any theory Λ we have a functor from $\mathbf{Mod}(\Lambda)$ to $\mathbf{Alg}(\mathsf{T}_\Lambda)$. We want to construct an inverse of such functor.

▶ **Definition 5.14.** *Let Λ be in $\mathsf{Th}(\mathcal{L})$ and $\xi : \mathsf{T}_\Lambda(M, \mu_M) \to (M, \mu_M)$ an object of $\mathbf{Alg}(\mathsf{T}_\Lambda)$, we define its* associated algebra *$\mathcal{H}(\xi) = \Big((M, \mu_M), \Sigma^{\mathcal{H}(\xi)}\Big)$ putting, for every $c \in C$ and $f \in O$:*

$$c^{\mathcal{H}(\xi)} := \xi\Big(c^{\mathcal{F}_\Lambda(X,\mu_X)}\Big) \qquad f^{\mathcal{H}(\xi)} := \xi \circ f^{\mathcal{F}_\Lambda(X,\mu_X)} \circ \eta^{\mathsf{ar}(f)}_{(M,\mu_M)}$$

▶ **Lemma 5.15.** *For any Eilenberg-Moore algebra $\xi : \mathsf{T}_\Lambda(M, \mu_M) \to (M, \mu_M)$, ξ itself is an arrow $\mathcal{F}_\Lambda(X, \mu_X) \to \mathcal{H}(\xi)$ of Σ-Alg. In particular, for every term t and assignment ι:*

$$t^{\mathcal{H}(\xi),\iota} = \xi\Big(t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\Big)$$

Proof. The proof of the first half is a straightforward calculation. The second half follows from point 2 of Proposition 4.7 applied to ξ noticing that $\iota = \xi \circ \eta_{(M,\mu_M)} \circ \iota$. ◀

In general $\mathcal{H}(\xi)$ is not a model of Λ, but we can restrict to a class of theories such this holds. As in [4, 15], we consider theories whose sequents' premises contain only variables.

▶ **Definition 5.16.** *A theory $\Lambda \in \mathsf{Th}(\mathcal{L})$ is* basic[1] *if, for any sequent $\Gamma \vdash \phi$ in it, all the formulae in Γ contain only variables.*

▶ **Example 5.17.** Fuzzy groups, fuzzy normal groups, fuzzy semigroups and left, right, bilateral ideals (Examples 3.11 and 3.12) are all examples of basic theories.

[1] In [3] such theories are called *simple*.

▶ **Lemma 5.18.** *$\mathcal{H}(\xi)$ is a model of Λ for any basic theory $\Lambda \in \mathsf{Th}(\mathcal{L})$ and Eilenberg-Moore algebra $\xi : \mathsf{T}_\Lambda(M, \mu_M) \to (M, \mu_M)$.*

Proof. We start observing that if $\Gamma \vdash \phi$ is in Λ and $\iota : X \to M$ is an assignment such that $\mathcal{H}(\xi) \vDash_\iota \Gamma$ then $\mathcal{F}_\Lambda(M, \mu_M) \vDash_{\eta_{(M,\mu_M)} \circ \iota} \Gamma$. Indeed, ψ in Γ can be or $x \equiv y$, and in such case $\iota(x) = \iota(y)$ implies the thesis, or ψ is $\mathsf{E}(l, x)$, but then we can conclude since the membership degree of $\eta_{(M,\mu_M)}(\iota(x))$ in $\mathsf{T}_\Lambda(M, \mu_M)$ is greater than $\mu_M(\iota(x))$. Therefore, we know that $\mathcal{F}_\Lambda(M, \mu_M) \vDash_{\eta_{(M,\mu_M)} \circ \iota} \phi$. Let us split again the two cases.

- ϕ is $t \equiv s$. In this case, $t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota} = s^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}$, therefore

$$t^{\mathcal{H}(\xi),\iota} = \xi\left(t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\right) = \xi\left(s^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\right) = s^{\mathcal{H}(\xi),\iota}$$

- ϕ is $\mathsf{E}(l, t)$. This means that $l \leq \mu_\Lambda\left(t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\right)$, hence, thus:

$$l \leq \mu_\Lambda\left(t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\right) \leq \mu_M\left(\xi\left(t^{\mathcal{F}_\Lambda(M,\mu_M),\eta_{(M,\mu_M)}\circ\iota}\right)\right) = \mu_M\left(t^{\mathcal{H}(\xi),\iota}\right)$$

  and we can conclude. ◀

▶ **Theorem 5.19.** *For any basic theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, the functor $\mathcal{K} : \mathbf{Mod}(\Lambda) \to \mathbf{Alg}(\mathsf{T}_\Lambda)$ has an inverse $\mathcal{H} : \mathbf{Alg}(\mathsf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$ sending $\xi : \mathsf{T}_\Lambda(M, \mu_M) \to (M, \mu_M)$ to $\mathcal{H}(\xi)$.*

Proof (sketches). We have already constructed the inverse $\mathcal{H}$ on objects. If Λ is basic it can be extended to a functor $\mathbf{Alg}(\mathsf{T}_\Lambda) \to \mathbf{Mod}(\Lambda)$ defining its action on arrows as the identity. A straightforward calculation now shows that $\mathcal{K} \circ \mathcal{H} = \mathrm{id}_{\mathbf{Alg}(\mathsf{T}_\Lambda)}$ and $\mathcal{H} \circ \mathcal{K} = \mathrm{id}_{\mathbf{Mod}(\Lambda)}$. ◀

▶ **Corollary 5.20.** *For any basic theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, $\mathbf{Alg}(\mathsf{T}_\Lambda)$ and $\mathbf{Mod}(\Lambda)$ are isomorphic, and thus equivalent, categories.*

6 Equational axiomatizations

In this section we prove two results for our calculus analogous to the classic HSP theorem [5], using the results by Milius and Urbat [17].

The abstract framework. Let us start recalling the tools introduced in [17], adapted to our situation. In the following we will fix a tuple[2] $\left(\mathbf{C}, \left(\mathcal{E}, \mathcal{M}\right), \mathcal{X}\right)$ where $\mathbf{C}$ is a category, $\left(\mathcal{E}, \mathcal{M}\right)$ is a proper factorization system on $\mathbf{C}$ and $\mathcal{X}$ is a class of objects of $\mathbf{C}$.

▶ **Definition 6.1.** *An object X of $\mathbf{C}$ is* projective with respect to an arrow $f : A \to B$ *if for any $h : X \to B$ there exists a $k : X \to A$ such that $f \circ k = h$.*

*We define $\mathcal{E}_\mathcal{X}$ as the class of $e \in \mathcal{E}$ such that for every $X \in \mathcal{X}$, X is projective with respect to e. A $\mathcal{E}_\mathcal{X}$-*quotient *is just an arrow in $\mathcal{E}_\mathcal{X}$.*

In the rest of the section, we assume that $\left(\mathbf{C}, \left(\mathcal{E}, \mathcal{M}\right), \mathcal{X}\right)$ satisfies the following requirements:

- $\mathbf{C}$ has all (small) products;
- for any $X \in \mathcal{X}$, the class $X{\downarrow}\mathbf{C}$ of all $e \in \mathcal{E}$ with domain X is essentially small, i.e. there is a set $\mathcal{J} \subset X{\downarrow}\mathbf{C}$ such that for any $e : X \to C \in X{\downarrow}\mathbf{C}$ there exists $e' : X \to D \in \mathcal{J}$ and an isomorphism ϕ such that $\phi \circ e = e'$;
- for every object C of $\mathbf{C}$ there exists $e : X \to C$ in $\mathcal{E}_\mathcal{X}$ with $X \in \mathcal{X}$.

[2] In their work Milius and Urbat additionaly require a full subcategory of $\mathbf{C}$ and a fixed class of cardinals, but we will not need this level of generality.

▶ **Definition 6.2.** *An $\mathcal{X}$-*equation *is an arrow $e \in X{\downarrow}\mathbf{C}$ with $X \in \mathcal{X}$. We say that an object A of $\mathbf{C}$* satisfies *$e : X \to C$, and we write $A \vDash_X e$, if for every $h : X \to A$ there exists $q : C \to A$ such that $q \circ e = h$. Given a class $\mathbb{E}$ of $\mathcal{X}$-equations, we define $\mathcal{V}(\mathbb{E})$ as the full subcategory of $\mathbf{C}$ given by objects that satisfy e for every $e \in \mathbb{E}$. A full subcategory $\mathbf{V}$ is* $\mathcal{X}$-equationally presentable *if there exists $\mathbb{E}$ such that $\mathbf{V} = \mathcal{V}(\mathbb{E})$.*

▶ Remark 6.3. The definition of equation in [17, Def. 3.3] is given in terms of suitable subclasses of $X{\downarrow}\mathbf{C}$. However in our setting Milius and Urbat's definition reduces to ours (cfr. [17, Remark 3.4]).

▶ **Theorem 6.4** ([17, Th. 3.15, 3.16])**.** *A full subcategory $\mathbf{V}$ of $\mathbf{C}$ is $\mathcal{X}$-equationally presentable if and only if it is closed under $\mathcal{E}_{\mathcal{X}}$-quotients, $\mathcal{M}$-subobjects and (small) products.*

Application to fuzzy algebras. In order to apply the results above to Σ-Alg, we need to define the required inputs, i.e., to specify a factorization system and a class of Σ-algebras.

▶ **Lemma 6.5.** *For any Σ, the classes $\mathcal{E}_\Sigma := \{e \text{ map of } \Sigma\text{-Alg} \mid \mathcal{U}_\Sigma(e) \text{ is epi}\}$ and $\mathcal{M}_\Sigma := \{m \text{ map of } \Sigma\text{-Alg} \mid \mathcal{U}_\Sigma(m) \text{ is strong mono}\}$ form a proper factorization system on Σ-Alg.*

▶ **Definition 6.6.** *We define the following two classes of Σ-algebras:*

$$\mathcal{X}_0 := \{\mathcal{F}_\Sigma^{\mathbf{Set}}(X) \mid X \in \mathbf{Set}\}$$
$$\mathcal{X}_{\mathsf{E}} := \{\mathcal{F}_\Sigma(X, \mu_X) \mid (X, \mu_X) \in \mathbf{Fuz}(H)\}$$

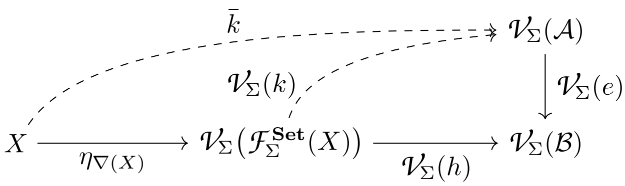
We will use $\mathcal{E}_{\Sigma,\mathcal{X}_0}$ (resp., $\mathcal{E}_{\Sigma,\mathcal{X}_\mathsf{E}}$) for the class of $e \in \mathcal{E}$ such that every $X \in \mathcal{X}_0$ (resp. $X \in \mathcal{X}_\mathsf{E}$) is projective with respect to e.

▶ Remark 6.7. $\mathcal{X}_0 = \{\mathcal{F}_\Sigma(X, \mu_X) \mid \mathsf{supp}(X, \mu_X) = \emptyset\}$.

We have now all the ingredients needed to use the results recalled above.

▶ **Lemma 6.8.** *With the above definitions:*
1. $\mathcal{E}_{\Sigma,\mathcal{X}_0} = \mathcal{E}_\Sigma$*;*
2. $\mathcal{E}_{\Sigma,\mathcal{X}_\mathsf{E}} = \{e \in \mathcal{E}_\Sigma \mid \mathcal{U}_\Sigma(e) \text{ is split}\}$*;*
3. $(\Sigma\text{-Alg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \mathcal{X}_0)$ *and* $(\Sigma\text{-Alg}, (\mathcal{E}_\Sigma, \mathcal{M}_\Sigma), \mathcal{X}_\mathsf{E})$ *satisfy the conditions of our settings.*

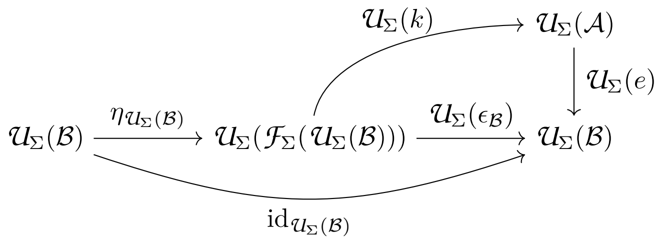
Proof.
1. Let $e : \mathcal{A} \to \mathcal{B}$ be an arrow in $\mathcal{E}_\Sigma$ and let $h : \mathcal{F}_\Sigma^{\mathbf{Set}}(X) \to \mathcal{B}$ be any morphism of Σ-Alg. By point 2 of Proposition 2.4 e is surjective so for any $x \in X$ we can take a $a_x \in e^{-1}(h(\eta_X(x)))$, where η is the unit of the adjunction $\mathcal{F}_\Sigma^{\mathbf{Set}} \dashv \mathcal{V}_\Sigma$ of Proposition 4.6, and define $\bar{k} : X \to A$ mapping x to a_x, where $\mathcal{A} = \big((A, \mu_A), \Sigma^{\mathcal{A}}\big)$, this induces $k : \mathcal{F}_\Sigma^{\mathbf{Set}}(X) \to \mathcal{A}$ and $(e \circ k) \circ \eta_X = e \circ \bar{k} = h \circ \eta_X$, thus $e \circ k = h$.



2. Let $e : \mathcal{A} \to \mathcal{B}$ be in $\mathcal{E}_\Sigma$ such that $\mathcal{U}_\Sigma(e)$ is split and let s be a section in $\mathbf{Fuz}(H)$, then, for any $h : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{B}$ we can consider the arrow $s \circ h \circ \eta_{(X,\mu_X)}$, which, by adjointness provides a $k : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{A}$, moreover:

$$e \circ k \circ \eta_{(X,\mu_X)} = e \circ s \circ h \circ \eta_{(X,\mu_X)} = (e \circ s) \circ \big(h \circ \eta_{(X,\mu_X)}\big) = h \circ \eta_{(X,\mu_X)}$$

so k is the wanted lifting. On the other hand, if e is in $\mathcal{E}_{\Sigma,X_1}$ we can take the diagram:



where $\epsilon_\mathcal{B}$ is the component of the counit $\epsilon : \mathcal{F}_\Sigma \circ \mathcal{U}_\Sigma \to \mathrm{id}_{\Sigma\text{-}\mathsf{Alg}}$ and k its lifting. Taking $\mathcal{U}_\Sigma(k) \circ \eta_{\mathcal{U}_\Sigma(\mathcal{B})}$ we get the desired section of $\mathcal{U}_\Sigma(e)$.

3. First, notice that $\mathbf{Fuz}(H)$ has all products by Lemma 2.6. Moeover, it can be shown that $X{\downarrow}\mathbf{C}$ is essentially small. For any fuzzy set (X, μ_X) we can consider the identity $\mathrm{id}_{(X,\mu_X)} : (X, \mu_X) \to (X, \mu_X)$ and the counit $\epsilon_{(X,\mu_X)} : \nabla(X) \to (X, \mu_X)$ of the adjunction $\nabla \dashv \mathcal{U}$ of Proposition 2.2. They induce arrows $e_0 : \mathcal{F}_\Sigma^{\mathsf{Set}}(X) \to (X, \mu_X)$ and $e_\mathsf{E} : \mathcal{F}_\Sigma(X, \mu_X) \to (X, \mu_X)$ such that $\mathcal{U}_\Sigma(e_0) \circ \eta_{\nabla(X)} = \epsilon_{(X,\mu_X)}$ and $\mathcal{U}_\Sigma(e_H) \circ \eta_{(X,\mu_X)} = \mathrm{id}_{(X,\mu_X)}$. So $\mathcal{U}_\Sigma(e_H)$ is split and, since $\epsilon_{(X,\mu_X)}$ is surjective, point 2 of Proposition 2.4 allows us to conclude that $\mathcal{U}_\Sigma(e_0)$ is an epimorphism. ◀

We want now to translate formulae of our sequent calculus into $\mathcal{X}_0$- and $\mathcal{X}_\mathsf{E}$-equations. To this end, we have to restrict to two classes of theories, which we introduce next.

▶ **Definition 6.9.** *Let $\mathcal{L} = (\Sigma, X)$ be a language, a theory $\Lambda \in \mathsf{Th}(\mathcal{L})$ is said to be:*

- unconditional *([17, App. B.5]) if any sequent in Λ is of the form $\vdash \phi$ for some formula ϕ;*
- of type E *if any sequent in Λ is of the form $\{\mathsf{E}(l_i, x_i)\}_{i\in I} \vdash \phi$ for some formula ϕ, $\{x_i\}_{i\in I} \subset X$ and $\{l_i\}_{i\in I} \subset H$.*

▶ **Lemma 6.10.** *For any signature Σ and $\mathcal{X}_\mathsf{E}$-equation $e : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{B}$ there exists a type E theory Λ_e such that, for every Σ-algebra $\mathcal{A}$, $\mathcal{A} \vDash_{X_1} e$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_e)$. Moreover $|\Gamma| \leq |\mathsf{supp}(X, \mu_X)|$ for any $\Gamma \vdash \phi \in \Lambda_e$.*

Proof. Let $\mathcal{L}_e$ be (Σ, X). We define $\Gamma_X := \{\mathsf{E}(\mu_X(x), x) \mid x \in \mathsf{supp}(X, \mu_X)\}$ and $\Lambda_e \in \mathsf{Th}(\mathcal{L})$ as $\Lambda_e^1 \cup \Lambda_e^2$ where

$$\Lambda_e^1 \coloneqq \{\Gamma_X \vdash \mathsf{E}(l, t) \mid l \leq \mu_B(e([t]))\}$$
$$\Lambda_e^2 \coloneqq \{\Gamma_X \vdash [s] \equiv [t] \mid e([t]) = e([s])\}$$

and (B, μ_B) is $\mathcal{U}_\Sigma(\mathcal{B})$. Let us show the two implications.

$\Rightarrow$ Any $\iota : X \to A$ such that $\mathcal{A} \vDash_\iota \Gamma_X$ defines an arrow $\bar{\iota}(X, \mu_X) \to \mathcal{U}_\Sigma(\mathcal{A})$. By adjointness we have a homomorphism $h_\iota : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{A}$ hence, by hypothesis, there exists $q_\iota : \mathcal{B} \to \mathcal{A}$ such that $q_\iota \circ e = h_\iota$. Now, notice that (see Theorem 5.9, and Proposition 5.7(4)) $h_\iota([t]) = t^{\mathcal{A},\iota}$. Take a sequent $\Gamma_X \vdash \psi$ in Λ_e, we have two cases, depending on ψ.

- If $\psi = \mathsf{E}(l, t) \in \Lambda_e^{me}$ we have

$$l \leq \mu_B(e([t])) \leq \mu_A(q_\iota(e([t]))) = \mu_A(h_\iota([t])) = t^{\mathcal{A},\iota}$$

therefore $\mathcal{A} \vDash_\iota \psi$.

- If $\phi = [s] \equiv [t] \in \Lambda_e^{eq}$ then

$$t^{\mathcal{A},\iota} = h_\iota([t]) = q_\iota(e([t])) = q_\iota(e([s])) = h_\iota([t]) = s^{\mathcal{A},\iota}$$

and even in this case $\mathcal{A} \vDash_\iota \psi$.

$\Leftarrow$ Take $h : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{A}$, $\mathcal{U}_\Sigma(h) \circ \eta_{\nabla(X)}$ is an arrow $(X, \mu_X) \to \mathcal{U}_\Sigma(\mathcal{A})$, so forgetting the fuzzy set structure too gives us an assignment $\iota_h : X \to A$ such that $\mathcal{A} \vDash_{\iota_h} \Gamma_X$. As before $h([t]) = t^{\mathcal{A},\iota_h}$ for every $[t] \in \mathcal{F}_\Sigma(X, \mu_X)$. Since $\mathcal{A} \in \mathbf{Mod}(\Lambda_e)$ we have
- $t^{\mathcal{A},\iota_h} = s^{\mathcal{A},\iota_h}$ for all terms t and s such that $e([t]) = e([s])$;
- $l \leq \mu_A(t^{\mathcal{A},\iota_h})$ for all terms t such that $l \leq \mu_B(e([t]))$.

So, the function $q : B \to A$ which sends $b \in B$ to $h([t])$ for some $[t] \in e^{-1}(b)$, provides us with an arrow $\mathcal{U}_\Sigma(\mathcal{B}) \to \mathcal{U}_\Sigma(\mathcal{A})$ such that $q \circ e = h$ and a straightforward computation shows that it is an arrow of Σ-Alg. ◀

▶ **Corollary 6.11.** *For any signature Σ and $\mathcal{X}_0$-equation $e : \mathcal{F}_\Sigma^{\mathbf{Set}}(X) \to \mathcal{B}$ there exists an unconditional theory Λ_e such that, for any Σ-algebra $\mathcal{A}$, $\mathcal{A} \vDash_{\mathcal{X}_0} e$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_e)$.*

Finally, from the results above we can easily conclude HSP-like results for Σ-Alg.

▶ **Theorem 6.12.** *Let $\mathbf{V}$ be a full subcategory of Σ-Alg, then*
1. *$\mathbf{V}$ is closed under epimorphisms, (small) products and strong monomorphisms if and only if there exists a class of unconditional theories $\{\Lambda_e\}_{e\in\mathbb{E}}$ such that $\mathcal{A} \in \mathbf{V}$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_e)$ for all $e \in \mathbb{E}$.*
2. *$\mathbf{V}$ is closed under split epimorphisms, (small) products and strong monomorphisms if and only if there exists a class of type E theories $\{\Lambda_e\}_{e\in\mathbb{E}}$ such that $\mathcal{A} \in \mathbf{V}$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda_e)$ for all $e \in \mathbb{E}$.*

Proof. Straightforward in light of Theorem 6.4, Lemma 6.10 and Corollary 6.11. ◀

▶ Remark 6.13. We cannot arrange the collection $\{\Lambda_e\}_{e\in\mathbb{E}}$ into a unique theory since in order to write down all the sequents we need a proper class of variables. A possible way to deal with this issue is to fix two Grothendieck universes ([25]) $\mathbf{U}_1 \subset \mathbf{U}_2$ and allow for a proper class (i.e., an element of $\mathbf{U}_2$) of variables in Definition 3.1. All the proofs of this paper can be repeated verbatim in this context carefully distinguishing between fuzzy *sets* (i.e., those defined on an element of $\mathbf{U}_1$) and fuzzy *classes* (i.e., those defined on an element of $\mathbf{U}_2$). Then the algebras of terms will be a fuzzy class in general but it is possible to show, using the explicit construction, that $\mathsf{T}_\Lambda(X, \mu_X)$ is a fuzzy set if $X \in \mathbf{U}_1$ and so we can retain all the results of Section 5.

The issue mentioned in the previous remark can be avoided if the family $\{\Lambda_e\}_{e\in\mathbb{E}}$ satisfies a boundedness property about the premises of the sequents belonging to each Λ_e.

▶ **Definition 6.14.** *Given a cardinal κ we say that a $\mathcal{X}_\mathsf{E}$-equation $e : \mathcal{F}_\Sigma(X, \mu_X) \to \mathcal{B}$ is* κ-supported *if $|\mathsf{supp}(X, \mu_X)| < \kappa$.*

▶ **Proposition 6.15.** *Let $\mathbf{V} = \mathcal{V}(\mathbb{E})$ be an $\mathcal{X}_\mathsf{E}$-equational defined subcategory of Σ-Alg and suppose every $e \in \mathbb{E}$ is κ-supported, then there exists a theory $\Lambda \in \mathsf{Th}(\mathcal{L})$, where $\mathcal{L} = (\Sigma, \kappa)$, such that $\mathbf{V} = \mathbf{Mod}(\Lambda)$.*

Proof. For any $e : \mathcal{F}_\Sigma(X_e, \mu_{X_e}) \to \mathcal{B}_e$ in $\mathbb{E}$ we can fix an injection $i_e : \mathsf{supp}(X_e, \mu_{X_e}) \to \kappa$ and an extension let $\bar{i}_e : X \to \kappa$ of it, fix also morphisms $\mathbf{I}^e : \mathcal{L}_e \to \mathcal{L}$ given by $(\mathrm{id}_\Sigma, \bar{i}_e)$. Let now $\{\Lambda_e\}_{e\in\mathbb{E}}$ be the collection of theories given by Corollary 6.11 and Theorem 6.12, since each $\Lambda_e \in \mathbf{Form}(\mathcal{L}_e)$ we can define:

$$\Lambda := \bigcup_{e\in\mathbb{E}} \mathbf{I}^e_*(\Lambda_e)$$

We have to show that $\mathcal{A} \in \mathbf{V}$ if and only if $\mathcal{A} \in \mathbf{Mod}(\Lambda)$.

$\Rightarrow$ Let $\mathsf{Form}(\mathbf{I}^e)(\Gamma_{X_e}) \vdash \mathsf{Form}(\mathbf{I}^e)(\psi)$ be a sequent in Λ and let $\iota : \kappa \to A$ an assignment such that $\mathcal{A} \vDash_\iota \mathsf{Form}(\mathbf{I}^e)(\Gamma_{X_e})$. By point 3 of Lemma 4.9 this implies $\mathcal{A} \vDash_{\iota \circ \bar{i}_e} \Gamma_{X_e}$, therefore $\mathcal{A} \vDash_{\iota \circ \bar{i}_e} \psi$ and we conclude applying lemma 4.9 again.

$\Leftarrow$ If $\mathcal{U}_\Sigma(\mathcal{A}) = (\emptyset, \mathsf{i}_H)$, ($\mathsf{i}_H$ being the empty map $\emptyset \to H$) then there are no assignment $\kappa \to A$ and so $\mathcal{A}$ is in $\mathbf{Mod}(\Lambda)$. In the other cases let $\Gamma_{X_e} \vdash \psi$ be in Λ_e and $\iota : X_e \to A$ such that $\mathcal{A} \vDash_\iota \Gamma_{X_e}$, since $A \neq \emptyset$ there exists $\hat{\iota} : \kappa \to A$ such that $\hat{\iota} \circ \bar{i}_e = \iota$ as in the previous point Lemma 4.9 implies $\mathcal{A} \vDash_{\hat{\iota}} \mathsf{Form}(\mathbf{I}^e)(\Gamma_{X_e})$, so $\mathcal{A} \vDash_{\hat{\iota}} \mathsf{Form}(\mathbf{I}^e)(\psi)$ and again this is equivalent to $\mathcal{A} \vDash_\iota \psi$. ◀

▶ **Corollary 6.16.** $\mathbf{V}$ *is closed under epimorphisms, (small) products and strong monomorphisms if and only if there exists a language* $\mathcal{L}$ *and an unconditional theory* $\Lambda \in \mathsf{Th}(\mathcal{L})$ *such that* $\mathbf{V} = \mathbf{Mod}(\Lambda)$.

7 Conclusions and future work

In this paper we have introduced a *fuzzy sequent calculus* to capture equational aspects of fuzzy sets. While equalities are captured by usual equations, information contained in the membership function is captured by *membership proposition* of the form $\mathsf{E}(l, t)$, to be interpreted as "the membership degree of t is at least l". We have used a natural concept of *fuzzy algebras* to provide a sound and complete semantics for such calculus, in the sense that a formula is satisfied by all the models of a given theory if and only if it is derivable from it using the rules of our sequent calculus.

As in the classical and quantitative contexts, there is a notion of *free model* of a theory Λ and thus an associated monad T_Λ on the category $\mathbf{Fuz}(H)$ of fuzzy sets over a frame H. However, in general Eilenberg-Moore algebras for such monad are not equivalent to models of Λ, but we have shown that this equivalence holds if Λ is *basic*. In this direction it would be interesting to better understand the categorical status of our approach, investigating possible links between our notion of fuzzy theory and $\mathbf{Fuz}(H)$-Lawvere theories as introduced in full generality by Nishizawa and Power in [20]. A difference between the two approaches is that for us arities are simply finite sets, while following [20] a $\mathbf{Fuz}(H)$-Lawvere theory arities would be given by finite fuzzy sets. A possible underlying concept to both approaches is that of *discrete Lawvere theories* [23, 10].

Finally, using the results provided in [17] we have proved that, given a signature Σ, subcategories of Σ-Alg which are closed under products, strong monomorphisms and epimorphic images correspond precisely with categories of models for *unconditional theories*, i.e. theories axiomatised by sequents without premises. Moreover, using the same results, we have also proved that the categories of models of *theories of type* E, i.e. those whose axioms' premises contain only membership propositions involving variables, are exactly those subcategories closed under products, strong monomorphisms and split epimorphisms.

Our category $\mathbf{Fuz}(H)$ of fuzzy sets has crisp arrows and crisp equality: arrows are ordinary functions between the underlying sets and equalities can be judged to be either true or false. A way to further "fuzzifying" concepts is to use the topos of *H-sets* over the frame H introduced in [9]: this is equivalent to the topos of sheaves over H and contains $\mathbf{Fuz}(H)$ as a (non full) subcategory. By construction, equalities and functions are "fuzzy". It would be interesting to study an application of our approach to this context. A promising feature is that in an H-set the membership degree function is built-in as simply the equality relation, so it would not be necessary to distinguish between equations and membership propositions. Even more generally, we can replace H with an arbitrary quantale $\mathcal{V}$ and consider the category of sets endowed with a "$\mathcal{V}$-valued equivalence relation" [6].

References

1 N. Ajmal. Homomorphism of fuzzy groups, correspondence theorem and fuzzy quotient groups. *Fuzzy sets and systems*, 61(3):329–339, 1994.
2 N. Ajmal and A. S. Prajapati. Fuzzy cosets and fuzzy normal subgroups. *Information sciences*, 64(1-2):17–25, 1992.
3 G. Bacci, R. Mardare, P. Panangaden, and G. Plotkin. An algebraic theory of Markov processes. In *33rd Symposium on Logic in Computer Science (LICS)*, pages 679–688, 2018.
4 G. Bacci, R. Mardare, P. Panangaden, and G. Plotkin. Quantitative equational reasoning. *Foundations of Probabilistic Programming*, page 333, 2020.
5 G. Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 10:433–454, 1935.
6 Filippo Bonchi, Barbara König, and Daniela Petrisan. Up-to techniques for behavioural metrics via fibrations. In *CONCUR*, volume 118 of *LIPIcs*, pages 17:1–17:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018.
7 F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures*, volume 2. Cambridge University Press, 1994.
8 Davide Castelnovo and Marino Miculan. Fuzzy algebraic theories. *CoRR*, abs/2110.10970, 2021. `arXiv:2110.10970`.
9 M. P Fourman and D. S. Scott. Sheaves and logic. In *Applications of sheaves*, pages 302–401. Springer, 1979.
10 Martin Hyland and John Power. Discrete lawvere theories and computational effects. *Theoretical Computer Science*, 366(1-2):144–162, 2006.
11 Martin Hyland and John Power. The category theoretic understanding of universal algebra: Lawvere theories and monads. *Electron. Notes Theor. Comput. Sci.*, 172:437–458, 2007. `doi:10.1016/j.entcs.2007.02.019`.
12 Peter T Johnstone. *Stone spaces*, volume 3. Cambridge University Press, 1982.
13 Gregory Maxwell Kelly. A note on relations relative to a factorization system. In *Category Theory*, pages 249–261. Springer, 1991.
14 S. MacLane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.
15 R. Mardare, P. Panangaden, and G. Plotkin. On the axiomatizability of quantitative algebras. In *32nd Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2017.
16 A. S. Mashour, M. H. Ghanim, and F. I. Sidky. Normal fuzzy subgroups. *Information Sciences*, 20:53–59, 1990.
17 S. Milius and H. Urbat. Equational axiomatization of algebras with structure. In *International Conference on Foundations of Software Science and Computation Structures*, pages 400–417. Springer, 2019.
18 Eugenio Moggi. Notions of computation and monads. *Inf. Comput.*, 93(1):55–92, 1991. `doi:10.1016/0890-5401(91)90052-4`.
19 J. N. Mordeson, D. S. Malik, and N. Kuroki. *Fuzzy semigroups*, volume 131. Springer, 2012.
20 K. Nishizawa and J. Power. Lawvere theories enriched over a general base. *Journal of Pure and Applied Algebra*, 213(3):377–386, 2009.
21 Gordon D. Plotkin and John Power. Notions of computation determine monads. In *FoSSaCS*, volume 2303 of *Lecture Notes in Computer Science*, pages 342–356. Springer, 2002.
22 Gordon D. Plotkin and John Power. Algebraic operations and generic effects. *Appl. Categorical Struct.*, 11(1):69–94, 2003. `doi:10.1023/A:1023064908962`.
23 John Power. Discrete lawvere theories. In *International Conference on Algebra and Coalgebra in Computer Science*, pages 348–363. Springer, 2005.
24 A. Rosenfeld. Fuzzy groups. *Journal of mathematical analysis and applications*, 35(3):512–517, 1971.
25 N. H. Williams. On Grothendieck universes. *Compositio Mathematica*, 21(1):1–3, 1969.
26 O. Wyler. *Lecture notes on topoi and quasitopoi*. World Scientific, 1991.
27 O. Wyler. Fuzzy logic and categories of fuzzy sets. In *Non-Classical Logics and Their Applications to Fuzzy Subsets*, pages 235–268. Springer, 1995.