Obstructions for Matroids of Path-Width at most k and Graphs of Linear Rank-Width at most k

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Abstract

Every minor-closed class of matroids of bounded branch-width can be characterized by a minimal list of excluded minors, but unlike graphs, this list could be infinite in general. However, for each fixed finite field $\mathbb{F}$, the list contains only finitely many $\mathbb{F}$-representable matroids, due to the well-quasi-ordering of $\mathbb{F}$-representable matroids of bounded branch-width under taking matroid minors [J. F. Geelen, A. M. H. Gerards, and G. Whittle (2002)]. But this proof is non-constructive and does not provide any algorithm for computing these $\mathbb{F}$-representable excluded minors in general.

We consider the class of matroids of path-width at most k for fixed k. We prove that for a finite field $\mathbb{F}$, every $\mathbb{F}$-representable excluded minor for the class of matroids of path-width at most k has at most $2^{|\mathbb{F}|^{O(k^2)}}$ elements. We can therefore compute, for any integer k and a fixed finite field $\mathbb{F}$, the set of $\mathbb{F}$-representable excluded minors for the class of matroids of path-width k, and this gives as a corollary a polynomial-time algorithm for checking whether the path-width of an $\mathbb{F}$-represented matroid is at most k. We also prove that every excluded pivot-minor for the class of graphs having linear rank-width at most k has at most $2^{2^{O(k^2)}}$ vertices, which also results in a similar algorithmic consequence for linear rank-width of graphs.

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1 Introduction

For a class $\mathcal{C}$ of graphs or matroids, a graph or a matroid is an *excluded minor* for $\mathcal{C}$ if it does not belong to $\mathcal{C}$ but all of its proper minors belong to $\mathcal{C}$.

Robertson and Seymour [20] proved that every minor-closed class of graphs has finitely many excluded minors. This deep theorem has many algorithmic consequences for minor-closed classes of graphs. One of the corollaries is that for each minor-closed class $\mathcal{I}$ of graphs, there exists a monadic second-order formula $\varphi_{\mathcal{I}}$ that expresses the membership in $\mathcal{I}$, as there is a formula to decide whether a graph has a minor isomorphic to a fixed graph. However, the proof of Robertson-Seymour theorem is non-constructive and provides no algorithm of constructing the list of excluded minors and therefore we only know the existence of $\varphi_{\mathcal{I}}$ and do not know how to construct $\varphi_{\mathcal{I}}$ in general.

The class of graphs of path-width at most k is minor-closed and therefore the list of excluded minors for the class of graphs of path-width at most k is finite for each k. Actually, this is also implied by an earlier theorem of Robertson and Seymour [19], stating that graphs of bounded tree-width are well-quasi-ordered under taking minors. But this is still non-constructive. In 1998, Lagergren [14] proved that each excluded minor for the class of graphs of path-width at most k has at most $2^{O(k^4)}$ edges. Therefore we can now construct a monadic second-order formula φ_k to decide whether the path-width of a graph is at most k for each k. Since Courcelle's theorem [3] allows us to decide φ_k on graphs of bounded tree-width in polynomial time, we obtain a polynomial-time algorithm to decide whether an input graph has path-width at most k for each fixed k, even though a direct algorithm was proposed by Bodlaender and Kloks [2].

We aim to prove analogous theorems for the class of matroids of path-width at most k and for the class of graphs of linear rank-width at most k. For a matroid M on the ground set $E(M)$, we define its connectivity function λ_M by

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) \quad \text{for } X \subseteq E(M),$$

where r_M is the rank function of M. The path-width of a matroid M is defined as the minimum *width* of linear orderings of its elements, called *path-decompositions* or *linear layouts*, where the width of a path-decomposition $e_1, e_2, \ldots, e_n$ is defined as the maximum of the values $\lambda_M(\{e_1, e_2, \ldots, e_i\})$ for all $i = 1, 2, \ldots, n$.

For matroid path-width, we do not yet know whether there are only finitely many excluded minors for the class of matroids of path-width at most k. Previously, Koutsonas, Thilikos, and Yamazaki [13] showed a lower bound, proving that the number of excluded minors for the class of matroids of path-width at most k is at least $(k!)^2$. We remark that a class of matroids of bounded path-width is not necessarily well-quasi-ordered under taking minors; Geelen, Gerards, and Whittle [6] showed that there is an infinite antichain of matroids of bounded path-width.

Geelen, Gerards, and Whittle [6] proved that for each finite field $\mathbb{F}$, $\mathbb{F}$-representable matroids of bounded branch-width are well-quasi-ordered under taking minors, as a generalization of the theorem of Robertson and Seymour [19] on graphs of bounded tree-width. This implies that for each finite field $\mathbb{F}$, there are only finitely many $\mathbb{F}$-representable excluded minors for the class of matroids of path-width at most k.

As a corollary, for each finite field $\mathbb{F}$ and an integer k, there exists a monadic second-order formula $\varphi_k^{\mathbb{F}}$ to decide whether an $\mathbb{F}$-representable matroid has path-width at most k, because one can write a monadic second-order formula to describe whether a matroid has a fixed matroid as a minor by Hliněný [7]. Hliněný [7] also proved an analog of Courcelle's theorem for $\mathbb{F}$-represented matroids, showing a fixed-parameter algorithm to decide a monadic second-order formula on $\mathbb{F}$-represented matroids of bounded branch-width, for a finite field $\mathbb{F}$. This allows us to conclude that there "exists" a fixed-parameter tractable algorithm to decide whether an input $\mathbb{F}$-represented matroid has path-width at most k by testing $\varphi_k^{\mathbb{F}}$.

However, the theorem of Geelen, Gerards, and Whittle [6] does not provide any method of constructing the list of $\mathbb{F}$-representable excluded minors and so we did not know how to find $\varphi_k^{\mathbb{F}}$. We are now ready to state our main theorem, showing an explicit upper bound of the size of every $\mathbb{F}$-representable excluded minor.

▶ **Theorem 1.** *For a finite field $\mathbb{F}$ and an integer k, each $\mathbb{F}$-representable excluded minor for the class of matroids of path-width at most k has at most $2^{|\mathbb{F}|^{O(k^2)}}$ elements.*

Thus, by Theorem 1, we "have" an algorithm to construct $\varphi_k^{\mathbb{F}}$ and we "have" a fixed-parameter algorithm to decide whether an input $\mathbb{F}$-represented matroid has path-width at most k. Note that there is a subtle difference between "have" and "there exist"; by Geelen, Gerards, and Whittle [6], we knew that there exists $\varphi_k^{\mathbb{F}}$, but we did not know how to construct it, because their proof is non-constructive. By Theorem 1 we can enumerate all matroids of small size to find the list of all $\mathbb{F}$-representable excluded minors and therefore we can finally construct $\varphi_k^{\mathbb{F}}$.

We remark that Geelen, Gerards, Robertson, and Whittle [5] showed an analogous theorem for branch-width of matroids; for each $k \geq 1$, every excluded minor for the class of matroids of branch-width at most k has at most $(6^{k+1}-1)/5$ elements.[1]

By extending our method slightly, we also prove a similar theorem for the linear rank-width of graphs as follows.

▶ **Theorem 2.** *Each excluded pivot-minor for the class of graphs of linear rank-width at most k has at most $2^{2^{O(k^2)}}$ vertices.*

Since every vertex-minor obstruction is also a pivot-minor obstruction, we deduce the following.

▶ **Corollary 3.** *Each excluded vertex-minor for the class of graphs of linear rank-width at most k has at most $2^{2^{O(k^2)}}$ vertices.*

The situation is very similar to that of matroids representable over a fixed finite field. Oum [16] showed that graphs of bounded rank-width are well-quasi-ordered under taking pivot-minors, which implies that the list of excluded pivot-minors for the class of graphs of linear rank-width at most k is finite. Again its proof is non-constructive and therefore it provides no algorithm to construct the list. Jeong, Kwon, and Oum [10, 11] proved that any list of excluded pivot-minors characterizing the class of graphs of linear rank-width at most k has at least $2^{\Omega(3^k)}$ graphs.

Corollary 3 answers an open problem of Jeong, Kwon, and Oum [11] on the number of vertices of each excluded vertex-minor for the class of graphs of linear rank-width at most k. Adler, Farley, and Proskurowski [1] characterized excluded vertex-minors for the class of graphs of linear rank-width at most 1. Theorem 6.1 of Kanté and Kwon [12] implies that distance-hereditary excluded vertex-minors for the class of graphs of linear rank-width at most k have at most $O(3^k)$ vertices.

[1] In [5], the connectivity function of matroids is defined to have $+1$, which makes $(6^k-1)/5$.

Previously, we only knew the existence of a modulo-2 counting monadic second-order formula Φ_k testing whether a graph has linear rank-width at most k. This is due to the theorem of Courcelle and Oum [4] stating that for each graph H, there is a modulo-2 counting monadic second-order formula to decide whether a graph has a pivot-minor isomorphic to H. As there is a polynomial-time algorithm to decide a modulo-2 counting monadic second-order formula for graphs of bounded rank-width (see [4, Proposition 5.7]), we can conclude that there "exists" a polynomial-time algorithm to decide whether an input graph has linear rank-width at most k. However, this algorithm is based on the existence of Φ_k, and we did not know how to construct Φ_k. Finally, by Theorem 2, we know how to construct Φ_k algorithmically.

Let us now explain the main ideas. We first observe that each excluded minor M has path-width $k+1$, admits a *linked path-decomposition*, which is a path-decomposition satisfying some Menger-like condition, and each proper minor of M has path-width at most k. Secondly, we show that each excluded minor of sufficiently large size has many nested cuts, all of the same value. We finally show that among those cuts of the same value, there are two nested cuts X and Y such that M has a minor on $X \cup (E(M) \setminus Y)$ of path-width $k+1$, contradicting that all proper minors of M have path-width at most k. One of the key ingredients in finding the minor is to use the data structure proposed by Jeong, Kim, and Oum [9]. Based on dynamic programming, they devised fixed-parameter algorithms to decide whether an $\mathbb{F}$-represented matroid has path-width at most k and to decide whether a graph has linear rank-width at most k without using the fact that there are only finitely many excluded minors. Their so-called *B-trajectories* encode partial solutions which may be extended to the full solutions. Here is the idea behind B-trajectories. If $\lambda_M(X) = k$, then the dimension of the vector space spanned by both X and $E(M) \setminus X$ is exactly k. Since the underlying field is finite, this intersection subspace has only finitely many subspaces. Combining this observation with the idea of *typical sequences* appearing in Bodlaender and Kloks [2], Jeong, Kim, and Oum [9] deduce that there are only finitely many collections, called the *full sets*, of meaningful partial solutions (*compact B-trajectories*) at every moment of the dynamic programming algorithm. We indeed prove that among all nested cuts ensured by the large size of M, there are two nested cuts X and Y such that the full set associated with Y can be obtained by applying the same linear transformation to all compact B-trajectories of the full set associated with X, where B is the vector space spanned by both X and $E(M) \setminus X$. The second key ingredient of our proof is the linking theorem for minors of matroids of Tutte [21] and a corresponding theorem for pivot-minors of graphs by Oum [16]; both are analogs of Menger's theorem. These linking theorems will ensure that when two nested cuts display the identical full set up to a certain linear transformation, one can obtain a proper minor or a proper pivot-minor having the same path-width or linear rank-width, respectively.

This paper is organized as follows. Section 2 reviews necessary definitions and known facts on matroids, branch-decompositions, path-decompositions, and Tutte's linking theorem. We review in Section 3 the data structure introduced in Jeong, Kim, and Oum [9]. Section 4 presents a lemma on finding many cuts of the same width inside a *linked* path-decomposition. We present the proof of the main theorem in Section 5. In Section 6, we present the proof for Theorem 2 on linear rank-width of graphs.

2 Preliminaries

For two sets A and B, we write $A \triangle B$ to denote $(A - B) \cup (B - A)$.

2.1 Matroids and minors

A *matroid* is a pair $(E, \mathcal{I})$ of a finite set E and a set $\mathcal{I}$ of subsets of E satisfying the following three properties:
(I1) $\emptyset \in \mathcal{I}$.
(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.
(I3) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there is $e \in Y - X$ such that $X \cup \{e\} \in \mathcal{I}$.
A subset of E is *independent* if it belongs to $\mathcal{I}$. The *ground set* of a matroid $M = (E, \mathcal{I})$ is the set E denoted by $E(M)$. A subset of E is *dependent* if it is not independent.

Let $M = (E, \mathcal{I})$ be a matroid on n elements. We write $\mathcal{I}(M)$ to denote the set of independent sets of a matroid M. A *base* of a matroid is a maximal independent set. A subset of E is *coindependent* if it is disjoint with some base. The *rank* of a set X in a matroid M, denoted by $r_M(X)$, is the size of a maximal independent subset of X in M. The *rank* of a matroid M is $r(M) := r_M(E(M))$. The *connectivity function* of a matroid M, denoted by λ_M is defined as

$$\lambda_M(X) := r_M(X) + r_M(E(M) - X) - r(M)$$

for all $X \subseteq E(M)$. It is easy to verify that λ_M is *submodular*, that is

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cup Y) + \lambda_M(X \cap Y)$$

for all $X, Y \subseteq E(M)$. Also observe that λ_M is *symmetric*, that is $\lambda_M(X) = \lambda_M(E(M) - X)$ for all $X \subseteq E(M)$.

For $X \subseteq E$, the *restriction* $M|_X$ of a matroid M on X is a matroid on the ground set X such that $I \subseteq X$ is an independent set of $M|_X$ if and only if it is an independent set of M. The *deletion* of X from M is the restriction of M on $E - X$, denoted as $M \setminus X$. Another matroid operation is a *contraction*. The contraction of M by X, denoted as M/X, is a matroid with the ground set $E - X$ such that a set $I \subseteq E - X$ is an independent set of M/X if and only if there exists a base B_X of $M|_X$ such that $I \cup B_X$ is an independent set of M. Note that for $Y \subseteq E - X$, $r_{M/X}(Y) = r_M(Y \cup X) - r_M(X)$, where r_M is the rank function of a matroid M. For two matroids M, N, we say that N is a *minor* of M if there exist disjoint subsets C and D of $E(M)$ such that $N = M \setminus D/C$. A minor N of M is *proper* if $E(N) \neq E(M)$.

The following lemma is obtained easily from the above equation on the rank of a minor.

▶ **Lemma 4** (Geelen, Gerards, and Whittle [6, (5.3)]). *Let $M = (E, \mathcal{I})$ be a matroid and let X, C, D be disjoint subsets of E. Then $\lambda_{M \setminus D/C}(X) \leq \lambda_M(X)$. Furthermore, the equality holds if and only if $r_M(X \cup C) = r_M(X) + r_M(C)$ and $r_M(E - X) + r_M(E - D) = r_M(E) + r_M(E - (X \cup D))$.*

2.2 Vector matroids

One of the key examples of matroids is the class of vector matroids. Let A be an $m \times n$ matrix over a field $\mathbb{F}$ whose columns are indexed by a set E of column labels. Then a matroid $M(A)$ on E can be defined from A so that X is independent in $M(A)$ if and only if the corresponding column vectors of A are linearly independent. Such a matroid $M(A)$ is called a *vector matroid* and A is called a *representation* of the matroid $M(A)$. We say that a matroid M is *representable* over $\mathbb{F}$, or equivalently $\mathbb{F}$-*representable* if there is a matrix A over $\mathbb{F}$ such that $M = M(A)$. We say a matroid M is $\mathbb{F}$-*represented* if it is given with its representation over $\mathbb{F}$.

Instead of using matrices, we may regard a vector matroid defined from a finite set of labeled vectors in a vector space, called a *configuration* as in [6]. For a configuration A, we write $M(A)$ to denote the matroid on A such that a subset of A is independent in $M(A)$ if and only if it is linearly independent in the underlying vector space. Note that vectors in a configuration may coincide as we allow two different labels to represent the same vector. We write $\langle A\rangle$ to denote the linear span of the vectors in A.

2.3 Path-width

Let E be a finite set with n elements. A function $f : 2^E \to \mathbb{Z}$ is *submodular* if $f(X)+f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq E$ and is *symmetric* if $f(X) = f(E - X)$ for all $X \subseteq E$. We say that a function $f : 2^E \to \mathbb{Z}$ is a *connectivity function* if it is submodular, symmetric, and $f(\emptyset) = 0$.

A *linear layout* of E is a permutation $\sigma = e_1, e_2, \dots, e_n$ of E. The *width* of a linear layout $\sigma = e_1, e_2, \dots, e_n$ with respect to f is $\max_{1\leq i<n} f(\{e_1, e_2, \dots, e_i\})$. The *path-width* of f is the minimum width of all possible linear layouts of E with respect to f.

If f is the matroid connectivity function λ_M of a matroid M, then the linear layout of $E(M)$ is called a *path-decomposition* of M and the path-width of M is defined as the path-width of λ_M.

A linear layout $\sigma = e_1, e_2, \dots, e_n$ is *linked* if for all $0 \leq i < j \leq n$,

$$\min_{\{e_1,e_2,\dots,e_i\}\subseteq X\subseteq\{e_1,e_2,\dots,e_j\}} f(X) = \min_{i\leq \ell\leq j} f(\{e_1, e_2, \dots, e_\ell\}).$$

Nagamochi [15] presented an algorithm that runs in polynomial time for fixed k to find a linear layout of width at most k if it exists for general connectivity functions. The key step of his algorithm implies the following theorem easily from [15, Lemma 2], which ensures that there always exists a linked linear layout of the optimum width. Actually, his algorithm outputs a linked linear layout.

▶ **Theorem 5** (Nagamochi [15]). *If a connectivity function f has path-width k, then it has a linked linear layout of width at most k.*

2.4 Tutte's linking theorem

▶ **Theorem 6** (Tutte [21]). *Let M be a matroid and A, B be disjoint subsets of $E(M)$. Then*

$\lambda_M(X) \geq k$ *for all* $A \subseteq X \subseteq E(M) - B$

if and only if M has a minor N on $A \cup B$ such that $\lambda_N(A) \geq k$.

For a configuration A and $X \subseteq A$, let

$\partial_A(X) := \langle X\rangle \cap \langle A - X\rangle.$

Observe that $\lambda_{M(A)}(X) = \dim \partial_A(X)$. The following proposition is essentially due to Geelen, Gerards, and Whittle [6, (5.7)] and we modified their statement with the almost same proof. Note that if $N = M/C \setminus D$ is a minor of M, then we can choose D as a coindependent set in M without changing N, see [18, Lemma 3.3.2]. Thus it is easy to satisfy the requirements of the following proposition from Tutte's linking theorem.

▶ **Proposition 7.** *Let A be a configuration over a field $\mathbb{F}$ and let S, T be subcollections of A such that $S \cap T = \emptyset$. Let C, D be disjoint subcollections of A such that $C \cup D = A - (S \cup T)$, D is coindependent in $M(A)$, and for the minor $N = M(A)/C \setminus D$ of $M(A)$ on $S \cup T$,*

$$\lambda_N(S) = \min_{S \subseteq X \subseteq A-T} \lambda_{M(A)}(X) = k.$$

Then for all subcollections Z of A, if $S \subseteq Z \subseteq A - T$ and $\lambda_{M(A)}(Z) = k$, then the following hold.

(i) *For all $x, y \in \langle Z \rangle$, $x - y \in \langle C \rangle$ if and only if $x - y \in \langle C \cap Z \rangle$.*

(ii) *For all $x, y \in \langle A - Z \rangle$, $x - y \in \langle C \rangle$ if and only if $x - y \in \langle C - Z \rangle$.*

(iii) *For all $x, y \in \partial_A(Z)$, $x - y \in \langle C \rangle$ if and only if $x = y$.*

(iv) *If Z' is also a subcollection of A such that $S \subseteq Z' \subseteq A - T$ and $\lambda_{M(A)}(Z') = k$, then for each $x \in \partial_A(Z')$, there is a unique $y \in \partial_A(Z)$ such that $x - y \in \langle C \rangle$. Moreover, $x - y \in \langle C \cap (Z \triangle Z') \rangle$.*

Proof. Let $M = M(A)$. Since D is coindependent, $r_M(A - D) = r_M(A)$. Let $C_1 = C \cap Z$, $D_1 = D \cap Z$, $C_2 = C - Z$, and $D_2 = D - Z$. By Lemma 4,

$$r_M(A - Z) + r_M(A - D_2) = r_M(A) + r_M(A - (Z \cup D_2)),$$
$$r_M(Z \cup C_2) = r_M(Z) + r_M(C_2).$$

As $r_M(A - D_2) = r_M(A)$, from the first equation, we have $r_M(A - Z) = r_M(A - (Z \cup D_2)) = r_M(T \cup C_2)$ and so

$$\langle A - Z \rangle = \langle T \cup C_2 \rangle. \tag{1}$$

From the second equation, we have

$$\langle Z \rangle \cap \langle C_2 \rangle = \{0\}. \tag{2}$$

By symmetry between S and T and between Z and $V - Z$, we have

$$\langle Z \rangle = \langle S \cup C_1 \rangle \text{ and } \langle A - Z \rangle \cap \langle C_1 \rangle = \{0\}. \tag{3}$$

Suppose that $x, y \in \langle Z \rangle$ and $x - y \in \langle C \rangle$. Let $c_1 \in \langle C_1 \rangle$ and $c_2 \in \langle C_2 \rangle$ such that $x - y = c_1 + c_2$. Then $x - y - c_1 \in \langle C_2 \rangle \cap \langle Z \rangle$. By (2), $x - y - c_1 = 0$ and so $x - y \in \langle C_1 \rangle$. This proves (i). By symmetry, (ii) is also proved.

By (i) and (ii), if $x, y \in \partial_A(Z)$ and $x - y \in \langle C \rangle$, then $x - y \in \langle C \cap Z \rangle \cap \langle C - Z \rangle$. By (2), $\langle C \cap Z \rangle \cap \langle C - Z \rangle = \{0\}$ and therefore $x = y$. This proves (iii).

To prove (iv), suppose that $x \in \partial_A(Z')$. By (1) applied to Z', there exist $t \in \langle T \rangle$ and $c_2 \in \langle C - Z' \rangle$ such that $x = t + c_2$. Similarly, by (3), there exist $s \in \langle S \rangle$ and $c_1 \in \langle C \cap Z' \rangle$ such that $x = s + c_1$. We can write $c_1 = c_{11} + c_{12}$ for $c_{11} \in \langle C \cap (Z \cap Z') \rangle$ and $c_{12} \in \langle C \cap (Z' - Z) \rangle$ and write $c_2 = c_{21} + c_{22}$ for $c_{21} \in \langle C \cap (Z - Z') \rangle$ and $c_{22} \in \langle C - (Z \cup Z') \rangle$. Let us define $y = s + c_{11} - c_{21} = t + c_{22} - c_{12}$. Then $y \in \partial_A(Z)$ because $s + c_{11} - c_{21} \in \langle Z \rangle$ and $t + c_{22} - c_{12} \in \langle A - Z \rangle$. Now observe that $x - y = c_{12} + c_{21} \in \langle C \cap (Z \triangle Z') \rangle$. This proves that the desired y exists. By (iii), such y is unique. ◀

3 Full sets

We review the concepts of B-trajectories and full sets introduced by Jeong, Kim, and Oum [9].

3.1 B-trajectories

Let B be a vector space. A *statistic* is a triple $a = (L, R, \lambda)$ of subspaces L, R of B and a non-negative integer λ. For convenience, we write $L(a) = L$, $R(a) = R$, and $\lambda(a) = \lambda$. A B*-trajectory* is a sequence $\Gamma = a_0, a_1, \dots, a_n$ of statistics for a non-negative integer n such that

- $R(a_0) = L(a_n)$,
- $L(a_0) \subseteq L(a_1) \subseteq \cdots \subseteq L(a_n) \subseteq B$,
- $R(a_n) \subseteq R(a_{n-1}) \subseteq \cdots \subseteq R(a_0) \subseteq B$.

The width of Γ is $\max_{0 \leq i \leq n} \lambda(a_i)$. We write $\Gamma(i)$ to denote a_i. The *length* of Γ, denoted by $|\Gamma|$, is $n+1$.

Let $A = \{e_1, e_2, \dots, e_n\}$ be a configuration over a field $\mathbb{F}$. From a path-decomposition $\sigma = e_1, e_2, \dots, e_n$ of a represented matroid $M = M(A)$, we can obtain its *canonical* B*-trajectory* as follows. For $i = 0, 1, 2, \dots, n$, let

$$\begin{aligned} L_i &= \langle e_1, e_2, \dots, e_i \rangle \cap B, \\ R_i &= \langle e_{i+1}, e_{i+2}, \dots, e_n \rangle \cap B, \text{ and} \\ \lambda_i &= \dim \langle e_1, e_2, \dots, e_i \rangle \cap \langle e_{i+1}, e_{i+2}, \dots, e_n \rangle - \dim L_i \cap R_i. \end{aligned}$$

Note that $L_0 = R_n = \{0\}$ and $\lambda_0 = \lambda_n = 0$. Let $a_i = (L_i, R_i, \lambda_i)$ for $i = 0, 1, 2, \dots, n$. Then it is easy to see that $\Gamma = a_0, a_1, a_2, \dots, a_n$ is a B-trajectory, which we call the *canonical* B*-trajectory* of σ. If Γ is a canonical B-trajectory of some path-decomposition σ of $M = M(A)$, then we say Γ is *realizable* in A.

For a B-trajectory $\Gamma = a_0, a_1, a_2, \dots, a_n$, the *compactification* of Γ, denoted by $\tau(\Gamma)$, is a B-trajectory obtained from Γ by applying the following operations repeatedly until no further operations can be applied.

- Remove an entry a_i if $a_{i-1} = a_i$.
- Remove a subsequence $a_{i+1}, a_{i+2}, \dots, a_{j-1}$ if $i+1 < j$, $L(a_i) = L(a_j)$, $R(a_i) = R(a_j)$, and either $\lambda(a_i) \leq \lambda(a_k) \leq \lambda(a_j)$ for all $k \in \{i+1, i+2, \dots, j-1\}$ or $\lambda(a_i) \geq \lambda(a_k) \geq \lambda(a_j)$ for all $k \in \{i+1, i+2, \dots, j-1\}$.

We say that a B-trajectory is *compact* if $\tau(\Gamma) = \Gamma$. Let $U_k(B)$ be the set of all compact B-trajectories of width at most k.

▶ **Lemma 8** (Jeong, Kim, and Oum [9, Lemma 11]). *Let B be a vector space over a finite field $\mathbb{F}$ with dimension θ. Then*

$$|U_k(B)| \leq 2^{9\theta+2} |\mathbb{F}|^{\theta(\theta-1)} 2^{2(2\theta+1)k}.$$

We can define binary relations which compare two B-trajectories as follows [9]. For two statistics a and b, we write $a \leq b$ if

$$L(a) = L(b),\ R(a) = R(b),\ \text{and } \lambda(a) \leq \lambda(b).$$

For two B-trajectories Γ_1 and Γ_2, we write $\Gamma_1 \leq \Gamma_2$ if the lengths of Γ_1 and Γ_2 are the same, say n, and $\Gamma_1(i) \leq \Gamma_2(i)$ for all $0 \leq i \leq n-1$. A B-trajectory Γ^* is called an *extension* of a B-trajectory Γ if Γ^* can be obtained by repeating some statistics of Γ. We say that $\Gamma_1 \preccurlyeq \Gamma_2$ if there are extensions Γ_1^* of Γ_1 and Γ_2^* of Γ_2 such that $\Gamma_1^* \leq \Gamma_2^*$.

3.2 A full set

We review the *full set* notion introduced by Jeong, Kim, and Oum [9] used for their algorithm to decide the path-width of represented matroids. Let A be a configuration of vectors in a vector space V over a field $\mathbb{F}$. Let B be a subspace of V.

The *full set* of A of width k with respect to B, denoted by $\mathrm{FS}_k(A, B)$, is the set of all compact B-trajectories Γ of width at most k such that there exists a B-trajectory Δ realizable in A with $\Delta \preccurlyeq \Gamma$. From the definition, it is clear that

$$\mathrm{FS}_k(A, \{0\}) \neq \emptyset \text{ if and only if } M(A) \text{ has path-width at most } k.$$

By Lemma 8, the number of B-trajectories in $\mathrm{FS}_k(A, B)$ is bounded by a function of $|\mathbb{F}|$, $\dim B$, and k.

The following lemma is an immediate consequence of Jeong, Kim, and Oum [9, Propositions 35 and 36].

▶ **Lemma 9.** *Let A, A' be configurations in a vector space V. Let k be a non-negative integer. Let B be a subspace of V. If $\mathrm{FS}_k(A, B) = \mathrm{FS}_k(A', B)$, then $\mathrm{FS}_k(A, \{0\}) = \mathrm{FS}_k(A', \{0\})$.*

▶ **Lemma 10.** *Let A_1, A_1', A_2, A_2' be configurations in a vector space V. Let k be a non-negative integer. Let B be a subspace of V such that $(\langle A_1\rangle + B) \cap (\langle A_2\rangle + B) = B$ and $(\langle A_1'\rangle + B) \cap (\langle A_2'\rangle + B) = B$. If $\mathrm{FS}_k(A_1, B) = \mathrm{FS}_k(A_1', B)$ and $\mathrm{FS}_k(A_2, B) = \mathrm{FS}_k(A_2', B)$, then $\mathrm{FS}_k(A_1 \cup A_2, B) = \mathrm{FS}_k(A_1' \cup A_2', B)$.*

For a configuration $A = \{e_1, e_2, \ldots, e_n\}$ and a linear transformation ϕ, we write $\phi(A)$ to denote a configuration $\{\phi(e_1), \phi(e_2), \ldots, \phi(e_n)\}$.

If B_1 and B_2 are subspaces of the same dimension and ϕ is a bijective linear transformation from B_1 to B_2, then for each B_1-trajectory Γ we can define a B_2-trajectory $\Delta := \phi(\Gamma)$ in the following way:

$$L(\Delta(i)) = \phi(L(\Gamma(i))), \quad R(\Delta(i)) = \phi(R(\Gamma(i))), \quad \lambda(\Delta(i)) = \lambda(\Gamma(i)),$$

for every $0 \leq i \leq |\Gamma| - 1$. For a set of B-trajectories $\mathcal{R}$, we define the set $\phi(\mathcal{R}) = \{\phi(\Gamma) : \Gamma \in \mathcal{R}\}$.

Observe that if ϕ is a linear transformation on $\langle A\rangle$ that is injective on $\langle A_1\rangle$ and B_1 is a subspace of $\langle A_1\rangle$, then

$$\phi(\mathrm{FS}_k(A_1, B_1)) = \mathrm{FS}_k(\phi(A_1), \phi(B_1)).$$

Here on the right-hand side, we use ϕ values for all vectors in $\langle A_1\rangle$ but on the left-hand side, we only use ϕ for vectors in B_1.

We can deduce the following lemma easily from Lemmas 9 and 10. We omit its proof.

▶ **Lemma 11.** *Let k be a non-negative integer and let $\mathbb{F}$ be a field. Let A be a configuration in a vector space V over $\mathbb{F}$ and let A' be a configuration in a vector space V' over $\mathbb{F}$. Let (A_1, A_2) be a partition of A and (A_1', A_2') be a partition of A'. If there is a bijective linear transformation $\phi : \partial_A(A_1) \to \partial_{A'}(A_1')$ such that*

$$\phi(\mathrm{FS}_k(A_1, \partial_A(A_1))) = \mathrm{FS}_k(A_1', \partial_{A'}(A_1')) \text{ and}$$
$$\phi(\mathrm{FS}_k(A_2, \partial_A(A_1))) = \mathrm{FS}_k(A_2', \partial_{A'}(A_1')),$$

then the path-width of $M(A)$ is at most k if and only if the path-width of $M(A')$ is at most k.

4 Finding many repeated cuts

The following lemma finds many cuts in the linked path-decomposition that are of the same width and linked each other.

▶ **Lemma 12.** *Let $\ell \ge 4$ be an integer. Let $a_0, a_1, a_2, \dots, a_n$ be a sequence of integers such that $a_i \ge a_0 = a_n$ for all $0 \le i \le n$ and $|a_i - a_{i+1}| \le 1$. If*

$$n \ge \left(\ell - 1 + \frac{2(\ell-2)}{\ell-3}\right)(\ell-2)^{\max_{0\le i\le n}(a_i - a_0)} - \frac{2(\ell-2)}{\ell-3},$$

then there exist $0 \le i_1 < i_2 < i_3 < \cdots < i_\ell \le n$ and w such that

$$a_{i_1} = a_{i_2} = \cdots = a_{i_\ell} = w \text{ and } a_i \ge w \text{ for all } i_1 \le i \le i_\ell.$$

Proof. We proceed by induction on $M = \max_{0\le i\le n}(a_i - a_0)$. It is trivial if $M = 0$. Let $m = |\{i \in \{0, 1, \dots, n\} : a_i = a_0\}|$. If $m \ge \ell$, then we are done. Thus we may assume that $m \le \ell - 1$. Then there exists a subsequence $a_p, a_{p+1}, \dots, a_q$ such that $a_i > a_0$ for all $p \le i \le q$, and $q - p + 1 \ge \frac{n}{m-1} - 1 \ge \frac{n}{\ell-2} - 1$. Equivalently, $q - p + \frac{2(\ell-2)}{\ell-3} \ge \frac{1}{\ell-2}\left(n + \frac{2(\ell-2)}{\ell-3}\right)$ and therefore

$$q - p \ge \left(\ell - 1 + \frac{2(\ell-2)}{\ell-3}\right)(\ell-2)^{M-1} - \frac{2(\ell-2)}{\ell-3}.$$

We may assume that $q - p$ is chosen as a maximum. Then by the assumption that $|a_i - a_{i+1}| \le 1$, we deduce that $a_p = a_q = a_0 + 1$. Now we apply the induction hypothesis to the subsequence $a_p, a_{p+1}, \dots, a_q$ to conclude the proof. ◀

We will apply Lemma 12 to a sequence $a_0, a_1, a_2, \dots, a_n$ obtained from a linked path-decomposition $\sigma = e_1, e_2, \dots, e_n$, where $a_i = \lambda_M(\{e_1, e_2, \dots, e_i\})$ for $i = 0, 1, 2, \dots, n$. It is easy to verify that any path-decomposition σ of a represented matroid meets the requirement that $|a_i - a_{i+1}| \le 1$ of Lemma 12. The next lemma is needed.

▶ **Lemma 13.** *Let M be a matroid. If $e \in X \subseteq E(M)$, then $|\lambda_M(X) - \lambda_M(X - \{e\})| \le 1$.*

Proof. By the submodularity of the connectivity function, we have $\lambda_M(X - \{e\}) + \lambda_M(\{e\}) \ge \lambda_M(X)$. Since $\lambda_M(\{e\}) \le 1$, we have $\lambda_M(X) \le \lambda_M(X - \{e\}) + 1$. Since λ_M is symmetric, we deduce that $\lambda_M(X - \{e\}) \le \lambda_M(X) + 1$. ◀

5 The proof

The following proposition proves Theorem 1.

▶ **Proposition 14.** *Let $\mathbb{F}$ be a finite field and k be a non-negative integer. Let M be an $\mathbb{F}$-representable matroid of path-width larger than k. Let $\ell = 2^{2^{9k+11}|\mathbb{F}|^{k(k+1)}2^{2(2k+3)k}} + 1$. If*

$$|E(M)| \ge \left(\ell - 1 + \frac{2(\ell-2)}{\ell-3}\right)(\ell-2)^{k+1} - \frac{2(\ell-2)}{\ell-3},$$

then there is $e \in E(M)$ such that M/e or $M \setminus e$ has path-width larger than k.

Proof. Let A be a configuration in a vector space over $\mathbb{F}$ such that $M = M(A)$. We may assume that $M \setminus e$ and M/e has path-width at most k for every $e \in E(M)$. This implies that M has path-width exactly $k+1$ and by Theorem 5, there is a linked path-decomposition $\sigma = e_1, e_2, \dots, e_n$ of M of width $k+1$. We identify e_i with a vector in A.

For $i = 0, 1, 2, \ldots, n$, let $a_i = \lambda_M(\{e_1, e_2, \ldots, e_i\})$. Then $0 \le a_i \le k+1$ for all i.

By Lemma 12, there exist integers $0 \le t_1 < t_2 < \cdots < t_\ell \le n$ and $0 \le \theta \le k+1$ such that $a_{t_1} = a_{t_2} = \cdots = a_{t_\ell} = \theta$ and $a_i \ge \theta$ for all $t_1 \le i \le t_\ell$. Let $A_i = \{e_1, e_2, \ldots, e_{t_i}\}$ and $B_i = \partial_A(A_i)$ for $1 \le i \le \ell$.

Since σ is a linked path-decomposition, $\lambda_M(X) \ge \theta$ for all $A_1 \subseteq X \subseteq A_\ell$. By Theorem 6, there are disjoint subcollections C, D of A such that $C \cup D = A - (A_1 \cup (A - A_\ell))$ and $\lambda_{M/C \setminus D}(A_1) = \theta$. We may assume that D is coindependent, see [18, Lemma 3.3.2]. Let $\pi : \langle A \rangle \to \langle A \rangle / \langle C \rangle$ be the linear transformation mapping $x \in \langle A \rangle$ to an equivalence class $[x]$ containing x where two vectors x and x' are equivalent if and only if $x - x' \in \langle C \rangle$. Let $B = \pi(\partial_A(A_1))$.

By (iii) and (iv) of Proposition 7, $\dim B = \theta$ and $\pi(\partial_A(A_i)) = \pi(\partial_A(A_j))$ for all $1 \le i < j \le \ell$.

Observe that $\pi(\mathrm{FS}_k(A_i, \partial_A(A_i))) \subseteq U_k(B)$. Since ℓ is big enough, by Lemma 8 and the pigeon-hole principle, there exist $1 \le i < j \le \ell$ such that $\pi(\mathrm{FS}_k(A_i, \partial_A(A_i))) = \pi(\mathrm{FS}_k(A_j, \partial_A(A_j)))$.

Let $C' = C \cap (A_j - A_i)$ and $D' = D \cap (A_j - A_i)$. Let $\phi : \langle A \rangle \to \langle A \rangle / \langle C' \rangle$ be the linear transformation mapping $x \in \langle A \rangle$ to an equivalence class containing x where two elements x, y are equivalent if and only if $x - y \in \langle C' \rangle$.

Let $B' = \phi(\partial_A(A_i))$. Since $C' \subseteq C$, by (iii) of Proposition 7, we have $\dim B' = \theta$. Furthermore, from (iv) of Proposition 7, we deduce that for $x \in \partial_A(A_i)$ and $y \in \partial_A(A_j)$, $\pi(x) = \pi(y)$ if and only if $\phi(x) = \phi(y)$. Therefore, $B' = \phi(\partial_A(A_j))$ and $\phi(\mathrm{FS}_k(A_i, \partial_A(A_i))) = \phi(\mathrm{FS}_k(A_j, \partial_A(A_j)))$.

We claim that ϕ is an injection on $\langle A_i \rangle$. Suppose that $x, y \in \langle A_i \rangle$ and $x - y \in \langle C' \rangle = \langle C \cap (A_j - A_i) \rangle \subseteq \langle A - A_i \rangle$. Then $x - y \in \langle C \rangle$ and by (i) of Proposition 7, we deduce that $x - y \in \langle C \cap A_i \rangle \subseteq \langle A_i \rangle$. This would imply that $x - y \in \partial_A(A_i)$ and therefore $x = y$ by (iii) of Proposition 7. By symmetry, we can also deduce that ϕ is an injection on $\langle A - A_j \rangle$.

Let $N = M(A)/C' \setminus D'$. Then $A' = \phi(A_i \cup (A - A_j))$ is a configuration in the vector space $\langle A \rangle / \langle C' \rangle$ such that $N = M(A')$. Since $B' \subseteq \langle \phi(A_i) \rangle$ and $B' \subseteq \langle \phi(A - A_j) \rangle$, we have $B' \subseteq \partial_{A'}(\phi(A_i))$. By Lemma 4, $\dim \partial_{A'}(\phi(A_i)) \le \theta$ and therefore $B' = \partial_{A'}(\phi(A_i))$.

Since ϕ is an injection on A_i, $\mathrm{FS}_k(\phi(A_i), \partial_{A'}(\phi(A_i))) = \phi(\mathrm{FS}_k(A_j), \partial_A(A_j))$. Since ϕ is an injection on $A - A_j$, trivially $\mathrm{FS}_k(\phi(A - A_j), \partial_{A'}(\phi(A - A_j))) = \phi(\mathrm{FS}_k(A - A_j), \partial_A(A - A_j))$.

Since N is a proper minor of M, the path-width of N is at most k. By Lemma 11, M has path-width at most k if and only if N has path-width at most k and therefore we deduce that the path-width of M is at most k, contradicting the assumption. ◀

6 Obstructions to linear rank-width

All graphs in this section are simple, having no loops and no parallel edges.

For a graph G, the *cut-rank* function ρ_G of G is defined as a function that maps a set X of vertices of G to the rank of the $X \times (V(G) - X)$ matrix over the binary field whose ab-entry is 1 if and only if $a \in X$ is adjacent to $b \in V(G) - X$. It is known that ρ_G is symmetric and submodular, see Oum and Seymour [17], and therefore it is a connectivity function. We remark that $\rho_G(\emptyset) = \rho_G(V(G)) = 0$. The *linear rank-width* of a graph G is defined to be the path-width of ρ_G.

For a pair (x, y) of distinct vertices of a graph G, *flipping* (x, y) is an operation that adds an edge xy if x, y are non-adjacent in G and deletes the edge xy otherwise. For an edge uv of a graph G, we write $G \wedge uv$ to denote the graph G' on $V(G)$ obtained by the following procedures.

1. For every pair $x \in N(u) \cap N(v)$ and $y \in N(u) - N(v)$, flip (x, y).
2. For every pair $x \in N(u) \cap N(v)$ and $y \in N(v) - N(u)$, flip (x, y).
3. For every pair $x \in N(u) - N(v)$ and $y \in N(v) - N(u)$, flip (x, y).
4. Swap the label of u and v.

This operation is called the *pivot*. We remark that the purpose of the last operation is to make $G \wedge uv \wedge vw = G \wedge uw$, see Oum [16]. Here is an important property of pivots with respect to the cut-rank function.

▶ **Proposition 15** (See Oum [16]). *If $H = G \wedge uv$, then $\rho_H(X) = \rho_G(X)$ for all $X \subseteq V(G)$.*

We say that a graph H is a *pivot-minor* of a graph G if H is an induced subgraph of a graph obtained from G by applying some sequence of pivots. We say that a pivot-minor H of G is *proper* if $V(H) \neq V(G)$. Since deleting a vertex never increases the cut-rank function, we deduce the following easily from the previous proposition.

▶ **Corollary 16.** *If H is a pivot-minor of G, then the linear rank-width of H is at most the linear rank-width of G.*

Oum [16] proved an analog of Tutte's linking theorem for pivot-minors.

▶ **Theorem 17.** *Let G be a graph and let S, T be disjoint vertex sets of G. Then there exists a pivot-minor H on $S \cup T$ such that $\rho_H(S) = \min_{S \subseteq X \subseteq V(G)-T} \rho_G(X)$.*

Let us now show how to represent a graph with a *subspace arrangement*. A *subspace arrangement* $\mathcal{V}$ over a field $\mathbb{F}$ is a finite set of subspaces of a finite-dimensional vector space over $\mathbb{F}$. We usually write a subspace arrangement as a family $\mathcal{V} = \{V_i\}_{i \in E}$ of subspaces indexed by a finite set E.

A *linear layout* of a subspace arrangement $\mathcal{V}$ is a permutation $\sigma = V_1, V_2, \ldots, V_n$ of $\mathcal{V}$. The *width* of a linear layout $\sigma = V_1, V_2, \ldots, V_n$ is equal to

$$\max_{1 \leq i < n} \dim(V_1 + V_2 + \cdots + V_i) \cap (V_{i+1} + V_{i+2} + \cdots + V_n).$$

Note that this function is a connectivity function on $\mathcal{V}$. The *path-width* of $\mathcal{V}$ is the minimum width of linear layouts of $\mathcal{V}$. If $|\mathcal{V}| \leq 1$, then we define the width of its linear layout to be 0 and its path-width to be 0.

As observed in [9, Section VII], for a matroid M represented by a configuration A, if we take $\mathcal{V} = \{\langle v \rangle : v \in A\}$, then the path-width of $\mathcal{V}$ is equal to the path-width of $M(A)$.

We are now going to review the construction, appeared in [9, Section VIII], of a subspace arrangement from graphs to relay the concept of linear rank-width to the path-width of its corresponding subspace arrangement. For a graph G on the vertex set $\{1, 2, \ldots, n\}$, let us define a subspace arrangement over the binary field as follows. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{F}_2^n$ where $\mathbb{F}_2$ is the binary field. Let $v_i = \sum_{j \in N_G(i)} e_j$, where $N_G(i)$ denotes the set of neighbors of i. Let $V_i = \langle e_i, v_i \rangle$ and let $\mathcal{V}_G = \{V_i\}_{i \in V(G)}$.

Here is the key observation.

▶ **Lemma 18** (Jeong, Kim, and Oum [9, Lemma 52]). *For $X \subseteq V(G)$,*

$$\dim\big((\sum_{i \in X} V_i) \cap (\sum_{j \in V(G)-X} V_j)\big) = 2\rho_G(X).$$

▶ **Corollary 19.** *The path-width of $\mathcal{V}_G$ is equal to twice the linear rank-width of G.*

For a subset X of $V(G)$, let $I_X = \{e_i : i \in X\}$, $A_X = \{v_i : i \in X\}$, and $\partial_X = \langle I_X \cup A_X\rangle \cap \langle I_{V(G)-X} \cup A_{V(G)-X}\rangle$. By Lemma 18, $\dim \partial_X = 2\rho_G(X)$. One can see that I_Z is a set of some column vectors in the $n \times n$ identity matrix and A_Z is a set of some column vectors in the adjacency matrix of G. Let M_G be the binary matroid represented by the matrix $(I_n\ A(G))$, where I_n is the $n \times n$ identity matrix and $A(G)$ is the adjacency matrix of G.

In Subsection 3.2, we reviewed the concept of full sets for the context of represented matroids or configurations. In fact, Jeong, Kim, and Oum [9] introduced full sets in more general form for subspace arrangements.

Here we are going to show the difference compared to Subsections 3.1 and 3.2. For a subspace arrangement $\mathcal{V}$ and its linear layout $\sigma = V_1, V_2, \ldots, V_n$, the *canonical B-trajectory* is defined as follows. For $i = 0, 1, \ldots, n$, let $L_i = (\sum_{j=1}^{i} V_j) \cap B$, $R_i = (\sum_{j=i+1}^{n} V_j) \cap B$, $\lambda_i = \dim(\sum_{j=1}^{i} V_j) \cap (\sum_{j=i+1}^{n} V_j) - \dim L_i \cap R_i$, and $a_i = (L_i, R_i, \lambda_i)$. Then $\Gamma = a_0, a_1, a_2, \ldots, a_n$ is the *canonical B-trajectory* of σ. We say that Γ is realizable in $\mathcal{V}$ if it is a canonical B-trajectory of some linear layout of $\mathcal{V}$.

For a subspace arrangement $\mathcal{V}$, $\mathrm{FS}_k(\mathcal{V}, B)$ is defined as the set of all compact B-trajectories Γ of width at most k such that there exists a B-trajectory Δ realizable in $\mathcal{V}$ with $\Delta \preceq \Gamma$.

Lemmas 9 and 10 are special cases of the following two lemmas easily deduced from the result of Jeong, Kim, and Oum [9].

▶ **Lemma 20.** *Let $\mathcal{V}$, $\mathcal{V}'$ be subspace arrangements over a field $\mathbb{F}$. Let k be a non-negative integer. Let B be a subspace of $\langle \mathcal{V} \cup \mathcal{V}'\rangle$. If $\mathrm{FS}_k(\mathcal{V}, B) = \mathrm{FS}_k(\mathcal{V}', B)$, then $\mathrm{FS}_k(\mathcal{V}, \{0\}) = \mathrm{FS}_k(\mathcal{V}', \{0\})$.*

▶ **Lemma 21.** *Let $\mathcal{V}_1$, $\mathcal{V}_1'$, $\mathcal{V}_2$, $\mathcal{V}_2'$ be subspace arrangements over a field $\mathbb{F}$. Let k be a non-negative integer. Let B be a subspace of $\langle \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_1' \cup \mathcal{V}_2'\rangle$ such that $(\langle \mathcal{V}_1\rangle + B) \cap (\langle \mathcal{V}_2\rangle + B) = B$ and $(\langle \mathcal{V}_1'\rangle + B) \cap (\langle \mathcal{V}_2'\rangle + B) = B$. If $\mathrm{FS}_k(\mathcal{V}_1, B) = \mathrm{FS}_k(\mathcal{V}_1', B)$ and $\mathrm{FS}_k(\mathcal{V}_2, B) = \mathrm{FS}_k(\mathcal{V}_2', B)$, then $\mathrm{FS}_k(\mathcal{V}_1 \cup \mathcal{V}_2, B) = \mathrm{FS}_k(\mathcal{V}_1' \cup \mathcal{V}_2', B)$.*

We can deduce the following lemma easily from Lemmas 20 and 21 by the same method of deducing Lemma 10 from Lemmas 9 and 10.

▶ **Lemma 22.** *Let k be a non-negative integer and let $\mathbb{F}$ be a field. Let $\mathcal{V}$ be a subspace arrangement over $\mathbb{F}$ and let $\mathcal{V}'$ be a subspace arrangement over $\mathbb{F}$. Let $(\mathcal{V}_1, \mathcal{V}_2)$ be a partition of $\mathcal{V}$ and $(\mathcal{V}_1', \mathcal{V}_2')$ be a partition of $\mathcal{V}'$. If there is a bijective linear transformation $\phi : \partial_{\mathcal{V}}(\mathcal{V}_1) \to \partial_{\mathcal{V}'}(\mathcal{V}_1')$ such that*

$$\phi(\mathrm{FS}_k(\mathcal{V}_1, \partial_{\mathcal{V}}(\mathcal{V}_1))) = \mathrm{FS}_k(\mathcal{V}_1', \partial_{\mathcal{V}'}(\mathcal{V}_1')) \text{ and } \phi(\mathrm{FS}_k(\mathcal{V}_2, \partial_{\mathcal{V}}(\mathcal{V}_1))) = \mathrm{FS}_k(\mathcal{V}_2', \partial_{\mathcal{V}'}(\mathcal{V}_1')),$$

then the path-width of $\mathcal{V}$ is at most k if and only if the path-width of $\mathcal{V}'$ is at most k.

The following proposition implies Theorem 2 and Corollary 3. We omit its proof.

▶ **Proposition 23.** *Let G be a graph of linear rank-width larger than k. Let $\ell = 2^{2^{18(k+1)+2+(2k+2)(2k+1)+2(4k+3)2k}} + 1$. If G has more than*

$$\left(\ell - 1 + \frac{2(\ell-2)}{\ell-3}\right)(\ell-2)^{k+1} - \frac{2(\ell-2)}{\ell-3},$$

vertices, then G has a proper pivot-minor H whose linear rank-width is larger than k.

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