Decidability for Sturmian Words

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Abstract

We show that the first-order theory of Sturmian words over Presburger arithmetic is decidable. Using a general adder recognizing addition in Ostrowski numeration systems by Baranwal, Schaeffer and Shallit, we prove that the first-order expansions of Presburger arithmetic by a single Sturmian word are uniformly ω-automatic, and then deduce the decidability of the theory of the class of such structures. Using an implementation of this decision algorithm called Pecan, we automatically reprove classical theorems about Sturmian words in seconds, and are able to obtain new results about antisquares and antipalindromes in characteristic Sturmian words.

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1 Introduction

It has been known for some time that, for certain infinite words $\mathbf{c} = c_0c_1c_2\cdots$ over a finite alphabet Σ, the first-order logical theory $\mathrm{FO}(\mathbb{N}, <, +, 0, 1, n \mapsto c_n)$ is decidable. In the case where $\mathbf{c}$ is a k**-automatic sequence** for $k \geq 2$, this is due to Büchi [5], although his original proof was flawed. The correct statement appears, for example, in Bruyère et al. [4]. Although the worst-case running time of the decision procedure is truly formidable (and non-elementary), it turns out that an implementation can, in many cases, decide the truth of interesting and nontrivial first-order statements about automatic sequences in a reasonable length of time. Thus, one can easily reprove known results, and obtain new ones, merely by translating the desired result into the appropriate first-order statement φ and running the decision procedure on φ. For an example of the kinds of things that can be proved, see, for example, Goč, Henshall, and Shallit [6].

More generally, the same ideas can be used for other kinds of sequences defined in terms of some numeration system for the natural numbers. Such a numeration system provides a unique (up to leading zeros) representation for n as a sum of terms of some other sequence $(s_n)_{n\geq 1}$. If the sequence $\mathbf{c} = c_0c_1c_2\cdots$ can be computed by a finite automaton taking the representation of n as input, and if further, the addition of represented integers is computable by another finite automaton, then once again the first-order theory $\mathrm{FO}(\mathbb{N}, <, +, 0, 1, n \mapsto c_n)$ is decidable. This is the case, for example, for the so-called Fibonacci-automatic sequences in Mousavi, Schaeffer, and Shallit [14] and the Pell-automatic sequences in Baranwal and Shallit [3].

More generally, the same kinds of ideas can handle Sturmian words. For quadratic numbers, this was first observed by Hieronymi and Terry [9]. In this paper we extend those results to all Sturmian characteristic words. Thus, the first-order theory of Sturmian characteristic words is decidable. As a result, many classical theorems about Sturmian words, which previously required intricate proofs, can be proved automatically by a theorem-prover in a few seconds. As examples, in Section 7 we reprove basic results such as the balanced property and the subword complexity of these words.

Let $\alpha, \rho \in \mathbb{R}$ be such that α is irrational. The **Sturmian word with slope $\boldsymbol{\alpha}$ and intercept $\boldsymbol{\rho}$** is the infinite $\{0,1\}$-word $\mathbf{c}_{\alpha,\rho} = c_{\alpha,\rho}(1)c_{\alpha,\rho}(2)\cdots$ such that for all $n \in \mathbb{N}$

$$c_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor - \lfloor \alpha \rfloor.$$

When $\rho = 0$, we call $\mathbf{c}_{\alpha,\mathbf{0}}$ the **characteristic word** of slope α. Sturmian words and their combinatorical properties have been studied extensively. We refer the reader to the survey by Berstel and Séébold [12, Chapter 2]. Note that $\mathbf{c}_{\alpha,\rho}$ can be understood as a function from $\mathbb{N}$ to $\{0,1\}$. Let $\mathcal{L}$ be the signature[1] of the first-order logical theory $\mathrm{FO}(\mathbb{N}, <, +, 0, 1)$ and let $\mathcal{L}_c$ denote the signature obtained by adding a single unary function symbol c to $\mathcal{L}$. Now let $\mathcal{N}_{\alpha,\rho}$ be the $\mathcal{L}_c$-structure $(\mathbb{N}, <, +, 0, 1, n \mapsto c_{\alpha,\rho}(n))$, where we expand Presburger arithmetic by a Sturmian word interpreted as a unary function. The main result of this paper is the decidability of the theory of the collection of such expansions. Set $\mathbf{Irr} := (0,1) \setminus \mathbb{Q}$. Let $\mathcal{K}_{\text{sturmian}} := \{\mathcal{N}_{\alpha,\rho} \ : \ \alpha \in \mathbf{Irr}, \rho \in \mathbb{R}\}$, and let $\mathcal{K}_{\text{char}} := \{\mathcal{N}_{\alpha,0} \ : \ \alpha \in \mathbf{Irr}\}$.

▶ **Theorem A.** *The first-order logical theories*[2] $\mathrm{FO}(\mathcal{K}_{\text{sturmian}})$ *and* $\mathrm{FO}(\mathcal{K}_{\text{char}})$ *are decidable.*

So far, decidability was only known for individual $\mathrm{FO}(\mathcal{N}_{\alpha,\rho})$, and only for very particular α. By [9] the logical theory $\mathrm{FO}(\mathcal{N}_{\alpha,0})$ is decidable when α is a quadratic irrational[3]. Moreover, if the continued fraction of α is not computable, it can be seen rather easily that $\mathrm{FO}(\mathcal{N}_{\alpha,0})$ is undecidable.

Theorem A is rather powerful, as it allows to automatically decide combinatorial statements about all Sturmian words. Consider the $\mathcal{L}_c$-sentence φ

$$\forall p \ (p > 0) \rightarrow \Big(\forall i \ \exists j \ j > i \wedge c(j) \neq c(j+p)\Big).$$

We observe that $\mathcal{N}_{\alpha,\rho} \models \varphi$ if and only if $\mathbf{c}_{\alpha,\rho}$ is not eventually periodic. Thus the decision procedure from Theorem A allows us to check that no Sturmian word is eventually periodic. Of course, it is well-known that no Sturmian word is eventually periodic, but this example indicates potential applications of Theorem A. We outline some of these in Section 7.

[1] In model theory this is usually called (or identified with) the language of the theory. However, here this conflicts with the convention of calling an arbitrary set of words a language.

[2] Given a signature $\mathcal{L}_0$ and a class $\mathcal{K}$ of $\mathcal{L}_0$-structures, the first-order logical theory of $\mathcal{K}$ is defined as the set of all $\mathcal{L}_0$-sentences that are true in all structures in $\mathcal{K}$. This theory is denoted by $\mathrm{FO}(\mathcal{K})$.

[3] A real number is **quadratic** if it is the root of a quadratic equation with integer coefficients.

We not only prove Theorem A, but instead establish a vastly more general theorem of which Theorem A is an immediate corollary. To state this general result, let $\mathcal{L}_m$ be the signature of $\mathrm{FO}(\mathbb{R},<,+,\mathbb{Z})$, and let $\mathcal{L}_{m,a}$ be the extension of $\mathcal{L}_m$ by a unary predicate. For $\alpha \in \mathbb{R}_{>0}$, we let $\mathcal{R}_\alpha$ denote $\mathcal{L}_{m,a}$-structure $(\mathbb{R},<,+,\mathbb{Z},\alpha\mathbb{Z})$. When $\alpha \in \mathbb{Q}$, it has long been known that $\mathrm{FO}(\mathcal{R}_\alpha)$ is decidable (arguably due to Skolem [19]). Recently this result was extended to quadratic numbers.

▶ **Fact 1** (Hieronymi [7, Theorem A])**.** *Let α be a quadratic irrational. Then $\mathrm{FO}(\mathcal{R}_\alpha)$ is decidable.*

See also Hieronymi, Nguyen and Pak [8] for a computational complexity analysis of this decision procedure. The proof of Fact 1 establishes that if α is quadratic, then $\mathcal{R}_\alpha$ is an ω-automatic structure; that is it can be represented by Büchi automata. Since every ω-automatic structure has a decidable first-order theory, so does $\mathcal{R}_\alpha$. See Khoussainov and Minnes [10] for a survey on ω-automatic structures. The key insight needed to prove ω-automaticity of $\mathcal{R}_\alpha$ is that addition in the Ostrowski-numeration system based on α is recognizable by a Büchi automaton when α is quadratic. See Section 2 for a definition of Ostrowski numeration systems.

As observed in [7], there are examples of non-quadratic irrationals α such that $\mathcal{R}_\alpha$ has an undecidable theory and hence is not ω-automatic. However, in this paper we show that the common theory of the $\mathcal{R}_\alpha$ is decidable. Let $\mathcal{K}$ denote the class of $\mathcal{L}_{m,a}$-structures $\{\mathcal{R}_\alpha \;:\; \alpha \in \mathbf{Irr}\}$.

▶ **Theorem B.** *The theory $\mathrm{FO}(\mathcal{K})$ is decidable.*

Indeed, we will even prove a substantial generalization of Theorem B. For each $\mathcal{L}_{m,a}$-sentence φ, we set $M_\varphi := \{\alpha \in \mathbf{Irr} \;:\; \mathcal{R}_\alpha \models \varphi\}$. Let $\mathbf{Irr}_{\mathrm{quad}}$ be the set of all quadratic irrational real numbers in $\mathbf{Irr}$. Define $\mathcal{M} = (\mathbf{Irr},<,(\mathrm{M}_\varphi)_\varphi,(\mathrm{q})_{\mathrm{q}\in\mathbf{Irr}_{\mathrm{quad}}})$ to be the expansion of the dense linear order $(\mathbf{Irr},<)$ by predicates for M_φ for each $\mathcal{L}_{m,a}$-sentence φ, and constant symbols for each quadratic irrational real number in $\mathbf{Irr}$.

▶ **Theorem C.** *The theory $\mathrm{FO}(\mathcal{M})$ is decidable.*

Observe that Fact 1 and Theorem B follow immediately from Theorem C. We outline how Theorem B implies Theorem A. Note that for every irrational α, the structure $\mathcal{R}_\alpha$ defines the usual floor function $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$, the singleton $\{\alpha\}$ and the successor function on $\alpha\mathbb{Z}$. Hence $\mathcal{R}_\alpha$ also defines the set $\{(\rho,\alpha n,c_{\alpha,\rho}(n)) \;:\; \rho \in \mathbb{R}, n \in \mathbb{N}\}$. From the definability of $\{\alpha\}$, we have that the function from $\alpha\mathbb{N}$ to $\{0,\alpha\}$ given by $\alpha n \mapsto \alpha c_{\alpha,\rho}(n)$ is definable in $\mathcal{R}_\alpha$. Thus the $\mathcal{L}_c$-structure $(\alpha\mathbb{N},<,+,0,\alpha,\alpha n \mapsto \alpha c_{\alpha,\rho}(n))$ can be defined in $\mathcal{R}_\alpha$, and this definition is uniform in α. Since the former structure is $\mathcal{L}_c$-isomorphic to $\mathcal{N}_{\alpha,\rho}$, we have that for every $\mathcal{L}_c$-sentence φ there is an $\mathcal{L}_{m,a}$-formula $\psi(x)$ such that

- $\varphi \in \mathrm{FO}(\mathcal{K}_{\mathrm{sturmian}})$ if and only if $\forall \mathrm{x}\ \psi(\mathrm{x}) \in \mathrm{FO}(\mathcal{K})$ and
- $\varphi \in \mathrm{FO}(\mathcal{K}_{\mathrm{char}})$ if and only if $\psi(0) \in \mathrm{FO}(\mathcal{K})$.

Even Theorem C is not the most general result we prove. Its statement is more technical and we postpone it until Section 6. However, we want to point out that we can add predicates for interesting subsets of $\mathbf{Irr}$ to $\mathcal{M}$ without changing the decidability of the theory. Examples of such subsets are the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are powers of 2, or the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are not in some fixed finite set. This means we can not only automatically prove theorems about all characteristic Sturmian words, but also prove theorems about all characteristic Sturmian words whose slope is one of these sets. There is a limit to this technique. If we add a predicate for the set of all $\alpha \in \mathbf{Irr}$ such that the terms of continued fraction expansion of α are bounded, or add a predicate for the set of elements in $\mathbf{Irr}$ whose continued fractions have strictly increasing terms, then our method is unable to conclude whether the resulting structure has a decidable theory. See Section 6 for a more precise statement about what kind of predicates can be added.

The proof of Theorem C follows closely the proof from [7] of the ω-automaticity of $\mathcal{R}_\alpha$ for fixed quadratic α. Here we show that the construction of the Büchi automata needed to represent $\mathcal{R}_\alpha$ is actually uniform in α. See Abu Zaid, Grädel, and Reinhardt [20] for a systematic study of uniformly automatic classes of structures. Deduction of Theorem C from this result is then rather straightforward. The key ingredient to establish the ω-automaticity of $\mathcal{R}_\alpha$ is an automaton that can perform addition in Ostrowski-numeration systems. By [9] there is an automaton that recognizes the addition relation for α-Ostrowski numeration systems for fixed quadratic α. So for a fixed quadratic number, there exists a 3-input automaton that accepts the α-Ostrowski representations of all triples of natural numbers x, y, z with $x + y = z$. In order to prove Theorem C, we need a uniform version of such an adder. This general adder is described in Baranwal, Schaeffer, and Shallit [2]. There a 4-input automaton is constructed that accepts 4-tuples consisting of an encoding of a real number α and three α-Ostrowski representations of natural numbers x, y, z with $x + y = z$. See Section 4 for details.

As mentioned above, an implementation of the decision algorithm provided by Theorem A can be used to study Sturmian words. We created a software program called Pecan [15] that includes such an implementation. Pecan is inspired by Walnut [13] by Mousavi, an automated theorem-prover for deciding properties of automatic words. The main difference is that Walnut is based on finite automata, while Pecan uses Büchi automata. In our setting it is more convenient to work with Büchi automata instead of finite automata, since the infinite families of words we want to consider – like Sturmian words – are indexed by real numbers. Section 7 provides more information about Pecan and contains further examples how Pecan is used prove statements about Sturmian words. Pecan's implementation is discussed in more detail in [16].

2 Preliminaries

Throughout, i, j, k, ℓ, m, n are used for natural numbers. Let X, Y be two sets and $Z \subseteq X \times Y$. For $x \in X$, we let Z_x denote the set $\{y \in Y : (x, y) \in Z\}$. Similarly, given a function $f : X \times Y \to W$ and $x \in X$, we write f_x for the function $f_x : Y \to W$ that maps $y \in Y$ to $f(x, y)$.

Given a (possibly infinite word) w over an alphabet Σ, we write w_i for the i-th letter of w, and $w|_n$ for $w_1 \cdots w_n$. We write $|w|$ for the length of w. We denote the set of infinite words over Σ by Σ^ω. If Σ is totally ordered by an $\prec$, we let $\prec_{\text{lex}}$ denote the corresponding lexicographic order on Σ^ω. Letting $u, v \in \Sigma^\omega$, we also write $u \prec_{\text{colex}} v$ if there is a maximal i such that $u_i \neq v_i$, and $u_i < v_i$ for this i. Note that while $\prec_{\text{lex}}$ is a total order on Σ^ω, the order $\prec_{\text{colex}}$ is only a partial order. However, for a given $\sigma \in \Sigma$, the order $\prec_{\text{colex}}$ is a total order on the set of all words $v \in \Sigma^\omega$ such that v_j is eventually equal to σ.

A **Büchi automaton (over an alphabet $\boldsymbol{\Sigma}$)** is a quintuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ where Q is a finite set of states, Σ is a finite alphabet, $\Delta \subseteq Q \times \Sigma \times Q$ is a transition relation, $I \subseteq Q$ is a set of initial states, and $F \subseteq Q$ is a set of accept states.

Let $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ be a Büchi automaton. Let $\sigma \in \Sigma^\omega$. A **run of $\boldsymbol{\sigma}$ from $\boldsymbol{p}$** is an infinite sequence s of states in Q such that $s_0 = p$, $(s_n, \sigma_n, s_{n+1}) \in \Delta$ for all $n < |\sigma|$. If $p \in I$, we say s is a **run of $\boldsymbol{\sigma}$**. Then σ is **accepted by $\boldsymbol{\mathcal{A}}$** if there is a run $s_0 s_1 \cdots$ of σ such that

$\{n : s_n \in F\}$ is infinite. We call this run an accepting run. We let $L(\mathcal{A})$ be the set of words accepted by $\mathcal{A}$. There are other types of ω-automata with different acceptance conditions, but in this paper we only consider Büchi automata.

Let Σ be a finite alphabet. We say a subset $X \subseteq \Sigma^\omega$ is ***ω*-regular** if it is recognized by some Büchi automaton. Let $u_1, \dots, u_n \in \Sigma^\omega$. We define the **convolution** $c(u_1, \dots, u_n)$ of $u_1, \dots, u_n$ as the element of $(\Sigma^n)^\omega$ whose value at position i is the n-tuple consisting of the values of $u_1, \dots, u_n$ at position i. We say that $X \subseteq (\Sigma^\omega)^n$ is ***ω*-regular** if $c(X)$ is ω-regular.

▶ **Fact 2.** *The collection of ω-regular sets is closed under union, intersection, complementation and projection.*

Closure under complementation is due to Büchi [5]. We refer the reader to Khoussainov and Nerode [11] for more information and a proof of Fact 2. As consequence of Fact 2, we have that for every ω-regular subset $W \subseteq (\Sigma^\omega)^{m+n}$ the set $\{s \in (\Sigma^\omega)^m \ : \ \forall t \in (\Sigma^\omega)^n \ (s,t) \in W\}$ is also ω-regular.

2.1 *ω*-regular structures

Let $\mathcal{U} = (U; R_1, \dots, R_m)$ be a structure, where U is a non-empty set and $R_1, \dots, R_m$ are relations on U. We say $\mathcal{U}$ is ***ω*-regular** if its domain and its relations are ω-regular.

Büchi's theorem [5] on the decidability of monadic second-order theory of one successor immediately gives the following well-known fact.

▶ **Fact 3.** *Let $\mathcal{U}$ be an ω-regular structure. Then the theory* $\mathrm{FO}(\mathcal{U})$ *is decidable.*

In this paper, we will consider families of ω-regular structures that are uniform in the following sense. Fix $m \in \mathbb{N}$ and a map $\mathrm{ar} : \{1, \dots, m\} \to \mathbb{N}$. Let Z be a set and for $z \in Z$ let $\mathcal{U}_z$ be a structure $(U_z; R_{1,z}, \dots, R_{m,z})$ such that $R_{i,z} \subseteq U_z^{\mathrm{ar}(i)}$. We say that $(\mathcal{U}_z)_{z \in Z}$ is a **uniform family of *ω*-regular structures** if

- $\{(z, y) \ : \ y \in U_z\}$ is ω-regular,
- $\{(z, y_1, \dots, y_{\mathrm{ar}(i)}) \ : \ (y_1, \dots, y_{\mathrm{ar}(i)}) \in R_{i,z}\}$ is ω-regular for each $i \in \{1, \dots, m\}$.

From Büchi's theorem, we immediately obtain the following.

▶ **Fact 4.** *Let $(\mathcal{U}_z)_{z \in Z}$ be a uniform family of ω-regular structures, and let φ be a formula in the signature of these structures. Then the set $\{(z, u) \ : \ z \in Z, u \in U_z, \ \mathcal{U}_z \models \varphi(u)\}$ is ω-regular, and the theory* $\mathrm{FO}(\{\mathcal{U}_z \ : \ z \in Z\})$ *is decidable.*

Proof. When φ is an atomic formula, the statement follows immediately from the definition of a uniform family of ω-regular structures and the ω-regularity of equality. By Fact 2, the statement holds for all formulas. ◀

2.2 Binary representations

For $k \in \mathbb{N}_{>1}$ and $b = b_0 b_1 b_2 \cdots b_n \in \{0,1\}^*$, we define $[b]_k := \sum_{i=0}^{n} b_i k^i$. For $N \in \mathbb{N}$ we say $b \in \{0,1\}^*$ is a **binary representation** of N if $[b]_2 = N$.

Throughout this paper, we will often consider infinite words over the (infinite) alphabet $\{0,1\}^*$. Let $[\cdot]_2 : (\{0,1\}^*)^\omega \to \mathbb{N}^\omega$ be the function that maps $u = u_1 u_2 \cdots \in (\{0,1\}^*)^\omega$ to $[u_1]_2 [u_2]_2 [u_3]_2 \cdots$ We will consider the following different relations on $(\{0,1\}^*)^\omega$.
Let $u, v \in (\{0,1\}^*)^\omega$. We write $u <_{\mathrm{lex},2} v$ if $[u]_2$ is lexicographically smaller than $[v]_2$. We write $u <_{\mathrm{colex},2} v$ if there is a maximal i such that $[u_i]_2 \neq [v_i]_2$, and $[u_i]_2 < [v_i]_2$. Note that while $<_{\mathrm{lex},2}$ is a total order on $(\{0,1\}^*)^\omega$, the order $<_{\mathrm{colex},2}$ is only a partial order. However,

$<_{\mathrm{colex},2}$ is a total order on the set of all words $v \in (\{0,1\}^*)^\omega$ such that $[v]_j$ is eventually 0. Let $u = u_1u_2\cdots, v = v_1v_2\cdots \in (\{0,1\}^*)^\omega$. Let k be minimal such that $[u_k]_2 \neq [v_k]_2$. We write $u <_{\mathrm{alex},2} v$ if either k is even and $[u_k]_2 < [v_k]_2$, or k is odd and $[u_k]_2 > [v_k]_2$.

2.3 Ostrowski representations

We now introduce Ostrowski representations based on the continued fraction expansions of real numbers. We refer the reader to Allouche and Shallit [1] and Rockett and Szüsz [18] for more details. A **finite continued fraction expansion** $[a_0; a_1, \dots, a_k]$ is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

For a real number α, we say $[a_0; a_1, \dots, a_k, \dots]$ is **the continued fraction expansion of $\boldsymbol{\alpha}$** if $\alpha = \lim_{k\to\infty}[a_0; a_1, \dots, a_k]$ and $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}_{>0}$ for $i > 0$. In this situation, we write $\alpha = [a_0; a_1, \dots]$. Every irrational number has precisely one continued fraction expansion. We recall the following well-known fact about continued fractions.

▶ **Fact 5.** *Let $\alpha = [a_0; a_1, \dots], \alpha' = [a_0'; a_1', \dots] \in \mathbb{R}$ be irrational. Let $k \in \mathbb{N}$ be minimal such that $a_k \neq a_k'$. Then $\alpha < \alpha'$ if and only if*
- *k is even and $a_k < a_k'$, or*
- *k is odd and $a_k > a_k'$.*

For the rest of this subsection, fix a positive irrational real number $\alpha \in (0,1)$ and let $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of α. Let $k \geq 1$. A quotient $p_k/q_k \in \mathbb{Q}$ is the **$\boldsymbol{k}$-th convergent of $\boldsymbol{\alpha}$** if $p_k \in \mathbb{N}$, $q_k \in \mathbb{Z}$, $\gcd(p_k, q_k) = 1$ and $\frac{p_k}{q_k} = [a_0; a_1, \dots, a_k]$. Set $p_{-1} := 1, q_{-1} := 0$ and $p_0 := a_0, q_0 := 1$. The convergents satisfy the following equations for $n \geq 1$:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

The **$\boldsymbol{k}$-th difference $\boldsymbol{\beta_k}$ of $\boldsymbol{\alpha}$** is defined as $\beta_k := q_k\alpha - p_k$. We use the following facts about k-th differences: for all $n \in \mathbb{N}$
1. $\beta_n > 0$ if and only if n is even,
2. $\beta_0 > -\beta_1 > \beta_2 > -\beta_3 > \beta_4 > \dots$, and
3. $-\beta_n = a_{n+2}\beta_{n+1} + a_{n+4}\beta_{n+3} + a_{n+6}\beta_{n+5} + \dots$.

We now recall a numeration system due to Ostrowski [17].

▶ **Fact 6** ([18, Ch. II-§4])**.** *Let $X \in \mathbb{N}$. Then X can be written uniquely as*

$$X = \sum_{n=0}^{N} b_{n+1} q_n \tag{1}$$

where $0 \leq b_1 < a_1$, $0 \leq b_{n+1} \leq a_{n+1}$ and $b_n = 0$ whenever $b_{n+1} = a_{n+1}$.

For $X \in \mathbb{N}$ satisfying (1) we write $X = [b_1b_2\cdots b_nb_{n+1}]_\alpha$ and call the word $b_1b_2\cdots b_{n+1}$ an α-Ostrowski representation of X. This representation is unique up to trailing zeros. Let $X, Y \in \mathbb{N}$ and let $b_1b_2\cdots b_{n+1}$ and $c_1c_2\cdots c_{n+1}$ be α-Ostrowski representations of X and

Y respectively. Since Ostrowski representations are obtained by a greedy algorithm, one can see easily that $X < Y$ if and only if $b_1 b_2 \cdots b_{n+1}$ is co-lexicographically smaller than $c_1 c_2 \cdots c_{n+1}$.

We now introduce a similar way to represent real numbers, also due to Ostrowski [17]. Let I_α be the interval $\big[\lfloor\alpha\rfloor - \alpha, 1 + \lfloor\alpha\rfloor - \alpha\big)$.

▶ **Fact 7** (cp. [18, Ch. II.6 Theorem 1]). *Let $x \in I_\alpha$. Then x can be written uniquely as*

$$\sum_{k=0}^{\infty} b_{k+1}\beta_k, \tag{2}$$

where $b_k \in \mathbb{Z}$ with $0 \le b_k \le a_k$, and $b_{k-1} = 0$ whenever $b_k = a_k$,(in particular, $b_1 \neq a_1$), and $b_k \neq a_k$ for infinitely many odd k.

For $x \in I_\alpha$ satisfying (2) we write $x = [b_1 b_2 \cdots]_\alpha$ and call the infinite word $b_1 b_2 \cdots$ the α-Ostrowski representation of x. This is closely connected to the integer Ostrowski representation. Note that for every real number there a unique element of I_α such that that their difference is an integer. We define $f_\alpha : \mathbb{R} \to I_\alpha$ to be the function that maps x to $x - u$, where u is the unique integer such that $x - u \in I_\alpha$.

▶ **Fact 8** ([7, Lemma 3.4]). *Let $X \in \mathbb{N}$ be such that $\sum_{k=0}^{N} b_{k+1} q_k$ is the α-Ostrowski representation of X. Then $f_\alpha(\alpha X) = \sum_{k=0}^{\infty} b_{k+1}\beta_k$ is the α-Ostrowski representation of $f_\alpha(\alpha X)$, where $b_{k+1} = 0$ for $k > N$.*

Since $\beta_k > 0$ if and only if k is even, the order of two elements in I_α can be determined by the Ostrowski representation as follows.

▶ **Fact 9** ([7, Fact 2.13]). *Let $x, y \in I_\alpha$ with $x \neq y$ and let $[b_1 b_2 \cdots]_\alpha$ and $[c_1 c_2 \cdots]_\alpha$ be the α-Ostrowski representations of x and y. Let $k \in \mathbb{N}$ be minimal such that $b_k \neq c_k$. Then $x < y$ if and only if*

(i) *$b_{k+1} < c_{k+1}$ if k is even;*

(ii) *$b_{k+1} > c_{k+1}$ if k is odd.*

3 #-binary coding

In this section, we introduce #-binary coding. Fix the alphabet $\Sigma_\# := \{0, 1, \#\}$. Let H_∞ denote the set of all infinite $\Sigma_\#$-words in which # appears infinitely many times. Clearly H_∞ is ω-regular.

Let $C_\# : (\{0,1\}^*)^\omega \to H_\infty$ map an infinite word $b = b_1 b_2 b_3 \cdots$ over $\{0,1\}^*$ to the infinite $\Sigma_\#$-word $\#b_1\#b_2\#b_3\#\cdots$ We note that the map $C_\#$ is a bijection. Let $u = u_1 u_2 u_3 \cdots, v = v_1 v_2 v_3 \cdots \in \Sigma_\#^\omega$. We say u and v are **aligned** if for all $i \in \mathbb{N}$ $u_i = \#$ if and only if $v_i = \#$. This defines an ω-regular equivalence relation on $\Sigma_\#^\omega$. We denote this equivalence relation by $\sim_\#$. The following fact follows easily.

▶ **Fact 10.** *The following sets are ω-regular:*

- $\{(u, v) \in H_\infty^2 \ : \ u \sim_\# v$ *and* $C_\#^{-1}(u) <_{\mathrm{lex},2} C_\#^{-1}(v)\}$,
- $\{(u, v) \in H_\infty^2 \ : \ u \sim_\# v$ *and* $C_\#^{-1}(u) <_{\mathrm{colex},2} C_\#^{-1}(v)\}$,
- $\{(u, v) \in H_\infty^2 \ : \ u \sim_\# v$ *and* $C_\#^{-1}(u) <_{\mathrm{alex},2} C_\#^{-1}(v)\}$.

3.1 #-binary coding of continued fractions

We now code the continued fraction expansions of real numbers as infinite $\Sigma_{\#}$-words.

▶ **Definition 11.** *Let $\alpha \in (0,1)$ be irrational such that $[0; a_1, a_2, \dots]$ is the continued fraction expansion of α. Let $u = u_1 u_2 \cdots \in (\{0,1\}^*)^\omega$ such that $u_i \in \{0,1\}^*$ is a binary representation of a_i for each $i \in \mathbb{Z}_{\geq 0}$. We say that $C_{\#}(u)$ is a* ***#-binary coding of the continued fraction of $\boldsymbol{\alpha}$****.*

Let R be the set of elements of $\Sigma_{\#}^\omega$ of the form $(\#(0|1)^*1(0|1)^*)^\omega$. Obviously, R is ω-regular.

▶ **Lemma 12.** *Let $w \in R$. Then there is a unique irrational number $\alpha \in [0,1]$ such that w is a #-binary coding of the continued fraction of α.*

The proof of Lemma 12 is in Appendix A. For $w \in R$, let $\alpha(w)$ be the real number given by Lemma 12. When $v = (v_1, \dots, v_n) \in R^n$, we write $\alpha(v)$ for $(\alpha(v_1), \dots, \alpha(v_n))$.

Even though continued fractions are unique, their #-binary codings are not, because binary representations can have trailing zeroes. This ambiguity is required in order to properly recognize relationships between multiple numbers, as one of the numbers involved may require more bits in a coefficient than the other(s). Occasionally we need to ensure that all possible representations of a given tuple of numbers are contained in a set. For this reason, we introduce the zero-closure of subsets of R^n.

▶ **Definition 13.** *Let $X \subseteq R^n$. The* ***zero-closure*** *of X is $\{u \in R^n \ : \ \exists v \in X \ \alpha(u) = \alpha(v)\}$.*

▶ **Lemma 14.** *Let $X \subseteq R^n$ be ω-regular. Then the zero-closure of X is also ω-regular.*

The essence of the proof is that trailing zeroes are the only way that #-binary codings of the same number can differ. The details of proof are technical and can be found in Appendix A.

▶ **Lemma 15.** *The set $\{(w_1, w_2) \in R^2 \ : \ w_1 \sim_{\#} w_2 \text{ and } \alpha(w_1) < \alpha(w_2)\}$ is ω-regular.*

Proof. Let $w_1, w_2 \in R$ be such that $w_1 \sim_{\#} w_2$. By Fact 5 we have that $\alpha(w_1) < \alpha(w_2)$ if only $C_{\#}^{-1}(w_1) <_{\text{alex},2} C_{\#}^{-1}(w_2)$. Thus ω-regularity follows from Fact 10. ◀

▶ **Lemma 16.** *Let $a \in [0,1)$ be a quadratic irrational. Then $\{w \in R \ : \ \alpha(w) = a\}$ is ω-regular.*

Proof. The continued fraction expansion of a is eventually periodic. Thus there is an eventually periodic $u \in (\{0,1\}^*)^\omega$ such that $C_{\#}(u)$ is a #-binary coding of the continued fraction of a. The singleton set containing an eventually periodic string is ω-regular. It remains to expand this set to contain all representations via Lemma 14. ◀

▶ **Lemma 17.** *The set $\{w \in R \ : \ \alpha(w) < \frac{1}{2}\}$ is ω-regular.*

Proof. Let $\alpha(w) = [0; a_1, a_2, \dots]$. It is easy to see that $\alpha(w) < \frac{1}{2}$ if and only if $a_1 > 1$. Thus we need only check that $a_1 \neq 1$. The set of $w \in R$ for which this true is just $R \setminus Y$, where $Y \subseteq \Sigma_{\#}^\omega$ is given by the regular expression $\#10^*(\#(0 \cup 1)^*)^\omega$. ◀

3.2 #-Ostrowski-representations

We now extend the #-binary coding to Ostrowski representations.

▶ **Definition 18.** *Let $v, w \in (\Sigma_\#)^\omega$, let $x = x_1x_2x_3\cdots \in \mathbb{N}^\omega$ and let $b = b_1b_2b_3\cdots \in (\{0,1\}^*)^\omega$ be such that $w = C_\#(b)$ and $[b_i]_2 = x_i$ for each i.*

- *For $N \in \mathbb{N}$, we say that w is a **#-v-Ostrowski representation of X** if v and w are aligned and x is an $\alpha(v)$-Ostrowski representation of N.*
- *For $c \in I_{\alpha(v)}$, we say that w is a **#-v-Ostrowski representation of c** if v and w are aligned and x is an $\alpha(v)$-Ostrowski representation of c.*

We let A_v denote the set of all words $w \in \Sigma_\#^\omega$ such that w is a #-v-Ostrowski representation of some $c \in I_{\alpha(v)}$, and similarly, by A_v^{fin} the set of all words $w \in \Sigma_\#^\omega$ such that w is a #-v-Ostrowski representation of some $N \in \mathbb{N}$.

▶ **Lemma 19.** *The sets*

$$A^{\text{fin}} := \{(v,w) \ : \ v \in R, w \in A_v^{\text{fin}}\}, \text{ and } A := \{(v,w) \ : \ v \in R, w \in A_v\}.$$

are ω-regular. Moreover, $A^{\text{fin}} \subseteq A$.

The straightforward proof is in Appendix A.

▶ **Definition 20.** *Let $v \in R$. We define $Z_v : A_v^{\text{fin}} \to \mathbb{N}$ to be the function that maps w to the natural number whose #-v-Ostrowski representation is w.*
Similarly, we define $O_v : A_v \to I_{\alpha(v)}$ to be the function that maps w to the real number whose #-v-Ostrowski representation is w.

▶ **Lemma 21.** *Let $v \in R$. Then $Z_v : A_v^{\text{fin}} \to \mathbb{N}$ and $O_v : A_v \to I_{\alpha(v)}$ are bijective.*

The proof is in Appendix A.

▶ **Definition 22.** *Let $v \in R$. We write $\mathbf{0}_v$ for $Z_v^{-1}(0)$, and $\mathbf{1}_v$ for $Z_v^{-1}(1)$.*

▶ **Lemma 23.** *The relations $\mathbf{0}_* = \{(v, \mathbf{0}_v) \ : \ v \in R\}$ and $\mathbf{1}_* = \{(v, \mathbf{1}_v) \ : \ v \in R\}$ are ω-regular.*

▶ **Lemma 24.** *Let $s \in A_v^{\text{fin}}$. Then $\alpha(v)Z_v(s) - O_v(s) \in \mathbb{Z}$ and*

$$O_v(\mathbf{1}_v) = \begin{cases} \alpha(v) & \text{if } \alpha(v) < \frac{1}{2}; \\ \alpha(v) - 1 & \text{otherwise.} \end{cases}$$

Again, proofs of these lemmas are in Appendix A.

▶ **Lemma 25.** *The sets*

$$\prec^{\text{fin}} := \{(v,s,t) \in \Sigma_\#^3 \ : \ s,t \in A_v^{\text{fin}} \wedge Z_v(s) < Z_v(t)\},$$
$$\prec := \{(v,s,t) \in \Sigma_\#^3 \ : \ s,t \in A_v \wedge O_v(s) < O_v(t)\}$$

are ω-regular.

Proof of Lemma 25. For $\prec^{\text{fin}}$, first recall that for $X, Y \in \mathbb{N}$ and α irrational, we have $X < Y$ if and only if the α-Ostrowski representation of X is co-lexicographically smaller than the α-Ostrowski representation of Y. Therefore, we need only recognize co-lexicographic ordering on the list of coefficients, with each coefficient ordered according to binary. This follows immediately from Fact 10.

For $\prec$, note that by Fact 9 the usual order on real numbers corresponds to the alternating lexicographic ordering on real Ostrowski representations. Therefore, we need only recognize the alternating lexicographic ordering on the list of coefficients, with each coefficient ordered according to binary. This follows immediately from Fact 10. ◀

We consider $\mathbb{R}^n$ as a topological space using the usual order topology. For $X \subseteq \mathbb{R}^n$, we denote its topological closure by $\overline{X}$.

▶ **Corollary 26.** *Let $W \subseteq (\Sigma_{\#}^{n+1})^*$ ω-regular be such that $W \subseteq \{(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^* \ : \ s_1, \dots, s_n \in A_v\}$. Then the following set is also ω-regular:*

$$\overline{W} := \{(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^* \ : \ s_1, \dots, s_n \in A_v \wedge (O_v(s_1), \dots, O_v(s_n)) \in \overline{O(W_v)}\}.$$

Proof. Let $(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^*$ be such that $s_1, \dots, s_n \in A_v$. Let $X_i = O_v(s_i)$. By the definition of the topological closure, we have that $(X_1, \dots, X_n) \in \overline{O(W_v)}$ if and only if for all $Y_1, \dots Y_n, Z_1, \dots, Z_n \in \mathbb{R}$ with $Y_i < X_i < Z_i$ for $i = 1, \dots, n$ there are $X' = (X'_1, \dots, X'_n) \in O(W_v)$ such that $Y_i < X'_i < Z_i$ for $i = 1, \dots, n$. Thus by Lemma 25, $(v, s_1, \dots, s_n) \in \overline{W}$ if and only if for all $t_1, \dots t_n, u_1, \dots, u_n \in A_v$ with $t_i \prec s_i \prec u_i$, there are $s' = (s'_1, \dots, s'_n) \in W_v$ such that $t_i \prec s'_i \prec Z_i$ for $i = 1, \dots, n$. The latter condition is ω-regular by Fact 2. ◀

4 Recognizing addition in Ostrowski numeration systems

The key to the rest of this paper is a general automaton for recognizing addition of Ostrowski representations uniformly. We will prove the following.

▶ **Theorem 27.** *The set $\oplus^{\text{fin}} := \{(v, s_1, s_2, s_3) \ : \ s_1, s_2, s_3 \in A_v^{\text{fin}} \wedge Z_v(s_1) + Z_v(s_2) = Z_v(s_3)\}$ is ω-regular.*

Proof. In [2, Section 3] the authors generate an automaton $\mathcal{A}$ over the alphabet $\mathbb{N}^4$ such that a finite word $(d_1, x_1, y_1, z_1)(d_2, x_2, y_2, z_2) \cdots (d_m, x_m, y_m, z_m) \in (\mathbb{N}^4)^*$ is accepted by $\mathcal{A}$ if and only if there are $d_{m+1}, \dots \in \mathbb{N}$ and $x, y, z \in \mathbb{N}$ such that for $\alpha = [0; d_1, d_2, \dots]$ we have $[x_1 x_2 \dots x_m]_\alpha = x$, $[y_1 y_2 \dots y_m]_\alpha = y$, $[z_1 z_2 \dots z_m]_\alpha = z$, and $x + y = z$.

We now describe how to adjust the the automaton $\mathcal{A}$ for our purposes. The input alphabet will be $\Sigma_{\#}^4$ instead of $\mathbb{N}^4$. The new automaton will take four inputs w, s_1, s_2, s_3, where $s_1, s_2, s_3 \in A_w^{\text{fin}}$. We can construct this automaton as follows:

1. Begin with the automaton $\mathcal{A}$ from [2].
2. Add an initial state that transitions to the original start state on $(\#, \#, \#, \#)$. This will ensure that the first character in each string is $\#$.
3. Replace each transition in the automaton with a sub-automaton that recognizes the corresponding relationship between w, s_1, s_2, s_3 in binary. As an example, one of the transitions in Figure 3 of [2] is given as "$-d_i + 1$," meaning that it represents all cases where, letting $w_i, s_{1i}, s_{2i}, s_{i3}$ be the ith letter of w, s_1, s_2, s_3 respectively, we have $s_{3i} - s_{1i} - s_{2i} = -w_i + 1$. This is an affine and hence an automatic relation. Thus it can be recognized by a sub-automaton.
4. The accept states in the resulting automaton are precisely the accept states from the original automaton.

The resulting automaton recognizes $\oplus^{\text{fin}}$. ◀

The automaton constructed above has 82 states. Using our software Pecan, we can formally check that this automaton recognizes the set in Theorem 27. Following a strategy already used in Mousavi, Schaeffer, and Shallit [14, Remark 2.1] we check that our adder satisfies the standard recursive definition of addition on the natural numbers; that is for all $x, y \in \mathbb{N}$

$$\begin{aligned} 0 + y &= y \\ s(x) + y &= s(x + y) \end{aligned}$$

where $x, y \in \mathbb{N}$ and $s(x)$ denotes the successor of x in $\mathbb{N}$. The successor function on $\mathbb{N}$ can be defined using only $<$ as follows:

$s(x) = y$ if and only if $(x < y) \wedge (\forall z\ (z \leq x) \vee (z \geq y))$.

Thus in Pecan we define `bco_succ(a,x,y)` as

```
bco_succ(a,x,y) := bco_valid(a,x) ∧ bco_valid(a,y) ∧ bco_leq(x,y)
∧ ¬bco_eq(x,y) ∧ ∀z. bco_valid(a,z) => (bco_leq(z,x) ∨ bco_leq(y,z))
```

where `bco_eq` recognizes $\{(x, y) : x = y\}$, `bco_leq` recognizes $\{(x, y) : x \leq_{\text{colex}} y\}$, and `bco_valid` recognizes A_{fin}. We now confirm that our adder satisfies the above equations using the following Pecan code:

```
Let x,y,z be ostrowski(a).
Theorem ("Addition base case (0 + y = y).", {
    ∀a. ∀x,y,z. bco_zero(x) => (bco_adder(a,x,y,z) ⇔ bco_eq(y,z))
}).
Theorem ("Addition inductive case (s(x) + y = s(x + y)).", {
    ∀a. ∀x,y,z,u,v. (bco_succ(a,u,x) ∧ bco_succ(a,v,z))
                    => (bco_adder(a,x,y,z) ⇔ bco_adder(a,u,y,v))
}).
```

In the above code `bco_adder` recognizes $\oplus^{\text{fin}}$, `bco_zero` recognizes $\mathbf{0}_*$, and `bco_succ` recognizes $\{(v, x, y) : x, y \in A_v^{\text{fin}}, Z_v(x) + 1 = Z_v(y)\}$. Pecan confirms both statements are true. This proves Theorem 27 modulo correctness of Pecan and the correctness of the implementations of the automata for `bco_eq`, `bco_leq`, `bco_valid` and `bco_zero`. For more details about Pecan, see Section 7.

Using Corollary 26 we can extend the automaton in Theorem 27 to an automaton for addition modulo 1 on I_α. The details are in Appendix B.

▶ **Lemma 28.** *The set* $\oplus := \{(v, s_1, s_2, s_3) : s_1, s_2, s_3 \in A_v \wedge O_v(s_1) + O_v(s_2) \equiv O_v(s_3) \pmod 1\}$ *is ω-regular. Moreover,* $\oplus^{\text{fin}} \subseteq \oplus$.

5 The uniform ω-regularity of $\mathcal{R}_\alpha$

In this section, we turn to the question of the decidability of the logical first-order theory of $\mathcal{R}_\alpha$. Recall that $\mathcal{R}_\alpha := (\mathbb{R}, <, +, \mathbb{Z}, \alpha\mathbb{Z})$ for $\alpha \in \mathbb{R}$. The main result of this section is the following:

▶ **Theorem 29.** *There is a uniform family of ω-regular structures $(\mathcal{D}_v)_{v \in R}$ such that $\mathcal{D}_v \simeq \mathcal{R}_{\alpha(v)}$ for each $v \in R$.*

Theorem 29 then hinges on the following lemma.

▶ **Lemma 30.** *There is a uniform family of ω-regular structures $(\mathcal{C}_a)_{a \in R}$ such that $\mathcal{C}_a \simeq ([-\alpha(a), \infty), <, +, \mathbb{N}, \alpha(a)\mathbb{N})$ for each $a \in R$.*

The proof of Lemma 30 is a uniform version of the argument given in [7] that also fixes some minor errors of the original proof. By Lemma 10 and Theorem 27, we already know that $Z_v : (A_v^{\text{fin}}, \prec_v^{\text{fin}}, \oplus_v^{\text{fin}}) \to (\mathbb{N}, <, +)$ is an isomorphism for every $v \in R$. As our eventual goal also requires us to define the set $\alpha\mathbb{N}$, it turns out to be much more natural to instead use the isomorphism $\alpha(v)Z_v : (A_v^{\text{fin}}, \prec_v^{\text{fin}}, \oplus_v^{\text{fin}}) \to (\alpha(v)\mathbb{N}, <, +)$ and recover $\mathbb{Z}$. We do so by following the argument in [7]. The full proof is availabe in Appendix C.

Proof of Theorem 29. We just observe that $([-\alpha, \infty), <, +, \mathbb{N}, \alpha\mathbb{N})$ defines (in a matter uniform in α) an isomorphic copy of $\mathcal{R}_\alpha$. Now apply Lemma 30. ◀

6 Decidability results

We are now ready to prove the results listed in the introduction. We first recall some notation. Let $\mathcal{L}_m$ be the signature of the first-order structure $(\mathbb{R}, <, +, \mathbb{Z})$, and let $\mathcal{L}_{m,a}$ be the extension of $\mathcal{L}_m$ by a unary predicate. For $\alpha \in \mathbb{R}_{>0}$, let $\mathcal{R}_\alpha$ denote the $\mathcal{L}_{m,a}$-structure $(\mathbb{R}, <, +, \mathbb{Z}, \alpha\mathbb{Z})$. For each $\mathcal{L}_{m,a}$-sentence φ, we set $R_\varphi := \{v \in R \;:\; \mathcal{R}_{\alpha(v)} \models \varphi\}$.

▶ **Theorem 31.** *Let φ be an $\mathcal{L}_{m,a}$-sentence. Then R_φ is ω-regular.*

Proof. By Theorem 29 there is a uniform family of ω-regular structures $(\mathcal{D}_v)_{v\in R}$ such that such that $\mathcal{D}_v \simeq \mathcal{R}_{\alpha(v)}$ for each $v \in R$. Then $R_\varphi = \{v \in R \;:\; \mathcal{D}_v \models \varphi\}$. This set is ω-regular by Fact 4. ◀

Let $\mathcal{N} = (R; (R_\varphi)_\varphi, (X)_{X\subseteq R^n \text{ } \omega\text{-regular}})$ be the relational structure on R with the relations R_φ for every $\mathcal{L}$-sentences φ and $X \subseteq R^n$ ω-regular. Because $\mathcal{N}$ is an ω-regular structure, the theory of $\mathcal{N}$ is decidable.

We now proceed towards the proof of Theorem C. Recall that $\mathbf{Irr} := (0, 1) \setminus \mathbb{Q}$.

▶ **Definition 32.** *Let $X \subseteq \mathbf{Irr}^{\mathrm{n}}$. Let X_R be defined by*

$$X_R := \{(v_1, \dots, v_n) \in R^n \;:\; v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n \wedge (\alpha(v_1), \dots, \alpha(v_n)) \in X\}$$

*We say X is **recognizable modulo $\sim_\#$** if X_R is ω-regular.*

▶ **Lemma 33.** *The collection of sets recognizable modulo $\sim_\#$ is closed under Boolean operations and coordinate projections.*

Proof. Let $X, Y \subseteq \mathbf{Irr}$ be recognizable modulo $\sim_\#$. It is clear that $(X \cap Y)_R = X_R \cap Y_R$. Thus $X \cap Y$ is recognizable modulo $\sim_\#$. Let X^c be $\mathbf{Irr}^{\mathrm{n}} \setminus \mathrm{X}$, the complement of X. For ease of notation, set $E := \{(v_1, \dots, v_n) \in R^n \;:\; v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n\}$. Then

$$\begin{aligned}(X^c)_R &= \{(v_1, \dots, v_n) \in R^n \;:\; v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n \wedge (\alpha(v_1), \dots, \alpha(v_n)) \notin X\} \\ &= E \cap \{(v_1, \dots, v_n) \in R^n \;: (\alpha(v_1), \dots, \alpha(v_n)) \notin X\} \\ &= E \cap \{(v_1, \dots, v_n) \in R^n \;: (\alpha(v_1), \dots, \alpha(v_n)) \notin X \vee \neg(v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n)\} \\ &= E \cap (R^n \setminus X_R).\end{aligned}$$

This set is ω-regular, and hence X^c is recognizable modulo $\sim_\#$.

For coordinate projections, it is enough to consider projections onto the first $n-1$ coordinates. Let $n > 0$ and let π be the coordinate projection onto first $n-1$ coordinates. Observe that $\pi(X) = \{(\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{R}^{n-1} \;:\; \exists \alpha_n \in \mathbb{R}\ (\alpha_1, \dots, \alpha_{n-1}, \alpha_n) \in X\}$. Thus $\pi(X)_R$ is equal to $\{(v_1, \dots, v_{n-1}) \in R^{n-1} : v_1 \sim_\# \cdots \sim_\# v_{n-1} \wedge \exists \alpha_n : (\alpha(v_1), \dots, \alpha(v_{n-1}), \alpha_n) \in X\}$. Note that $v \mapsto \alpha(v)$ is a surjection $R \twoheadrightarrow (0, 1) \setminus \mathbb{Q}$. Thus $\pi(X)_R$ is also equal to:

$$\{(v_1, \dots, v_{n-1}) \in R^{n-1} : v_1 \sim_\# \cdots \sim_\# v_{n-1} \wedge \exists v_n : (\alpha(v_1), \dots, \alpha(v_n)) \in X\}.$$

Unfortunately, this set is not necessarily equal to $\pi(X_R)$. There might be tuples $(v_1, \dots, v_{n-1})$ such that no v_n can be found, because it would require more bits in one of its coefficients than $v_1, \dots, v_{n-1}$ have for that coefficient. But $\pi(X_R)$ always contains *some* representation of $\alpha(v_1), \dots, \alpha(v_{n-1})$ with the appropriate number of digits. We need only ensure that removal of trailing zeroes does not affect membership in the language. Thus $\pi(X)_R$ is just the zero-closure of $\pi(X_R)$. Thus $\pi(X)_R$ is ω-regular by Lemma 14. ◀

▶ **Theorem 34.** *Let $X_1, \ldots, X_n$ be recognizable modulo $\sim_\#$ by Büchi automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and let $\mathcal{Q}$ be the structure $(\boldsymbol{Irr}; \mathrm{X}_1, \ldots, \mathrm{X}_n)$. Then the theory of $\mathcal{Q}$ is decidable.*

Proof. By Lemma 33 every set definable in $\mathcal{Q}$ is recognizable modulo $\sim_\#$. Moreover, for each definable set Y the automaton that recognizes Y modulo $\sim_\#$, can be computed from the automata $\mathcal{A}_\infty, \ldots, \mathcal{A}_\backslash$. Let ψ be a sentence in the signature of $\mathcal{Q}$. Without loss of generality, we can assume that ψ is of the form $\exists x\ \chi(x)$. Set $Z := \{a \in \mathbf{Irr}^{\mathrm{n}} : \mathcal{Q} \models \chi(\mathrm{a})\}$. Observe that $\mathcal{Q} \models \psi$ if and only if Z is non-empty. Note for every $a \in \mathbf{Irr}^{\mathrm{n}}$ there are $v_1, \ldots, v_n \in R$ such that $v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n$ and $(\alpha(v_1), \ldots, \alpha(v_n)) = a$. Thus Z is non-empty if and only if $\{(v_1, \ldots, v_n) \in R^n : v_1 \sim_\# v_2 \sim_\# \cdots \sim_\# v_n \wedge (\alpha(v_1), \ldots, \alpha(v_n)) \in Z\}$ is non-empty. Thus to decide whether $\mathcal{Q} \models \psi$, we first compute the automaton $\mathcal{B}$ that recognizes Z modulo $\sim_\#$, and then check whether the automaton accepts any word. ◀

We are now ready to prove Theorem C; that is decidability of the theory of the structure $\mathcal{M} = (\mathbf{Irr}, <, (\mathrm{M}_\varphi)_\varphi, (\mathrm{q})_{\mathrm{q} \in \mathbf{Irr}_{\mathrm{quad}}})$, where M_φ is defined for each $\mathcal{L}_{m,a}$-formula as $M_\varphi := \{\alpha \in \mathbf{Irr} : \mathcal{R}_\alpha \models \varphi\}$.

Proof of Theorem C. We just need to check that the relations we are adding are all recognizable modulo $\sim_\#$. By Lemma 15 the ordering $<$ is recognizable modulo $\sim_\#$. By Lemma 16, the singleton $\{q\}$ is is recognizable modulo $\sim_\#$ for every $q \in \mathbf{Irr}_{\mathrm{quad}}$. Since $M_\varphi = \alpha(R_\varphi)$, recognizability of M_φ modulo $\sim_\#$ follows from Theorem 31. ◀

We can add to $\mathcal{M}$ a predicate for every subset of $\mathbf{Irr}^{\mathrm{n}}$ that is recognizable modulo $\sim_\#$, and preserve the decidability of the theory. The reader can check that examples of subsets of $\mathbf{Irr}$ recognizable modulo $\sim_\#$ are the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are powers of 2, the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are in (or are not in) some fixed finite set, and the set of all $\alpha \in \mathbf{Irr}$ such that all even (or odd) terms in their continued fraction expansion are 1.

7 Automatically Proving Theorems about Sturmian Words

We have created an automatic theorem-prover based on the ideas and the decision algorithms outlined above, called Pecan [15]. We use Pecan to provide proofs of known and unknown results about characteristic Sturmian words. We begin by giving automated proofs for several classical result result about Sturmian words. We refer the reader to [12] for more information and traditional proofs of these results.

In the following, we assume that $a \in \mathbb{R}$ and i, j, k, n, m, p, s are a-Ostrowski representations. This can be expressed in Pecan as

```
Let a ∈ bco_standard.
Let i,j,k,n,m,p,s ∈ ostrowski(a).
```

We write $c_{a,0}(i)$ as `$C[i]` in Pecan.

▶ **Theorem 35.** *Characteristic Sturmian words are balanced and aperiodic.*

Proof of Theorem 35. To show that a characteristic Sturmian word $c_{\alpha,0}$ is balanced, note that it is sufficient to show that there is no palindrome w in $c_{\alpha,0}$ such that $0w0$ and $1w1$ are in $c_{\alpha,0}$ (see [12, Proposition 2.1.3]). We encode this in Pecan as follows. The predicate `palindrome(a,i,n)` is true when $c_{a,0}[i..i+n] = c_{a,0}[i..i+n]^R$. The predicate `factor_len(a,i,n,j)` is true when $c_{a,0}[i..i+n] = c_{a,0}[j..j+n]$.

```
Theorem ("Balanced", {
∀a. ¬(∃i,n. palindrome(a,i,n) ∧
            (∃j. factor_len(a,i,n,j) ∧ $C[j-1] = 0 ∧ $C[j+n] = 0) ∧
            (∃k. factor_len(a,i,n,k) ∧ $C[k-1] = 1 ∧ $C[k+n] = 1))
}).
```

Pecan takes `321.73` seconds to prove the theorem.

Encoding the property that a word is eventually periodic is straightforward:

```
eventually_periodic(a, p) :=
    p > 0 ∧ ∃n. ∀i. if i > n then $C[i] = $C[i+p]
```

The resulting automaton has 4941 states and 35776 edges, and takes `117.78` seconds to build. We then state the theorem in Pecan, which confirms the theorem is true.

```
Theorem ("Aperiodic", {
    ∀a. ∀p. if p > 0 then ¬eventually_periodic(a, p)
}) .
```

◀

Let $w \in \{0,1\}^*$. We let $\overline{w}$ denote the $\{0,1\}$-word obtained by replacing each 1 in w by 0 and each 0 in w by 1. A word $w \in \{0,1\}^*$ is an **antisquare** if $w = v\overline{v}$ for some $v \in \{0,1\}^*$. We define $A_O : (0,1) \setminus \mathbb{Q} \to \mathbb{N} \cup \{\infty\}$ to map an irrational α to the maximum order of any antisquare in $c_{\alpha,0}$ if such a maximum exists, and to ∞ otherwise. We let $A_L : (0,1) \setminus \mathbb{Q} \to \mathbb{N} \cup \{\infty\}$ map α to the maximum length of any antisquare in $c_{\alpha,0}$ if such a maximum exists and ∞ otherwise. Note that $A_L(\alpha) = 2A_O(\alpha)$.

We let w^R denote the reversal of a word w. We say a word w is a **palindrome** if $w = w^R$. A word $w \in \{0,1\}^*$ is an **antipalindrome** if $w = \overline{w^R}$. We set $A_P : (0,1) \setminus \mathbb{Q} \to \mathbb{N} \cup \{\infty\}$ to be the map that takes an irrational α to the maximum length of any antipalindrome in $c_{\alpha,0}$ if such a maximum, and to ∞ otherwise. We will use Pecan to prove that $A_O(\alpha)$, $A_L(\alpha)$ and $A_P(\alpha)$ are finite for every α. While the quantities $A_O(\alpha)$, $A_P(\alpha)$ and $A_L(\alpha)$ can be arbitrarily large, we prove the new results that the length of the Ostrowski representations of these quantities is bounded, independent of α.

Let $\alpha \in (0,1)$ be irrational and $N \in \mathbb{N}$. Let $|N|_\alpha$ denote the length of the α-Ostrowski representation of N, that is the index of the last nonzero digit of α-Ostrowski representation of N, or 0 otherwise.

▶ **Theorem 36.** *For every irrational $\alpha \in (0,1)$,*

$$|A_O(\alpha)|_\alpha \le 4,\ \ |A_P(\alpha)|_\alpha \le 4,\ \ |A_L(\alpha)|_\alpha \le 6, A_O(\alpha) \le A_P(\alpha) \le A_L(\alpha) = 2A_O(\alpha).$$

There are irrational numbers $\alpha, \beta \in (0,1)$ such that $A_O(\alpha) = A_P(\alpha)$ and $A_P(\beta) = A_L(\beta)$.

Proof. Using Pecan, we create automata which compute A_O, A_P, and A_L:

$$A_O(\alpha, n) := \texttt{has\_antisquare}(\alpha, n) \wedge \forall m.\texttt{has\_antisquare}(\alpha, m) \implies m \le n$$
$$A_P(\alpha, n) := \texttt{has\_antipalindrome}(\alpha, n) \wedge \forall m.\texttt{has\_antipalindrome}(\alpha, m) \implies m \le n$$
$$A_L(\alpha, n) := \texttt{has\_antisquare\_len}(\alpha, n) \wedge \forall m.\texttt{has\_antisquare\_len}(\alpha, m) \implies m \le n$$

We build automata recognizing α-Ostrowski representations of at most 4 and 6 nonzero digits, called `has_4_digits`(n) and `has_6_digits`(n). Then we use Pecan to prove all the parts of the theorem by checking the following statement.

```
Theorem ("(i), (ii), (iii), and (iv)", {
∀a. has_4_digits(max_antisquare(a)) ∧
    has_4_digits(max_antipalindrome(a)) ∧
    has_6_digits(max_antisquare_len(a)) ∧
    max_antisquare(a) <= max_antipalindrome(a) ∧
    max_antipalindrome(a) <= max_antisquare_len(a)
}).
```

We also use Pecan to find examples of the equality: when $\alpha = [0; 3, 3, \overline{1}]$, we have $A_O(\alpha) = A_P(\alpha) = 2$, and when $\alpha = [0; 4, 2, \overline{1}]$, we have $A_P(\alpha) = A_L(\alpha) = 2$. ◀

▶ **Theorem 37.** *For every irrational $\alpha \in (0, 1)$, all antisquares and antipalindromes in $c_{\alpha,0}$ are either of the form $(01)^*$ or of the form $(10)^*$.*

Proof. We begin by creating a predicate called `is_all_01` stating that a subword $c_{\alpha,0}[i..i+n]$ is of the form $(01)^*$ or $(10)^*$. We do this simply stating that $c_{\alpha,0}[k] \neq c_{\alpha,0}[k+1]$ for all k with $i \leq k < i + n - 1$.

```
is_all_01(a,i,n) :=
    ∀k. if i <= k ∧ k < i+n-1 then $C[k] ≠ $C[k+1]
```

We can now directly state both parts of the theorem; Pecan proves both in `76.1` seconds.

```
Theorem ("All antisquares are of the form (01)^* or (10)^*", {
∀a. ∀i,n. if antisquare(a,i,n) then is_all_01(a,i,n)
}).

Theorem ("All antipalindromes are of the form (01)^* or (10)^*", {
∀a. ∀i,n. if antipalindrome(a,i,n) then is_all_01(a,i,n)
}).
```

◀

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A Proofs from Section 2

Proof of Lemma 12. By the definition of R, there is $w_1w_2\cdots \in ((0|1)^*1(0|1)^*)^\omega$ such that $w = \#w_1\#w_2\#\cdots$. Since $w_i \in (0|1)^*1(0|1)^*$, we have that w_i is a $\{0,1\}$-word containing at least one 1. Let a_i be the natural number that $a_i = [w_i]_2$. Because w_i contains a 1, we must have $a_i \neq 0$. Thus w is a $\#$-binary coding of the infinite continued fraction of the irrational $\alpha = [0; a_1, a_2, \dots]$. Uniqueness follows directly from the fact that both binary expansions and continued fraction expansions only represent one number. ◀

Proof of Lemma 14. Let $\mathcal{A}$ be a Büchi automaton recognizing X. We use Q to denote the set of states of $\mathcal{A}$. We create a new automaton $\mathcal{A}'$ that recognizes the zero-closure of X, as follows:

(Step 1) Start with the automata $\mathcal{A}$.

(Step 2) For each transition on the n-tuple $(\#,\dots,\#)$ from a state p to a state q, we add a new state $\mu(p,q)$ that loops to itself on the n-tuple $(0,\dots,0)$ and transitions to state q on $(\#,\dots,\#)$. We add a transition from p to $\mu(p,q)$ on $(0,\dots,0)$.

(Step 3) For every pair p, q of states of $\mathcal{A}$ for which p has a run to q on a word of the form $(0,\ldots,0)^m(\#,\ldots,\#)$ for some m, we add a transition from state p to a new state $\nu(p,q)$ on $(\#,\ldots,\#)$, and for every transition out of state q, we create a copy of the transition that starts at state $\nu(p,q)$ instead. If any original run from state p to state q passes through a final state, we make $\nu(p,q)$ a final state.

(Step 4) Denote the resulting automaton by $\mathcal{A}'$ and its set of states by Q'.

We now show that $L(\mathcal{A}')$ is the zero-closure of X. We first show that the zero-closure is contained in $L(\mathcal{A}')$. Let $v \in X$ and $w \in R$ be such that $\alpha(v) = \alpha(w)$. Let $b = b_1b_2\cdots, c = c_1c_2 \in (\{0,1\}^*)^\omega$ such that $C_\#(b) = v$ and $C_\#(c) = w$. Since $\alpha(v) = \alpha(w)$, we have that $[b_i]_2 = [c_i]_2$ for $i \in \mathbb{N}$. Therefore, for each $i \in \mathbb{N}$, the words b_i and c_i only differ by trailing zeroes. Let $s = s_1s_2\cdots \in Q^\omega$ be an accepting run of v on $\mathcal{A}$. We now transfer this run into an accepting run $s' = s'_1s'_2\cdots$ of w on $\mathcal{A}'$. For $i \in \mathbb{N}$, let $y(i)$ be the position of the i-th $(\#,\ldots,\#)$ in v and let $z(i)$ be the position of the i-th $(\#,\ldots,\#)$ in w. For each $i \in \mathbb{N}$, we define a sequence $s'_{z(i)+1}\cdots s'_{z(i+1)}$ of states of $\mathcal{A}'$ as follows:

1. If $|c_i| = |b_i|$, then $c_i = b_i$. We set
$$s'_{z(i)+1}\cdots s'_{z(i+1)} := s_{y(i)+1}\cdots s_{y(i+1)}.$$
2. If $|c_i| > |b_i|$, then $c_i = b_i(0,\ldots,0)^{|c_i|-|b_i|}$. We set
$$s'_{z(i)+1}\cdots s'_{z(i+1)}$$
$$:= s_{y(i)+1}\cdots s_{y(i+1)-1}\underbrace{\mu(s_{y(i+1)-1}, s_{y(i+1)})\cdots\mu(s_{y(i+1)-1}, s_{y(i+1)})}_{(|c_i|-|b_i|)\text{-times}}s_{y(i+1)}$$
Thus the new run follows the old run up to $s_{y(i+1)-1}$ and then transitions to one of the newly added states in the Step 2. It loops on $(0,\ldots,0)$ for $|c_i| - |b_i| - 1$-times before moving to $s_{y(i+1)}$.
3. If $|c_i| < |b_i|$, then $b_i = c_i(0,\ldots,0)^{|b_i|-|c_i|}$. We set
$$s'_{z(i)+1}\cdots s'_{z(i+1)} := s_{y(i)+1}\cdots s_{y(i)+|c_i|}\nu(s_{y(i)+|c_i|}, s_{y(i+1)})$$
The new run utilizes one of the newly added $(\#,\ldots,\#)$ transitions and corresponding states added in Step 3.

The reader can now easily check that s' is an accepting run of w on $\mathcal{A}'$.

We now show that $L(\mathcal{A}')$ is contained in the zero-closure of X. We prove that the only accepting runs on $\mathcal{A}'$ are based on accepting runs on $\mathcal{A}$ with trailing zeroes either added or removed. Let $w = w_1w_2\cdots \in L(\mathcal{A}')$ and let $c = c_1c_2\cdots \in (\{0,1\}^*)^\omega$ be such that $C_\#(c) = w$. Let $s' = s'_1s'_2\cdots \in Q'^\omega$ be an accepting run of w on $\mathcal{A}'$. We construct $v \in X$ and a run $s = s_1s_2\cdots \in Q^\omega$ of w_2 on $\mathcal{A}$ such that $\alpha(v) = \alpha(w)$ and s is an accepting run of v. We start by setting $v := w_1w_2\cdots$ and $s := s'_1s'_2\cdots$. For each $i \in \mathbb{N}$, we replace w_i in v and s'_i in s as follows:

1. If $s'_i \in Q$, then we make no changes to s'_i and w_i.
2. If $s'_i = \mu(p,q)$ for some $p, q \in Q$, we delete the s'_i in s and delete w_i in v.
3. If $s_i = \nu(p,q)$ for some $p, q \in Q$, then we replace
   (a) s'_i by a run $t = t_1\cdots t_{n+1}$ of $(0,\ldots,0)^n(\#,...,\#)$ from p to q, and
   (b) w_i by $(0,\ldots,0)^n(\#,...,\#)$.
   If $\nu(p,q)$ is a final state of $\mathcal{A}'$, we choose t such that it passed through a final state of $\mathcal{A}$.

It is clear that the resulting s is in Q^ω. The reader can check s is an accepting run of v on $\mathcal{A}$ and that $\alpha(v) = \alpha(w)$. Thus w is in the zero-closure of X. ◀

Proof of Lemma 19. The statement that $A^{\text{fin}} \subseteq A$, follows immediately from the definitions of A^{fin} and A and Fact 8. It is left to establish the ω-regularity of the two sets.

For (1): Let $B \supseteq A^{\text{fin}}$ be the set of all pairs (v, w) such that $v \in R$ and $v \sim_{\#} w$. Note that B is ω-regular. Let $(v, w) \in B$. Since v and w have infinitely many # characters and are aligned, there are unique $a = a_1 a_2 \cdots, b = b_1 b_2 \cdots \in (\{0,1\}^*)^\omega$ such that $C_{\#}(a) = v$, $C_{\#}(b) = w$ and $|a_i| = |b_i|$ for each $i \in \mathbb{N}$. Then by Fact 6, $(v, w) \in A^{\text{fin}}$ if and only if

(a) b has finitely many 1 characters;
(b) $b_1 <_{\text{colex}} a_1$;
(c) $b_i \leq_{\text{colex}} a_i$ for all $i > 1$;
(d) if $b_i = a_i$, then $b_{i-1} = 0$.

It is easy to check that all four conditions are ω-regular.

For (2): As above, let $(v, w) \in B$. Since v and w have infinitely many # characters and are aligned, there are unique $a = a_1 a_2 \cdots, b = b_1 b_2 \cdots \in (\{0,1\}^*)^\omega$ such that $C_{\#}(a) = v$, $C_{\#}(b) = w$ and $|a_i| = |b_i|$ for each $i \in \mathbb{N}$. Then by Fact 7, $(v, w) \in A$ if and only if

(e) $b_1 <_{\text{colex}} a_1$;
(f) $b_i \leq_{\text{colex}} a_i$ for all $i > 1$;
(g) if $b_i = a_i$, then $b_{i-1} = 0$;
(h) $b_i \neq a_i$ for infinitely many odd i.

Again, it is easy to see that all for conditions are ω-regular. ◀

Proof of Lemma 21. We first consider injectivity. By Fact 6 and Fact 7 a number in $\mathbb{N}$ or in $I_{\alpha(v)}$ only has one $\alpha(v)$-Ostrowski representation. So we need only explain why such a representation will only have one encoding in A_v^{fin} (respectively A_v). This follows from the uniqueness of binary representations up to the length of the representation, and from the fact that the requirement of having the # characters aligned with v determines the length of each binary-encoded coefficient.

For surjectivity we need only explain why an $\alpha(v)$-Ostrowski representation can always be encoded into a string in A_v^{fin} (respectively A_v). It suffices to show that the requirement of having the # characters aligned with v will never result in needing to fit the binary encoding of a number into too few characters, i.e. that it will never result in having to encode a natural number n in binary in fewer than $1 + \lfloor \log_2 n \rfloor$ characters. Since the function $1 + \lfloor \log_2 n \rfloor$ is monotone increasing, we can encode any natural number below n in k characters if we can encode n in binary in k characters. However, by Fact 6 and Fact 7, the coefficients in an $\alpha(v)$-Ostrowski representation never exceed the corresponding coefficients in the continued fraction for $\alpha(v)$, i.e. $b_n \leq a_n$. ◀

Proof of Lemma 23. Recognizing $\mathbf{0}_*$ is trivial, as the Ostrowski representations of 0 are of the form $0 \cdots 0$ for all irrational α. Thus $\mathbf{0}_*$ is just the relation

$$\{(v, w)\ : v \in R, w \text{ is } v \text{ with all 1 bits replaced by 0 bits}\}.$$

This is clearly ω-regular.

We now consider $\mathbf{1}_*$. Let $\alpha = [0; a_1, a_2, \ldots]$ be an irrational number. If $a_1 > 1$, the α-Ostrowski representations of 1 are of the form $10 \cdots 0$. If $a_1 = 1$, the α-Ostrowski representations of 1 are of the form $010 \cdots 0$. Thus, in order to recognize $\mathbf{1}_*$, we only need to be able to recognize if a number in binary representation is 0, 1, or greater than 1. Of course, this is easily done on a Büchi automaton. ◀

Proof of Lemma 24. By Fact 8, $O_v(s) = f_\alpha(\alpha(v)Z_v(s))$. Thus

$$\alpha(v)Z_v(s) - O_v(s) = \alpha(v)Z_v(s) - f_\alpha(\alpha(v)Z_v(s)),$$

which is an integer by the definition of f. By the definition of $\mathbf{1}_v$ and by Fact 8, we know $O_v(\mathbf{1}_v) = f_\alpha(\alpha)$ is the unique element of $I_{\alpha(v)}$ that differs from $\alpha(v)$ by an integer. If $0 < \alpha(v) < \frac{1}{2}$, then $-\alpha(v) < \alpha(v) < 1 - \alpha(v)$. Thus in this case, $\alpha(v) \in I_{\alpha(v)}$ and $O_v(\mathbf{1}_v) = \alpha(v)$. When $\frac{1}{2} < \alpha(v) < 1$, then $-\alpha < \alpha - 1 < 1 - \alpha$. Therefore $\alpha(v) - 1 \in I_{\alpha(v)}$ and $O_v(\mathbf{1}_v) = \alpha(v) - 1$. ◀

B Proofs from Section 3

Proof of Lemma 28. First, let v, s_1, s_2, s_3 be such that $s_1, s_2, s_3 \in A_v^{\text{fin}}$. We claim that on this domain, $(s_1, s_2, s_3) \in \oplus_v$ if and only if $(s_1, s_2, s_3) \in \oplus_v^{\text{fin}}$. By Fact 8 we know that for all $s \in A_v^{\text{fin}}$

$$\alpha(v)Z_v(s) - O_v(s) \equiv 0 \pmod 1. \tag{3}$$

Let $(s_1, s_2, s_3) \in \oplus_v^{\text{fin}}$. Then by (3)

$$\begin{aligned} O_v(s_3) &\equiv \alpha(v)Z_v(s_3) \pmod 1 \\ &= \alpha(v)Z_v(s_1) + \alpha(v)Z_v(s_2) \\ &\equiv O_v(s_1) + O_v(s_2) \pmod 1. \end{aligned}$$

Thus $(s_1, s_2, s_3) \in \oplus_v$.

Now suppose that $(s_1, s_2, s_3) \in \oplus_v$. Then by (3) and the definition of $\oplus$, we obtain that $\alpha(v)Z(s_1) + \alpha(v)Z(s_2) \equiv \alpha(v)Z(s_3) \pmod 1$. However, then $\alpha(v)(Z(s_1) + Z(s_2) - Z(s_3)) \equiv 0 \pmod 1$. Since α is irrational, we obtain $Z(s_1) + Z(s_2) - Z(s_3) = 0$. Thus $(s_1, s_2, s_3) \in \oplus_v^{\text{fin}}$.

Thus for each $v \in R$, we have $\oplus_v \cap (A_v^{\text{fin}})^3 = \oplus_v^{\text{fin}}$. Let $v \in R$. We observe that the set $O_v(A_v^{\text{fin}})$ is dense in $O_v(A_v)$. Since addition is continuous, it follows that $O_v(\oplus_v^{\text{fin}})$ is dense in $O_v(\oplus_v)$. Since the graph of a continuous function is closed, the topological closure of $O_v(\oplus_v^{\text{fin}})$ is $O_v(\oplus_v)$. Thus $\oplus$ is ω-regular by Corollary 26. ◀

C Proofs from Section 4

In this section we present the proof of Lemma 30. We first state and prove three lemmas used in the proof.

▶ **Lemma 38.** *Let $v \in R$, and let $t_1, t_2, t_3 \in A_v$ be such that $t_1 \oplus_v t_2 = t_3$. Then*

$$O_v(t_1) + O_v(t_2) = \begin{cases} O_v(t_3) + 1 & \text{if } \mathbf{0}_v \prec_v t_1 \text{ and } t_3 \prec_v t_2; \\ O_v(t_3) - 1 & \text{if } t_1 \prec_v \mathbf{0}_v \text{ and } t_2 \prec_v t_3; \\ O_v(t_3) & \text{otherwise.} \end{cases}$$

Proof. For ease of notation, let $\alpha = \alpha(v)$, and set $x_i = O_v(t_i)$ for $i = 1, 2, 3$. By definition of $\oplus_v$, we have that $x_1, x_2, x_3 \in I_{\alpha(v)}$ with $x_1 + x_2 \equiv x_3 \pmod 1$. Note that $t_i \prec_v t_j$ if and only if $x_i < x_j$.

We first consider the case that $0 < x_1$ and $x_3 < x_2$. Thus $x_1 + x_2 > 1 - \alpha$. Note that

$$-\alpha = 1 - \alpha - 1 < x_1 + x_2 - 1 < (1 - \alpha) + (1 - \alpha) - 1 = 1 - 2\alpha < 1 - \alpha.$$

Thus $x_1 + x_2 - 1 \in I_\alpha$ and $x_3 = x_1 + x_2 - 1$.

Now assume that $x_1 < 0$ and $x_2 < x_3$. Then $x_1 + x_2 < -\alpha$, and therefore

$$1 - \alpha > x_1 + x_2 + 1 \geq (-\alpha) + (-\alpha) + 1 = (1 - \alpha) - \alpha > -\alpha.$$

Thus $x_1 + x_2 + 1 \in I_\alpha$ and hence $x_3 = x_1 + x_2 + 1$.

Finally consider that $0, x_1$ are ordered the same way as x_2, x_3. Since $x_1 + x_2 \equiv x_3 \pmod 1$, we know that $|x_1 - 0|$ and $|x_3 - x_2|$ differ by an integer k. If $k > 0$, would imply that one of these differences is at least 1, which is impossible within the interval I_α. Therefore $x_1 - 0 = x_3 - x_2$ and hence $x_3 = x_1 + x_2$. ◀

For $i \in \mathbb{N}$, set $\mathbf{i}_v := \underbrace{\mathbf{1}_v \oplus \cdots \oplus \mathbf{1}_v}_{i \text{ times}}$.

▶ **Lemma 39.** *The set $F := \{(v, s) \in A^{\text{fin}} \ : \ Z_v(s)\alpha(v) < 1\}$ is ω-regular, and for each $(v, s) \in F$*

$$O_v(s) = \begin{cases} \alpha(v)Z_v(s) & \textit{if } (\alpha(v) + 1)Z_v(s) < 1; \\ \alpha(v)Z_v(s) - 1 & \textit{otherwise.} \end{cases}$$

Proof. By Lemma 17, we can first consider the case that $\alpha(v) > \frac{1}{2}$. In this situation, F_v is just the set $\{\mathbf{0}_v, \mathbf{1}_v\}$, and hence obviously ω-regular.

Now assume that $\alpha(v) < \frac{1}{2}$. Let w be the $\prec_v^{\text{fin}}$-minimal element of A_v^{fin} with $w \prec_v \mathbf{0}_v$. We will show that

$$F_v = \{s \in A_v^{\text{fin}} \ : \ s \preceq_v^{\text{fin}} w\}.$$

Then ω-regularity of F follows then immediately.

Let $n \in \mathbb{N}$ be maximal such that $n\alpha(v) < 1$. It is enough to show that $Z_v(w) = n$. By Lemma 24, $O_v(\mathbf{1}_v) = \alpha(v)$. Hence $1\alpha(v), 2\alpha(v), \ldots, (n-1)\alpha(v) \in I_{\alpha(v)}$, but $n\alpha(v) > 1 - \alpha(v)$. Then for $i = 1, \ldots, n - 1$

$$O_v(\mathbf{i}_v) = i\alpha(v), \ O_v(\mathbf{n}_v) = n\alpha(v) - 1 < 0.$$

So $\mathbf{i}_v \succeq \mathbf{0}_v$ for $i = 1, \ldots, n$, but $\mathbf{n}_v \prec \mathbf{0}_v$. Thus $\mathbf{n}_v = w$ and $Z_v(w) = n$. ◀

▶ **Lemma 40.** *Let $v \in R$ and $t \in A_v^{\text{fin}}$. Then there is an $s \in F_v$ and $t' \in A_v^{\text{fin}}$ such that $t' \preceq_v 0$ and $t = t' \oplus_v s$. In particular, $A_v^{\text{fin}} = \{t \in A_v^{\text{fin}} \ : \ t \preceq_v \mathbf{0}_v\} \oplus_v F_v$.*

Proof of Lemma 40. Let $n \in \mathbb{N}$ be maximal such that $n\alpha < 1$. Let $t \in A_v^{\text{fin}}$. We need to find $s \in A_v^{\text{fin}}$ and $u \in F_v$ such that $t = s \oplus_v^{\text{fin}} v$. We can easily reduce to the case that $t \succ \mathbf{0}_v$ and $Z_v(t) > n$.

Let $i \in \{0, \ldots, n\}$ be such that $0 \geq O_v(t) - i\alpha > -\alpha$. Then let $s \in A_v^{\text{fin}}$ be such that $Z_v(s) = Z_v(t) - i$. Note $t = s \oplus_v^{\text{fin}} \mathbf{i}_v$. Thus we only need to show that $s \preceq \mathbf{0}_v$.

To see this, observe that by Lemma 39

$$O_v(s) + \alpha i \equiv O_v(s) + O_v(\mathbf{i}_v) \equiv O_v(t) \pmod 1.$$

Since $O_v(t) - i\alpha(v) \in I_{\alpha(v)}$, we know that $O_v(s) = O_v(t) - i\alpha(v) \leq 0$. Therefore $O_v(s) \preceq \mathbf{0}_v$. ◀

Proof of Lemma 30. Define $B \subseteq A^{\text{fin}}$ to be $\{(v, s) \in A^{\text{fin}} \ : \ s \preceq_v \mathbf{0}_v\}$. Clearly, B is ω-regular. We now define $\prec^B$ and $\oplus^B$ such that for each $v \in R$, the structure $(B_v, \prec_v^B, \oplus_v^B)$ is isomorphic to $(\mathbb{N}, <, +)$ under the map g_v defined as $g_v(s) = \alpha(v)Z_v(s) - O_v(s)$.

We define $\prec^B$ to be the restriction of $\prec^{\text{fin}}$ to B. That is, for $(v,s_1),(v,s_2) \in B$ we have

$(v,s_1) \prec^B (v,s_2)$ if and only if $(v,s_1) \prec^{\text{fin}} (v,s_2)$.

It is immediate that $\prec^B$ is ω-regular, since both B and $\prec^{\text{fin}}$ are ω-regular.

We define $\oplus^B$ as follows:

$$(v,s_1) \oplus^B (v,s_2) = \begin{cases} (v, s_1 \oplus_v s_2) & \text{if } s_1 \oplus_v^{\text{fin}} s_2 \preceq_v \mathbf{0}_v; \\ (v, s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) & \text{otherwise.} \end{cases}$$

We now show that $g_v(s_1 \oplus_v^B s_2) = g_v(s_1) + g_v(s_2)$ for every $s_1, s_2 \in B_v$.

Let $(v,s_1),(v,s_2) \in B$. We first consider the case that $s_1 \oplus_v s_2 \preceq_v \mathbf{0}_v$. By Lemma 38, $O_v(s_1 \oplus_v s_2) = O_v(s_1) + O_v(s_2)$. Thus

$$\begin{aligned} g_v(s_1 \oplus_v^B s_2) &= g_v(s_1 \oplus_v s_2) \\ &= \alpha(v) Z_v(s_1 \oplus_v s_2) - O_v(s_1 \oplus_v s_2) \\ &= \alpha Z_v(s_1) + \alpha Z_v(s_2) - O_v(s_1) - O_v(s_2) \\ &= g_v(s_1) + g_v(s_2). \end{aligned}$$

Now suppose that $s_1 \oplus_v s_2 \succ_v \mathbf{0}_v$. Since $-\alpha(v) \le O_v(s_1), O_v(s_2) \le 0$, we get that

$$1 - \alpha(v) > O_v(s_1) + O_v(s_2) + \alpha(v) \ge -\alpha(v).$$

Thus by Lemma 24,

$$O_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) = O_v(s_1) + O_v(s_2) + \alpha(v).$$

We obtain

$$\begin{aligned} g_v(s_1 \oplus_v^B s_2) &= g_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) \\ &= \alpha Z_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) - O_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) \\ &= \alpha(v)\big(Z_v(s_1) + Z_v(s_2)\big) + \alpha(v) - O_v(s_1) - O_v(s_2) - \alpha(v) \\ &= g_v(s_1) + g_v(s_2). \end{aligned}$$

Since $s_1 \prec_v s_2$ if and only if $Z_v(s_1) < Z_v(s_2)$, we get that g_v is an isomorphism between $(B_v, \prec_v^B, \oplus_v^B)$ and $(\mathbb{N}, <, +)$.

Let C be defined by

$$\{(v,s,t) \in (\Sigma_\#^\omega)^3 \ : \ (v,s) \in B \wedge (v,t) \in A\}.$$

Clearly C is ω-regular. Let $T_v : C_v \to [-\alpha(v), \infty) \subseteq \mathbb{R}$ map $(s,t) \mapsto g_v(s) + O_v(t)$.

Note that T_v is bijective for each $v \in R$, since every real number decomposes uniquely into a sum $n + y$, where $n \in \mathbb{Z}$ and $y \in I_v$.

We define an ordering $\prec_v^C$ on C_v lexicographically: $(s_1,t_1) \prec_v^C (s_2,t_2)$ if either

Table 1 Definitions of sets used in the proof.

Name	Definition
A	$\{(v,w) \ : \ v \in R,\ w \text{ is a } \#\text{-}v\text{-Ostrowski representation}\}$
A^{fin}	$\{(v,w) \ : \ v \in R,\ w \text{ is a } \#\text{-}v\text{-Ostrowski representation and eventually } 0\}$
B	$\{(v,s) \in A^{\text{fin}} \ : \ s \preceq_v \mathbf{0}_v\}$
C	$\{(v,s,t) \ : \ (v,s) \in B \wedge (v,t) \in A\}$

- $s_1 \prec_v^B s_2$, or
- $s_1 = s_2$ and $t_1 \prec_v t_2$.

The set

$$\{(v, s_1, t_1, s_2, t_2) \ : \ (s_1, t_1), (s_2, t_2) \in C_v \land (s_1, t_1) \prec_v^C (s_2, t_2)\}$$

is ω-regular. We can easily check that $(s_1, t_1) \prec_v^C (s_2, t_2)$ if and only if $T_v(s_1, t_1) < T_v(s_2, t_2)$.

Let $\mathbf{0}^B$ be $g_v^{-1}(0)$ and $\mathbf{1}^B$ be $g_v^{-1}(1)$. Let $\ominus^B$ be the (partial) inverse of $\oplus^B$. We define $\oplus^C$ for $(s_1, t_1), (s_2, t_2) \in C$ as follows:

$$(s_1, t_1) \oplus_v^C (s_2, t_2) = \begin{cases} (s_1 \oplus_v^B s_2 \ominus^B \mathbf{1}^B, t_1 \oplus_v t_2) & \text{if } t_1 \prec \mathbf{0}_v \land t_2 \prec_v t_1 \oplus_v t_2; \\ (s_1 \oplus_v^B s_2 \oplus_v^B \mathbf{1}^B, t_1 \oplus_v t_2) & \text{if } \mathbf{0}_v \prec t_1 \land t_1 \oplus_v t_2 \prec_v t_2; \\ (s_1 \oplus_v^B s_2, t_1 \oplus_v t_2) & \text{otherwise.} \end{cases}$$

(Note that $\oplus^C$ is only a partial function, as the case where $s_1 = s_2 = \mathbf{0}^B$ and $t_1 \prec \mathbf{0}_v \land t_2 \prec_v t_1 \oplus_v t_2$ is outside of the domain of $\ominus^B$.) It is easy to check that $\oplus^C$ is ω-regular. It follows directly from Lemma 38 that

$$T_v((s_1, t_1) \oplus_v^C (s_2, t_2)) = T_v((s_1, t_1)) + T_v((s_2, t_2)).$$

Thus for each $v \in R$, the function T_v is an isomorphism between $(C_v, \prec_v^C, \oplus_v^C)$ and $([-\alpha(v), \infty), <, +)$. To finish the proof, it is left to establish the ω-regularity of the following two sets:

1. $\{(v, s, t) \in C \ : \ T_v(s, t) \in \mathbb{N}\}$,
2. $\{(v, s, t) \in C \ : \ T_v(s, t) \in \alpha(u)\mathbb{N}\}$.

For (1), observe that the set $T_v^{-1}(\mathbb{N})$ is just the set $\{(s, t) \in C_v : t = \mathbf{0}_v\}$.

For (2), consider the following two sets:

- $U_1 = \{(v, s, t) \in C \ : \ s = t\}$,
- $U_2 = \{(v, \mathbf{0}_v, t) \in C \ : \ t \in F_v\}$.

Let $\mathbf{1}_v^C$ be $T_v^{-1}(1)$. Set

$$\begin{aligned} U := \{(v, (s_1, t_1) \oplus_v^c (\mathbf{0}_v, t_2)) \ : \ (v, s_1, t_1) \in U_1, (v, \mathbf{0}_v, t_2) \in U_2, t_2 \succeq 0\} \\ \cup \{(v, (s_1, t_1) \oplus_v^c (\mathbf{0}_v, t_2) \oplus \mathbf{1}_v^C) \ : \ (v, s_1, t_1) \in U_1, (v, \mathbf{0}_v, t_2) \in U_2, t_2 \prec 0\} \end{aligned}$$

The set U is clearly ω-regular, since both U_1 and U_2 are ω-regular. We now show that $T_v(U) = \alpha(v)\mathbb{N}$.

Let $(v, s, s) \in U_1$ and $(v, \mathbf{0}_v, t) \in U_2$. If $t \succeq \mathbf{0}_v$, then by Lemma 39

$$\begin{aligned} T_v((s, s) \oplus_C (\mathbf{0}_v, t)) &= T_v(s, s) + T_v(\mathbf{0}_v, t) \\ &= \alpha(v) Z_v(s) - O_v(s) + O_v(s) + O_v(t) \\ &= \alpha(v) Z_v(s) + \alpha(v) Z_v(t) = \alpha(v) Z_v(s \oplus_v t). \end{aligned}$$

Table 2 A list of the maps and their domains and codomains.

Map	Domain	Codomain
α	R	$\mathbf{Irr}$
O_v	A_v	$I_{\alpha(v)}$
Z_v	A_v^{fin}	$\mathbb{N}$
$g_v := \alpha(v)Z_v - O_v$	B_v	$\mathbb{N}$
$T_v := g_v + O_v$	C_v	$[-\alpha(v), \infty) \subseteq \mathbb{R}$

If $t \prec \mathbf{0}_v$, then by Lemma 39

$$\begin{aligned} T_v((s,s) \oplus_v^C (\mathbf{0}_v, t) \oplus_v^C \mathbf{1}_v^C) &= T_v(s,s) + T_v(\mathbf{0}_v, t) + 1 \\ &= \alpha(v) Z_v(s) - O_v(s) + O_v(s) + O_v(t) + 1 \\ &= \alpha(v) Z_v(s) + \alpha(v) Z_v(t) = \alpha(v) Z_v(s \oplus_v t). \end{aligned}$$

Thus $T_v(U) \subseteq \alpha(v)\mathbb{N}$. By Lemma 40, $T_v(U) = \alpha(v)\mathbb{N}$. ◀