Parking Functions, Multi-Shuffle, and Asymptotic Phenomena

Mei Yin

Department of Mathematics, University of Denver, Denver, CO, USA

Abstract

Given a positive integer-valued vector $\mathbf{u} = (u_1, \dots, u_m)$ with $u_1 < \cdots < u_m$, a $\mathbf{u}$-parking function of length m is a sequence $\boldsymbol{\pi} = (\pi_1, \dots, \pi_m)$ of positive integers whose non-decreasing rearrangement $(\lambda_1, \dots, \lambda_m)$ satisfies $\lambda_i \leq u_i$ for all $1 \leq i \leq m$. We introduce a combinatorial construction termed a parking function multi-shuffle to generic $\mathbf{u}$-parking functions and obtain an explicit characterization of multiple parking coordinates. As an application, we derive various asymptotic probabilistic properties of a uniform $\mathbf{u}$-parking function of length m when $u_i = cm + ib$. The asymptotic scenario in the generic situation $c > 0$ is in sharp contrast with that of the special situation $c = 0$.

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1 Introduction

Parking functions were introduced by Konheim and Weiss [10], under the name of "parking disciplines," to study the following problem. Consider a parking lot with n parking spots placed sequentially along a one-way street. In order, a line of $m \leq n$ cars enters the lot. The ith car drives to its preferred spot π_i and parks there if possible, and otherwise takes the next available spot if it exists. The sequence of preferences $\boldsymbol{\pi} = (\pi_1, \dots, \pi_m)$ is called a *parking function* if all cars successfully park. We denote the set of parking functions by $\mathrm{PF}(m, n)$, where m is the number of cars and n is the number of parking spots.

Nowadays, parking functions are an established area of research in combinatorics, with connections to labeled trees and forests [3], hyperplane arrangements, interval orders, and plane partitions [16, 17], diagonal harmonics and (q, t)-analogs of Catalan numbers [7], abelian sandpiles [4], to mention a few. Properties of random parking functions have also been of interest to statisticians and probabilists [5]. We refer to Knuth [9, Section 6.4] and Yan [20] for a comprehensive survey.

Given a positive integer-valued vector $\mathbf{u} = (u_1, \dots, u_m)$ with $u_1 < \cdots < u_m$, a $\mathbf{u}$-*parking function* of length m is a sequence $\boldsymbol{\pi} = (\pi_1, \dots, \pi_m)$ of positive integers whose non-decreasing rearrangement $(\lambda_1, \dots, \lambda_m)$ satisfies $\lambda_i \leq u_i$ for all $1 \leq i \leq m$. Denote the set of $\mathbf{u}$-parking functions by $\mathrm{PF}(\mathbf{u})$. There is a similar interpretation for $\mathbf{u}$-parking functions in terms of the parking scenario depicted above: One wishes to park m cars in a one-way street with u_m spots, but only m spots, at positions $u_1, \dots, u_m$, are still empty [12]. We recognize that the parking function $\mathrm{PF}(m, n)$ is a special case of the more general $\mathbf{u}$-parking functions with $u_i = n - m + i$.

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In our previous work on $\mathrm{PF}(m,n)$ [8] [21], we introduced an original combinatorial construction which we term a *parking function multi-shuffle*, and it facilitated an investigation of the properties of a parking function chosen uniformly at random from $\mathrm{PF}(m,n)$. This paper will delve into the essence of the multi-shuffle combinatorial construction on parking functions and introduce the concept to generic $\mathbf{u}$-parking functions, thus allowing for an explicit characterization of multiple coordinates of $\mathbf{u}$-parking functions. As an application, we will derive various asymptotic probabilistic properties of a uniform (a,b)-parking function, which is a class of $\mathbf{u}$-parking functions where $u_i = a + (i-1)b$ for some positive integers a and b. Denote the set of (a,b)-parking functions of length m by $\mathrm{PF}(a,b,m)$. It coincides with $\mathrm{PF}(m,n)$ when $a = n-m+1$ and $b = 1$. In view of and generalizing this correspondence, we will study the asymptotics of $\mathrm{PF}(a,b,m)$ when $b \geq 1$ is any integer and $a = cm + b$ for some $c \geq 0$. We find that for large m, various probabilistic quantities display strikingly different asymptotic tendencies in the generic situation $c > 0$ (corresponding to $m \lesssim n$) vs. the special situation $c = 0$ (corresponding to $m = n$), including the boundary behavior of a single coordinate and all moments of multiple coordinates.

Our asymptotic calculation utilizes the multi-dimensional Cauchy product of the *tree function* $F(z)$, which is a variant of the Lambert function, the *987Gončarov polynomials* $g_m(x; a_0, a_1, \dots, a_{m-1})$ for $m = 0, 1, \dots$, which form a natural basis for working with $\mathbf{u}$-parking functions, as well as *Abel's multinomial theorem*. In particular, our new perspective on parking functions leads to asymptotic moment calculations for multiple coordinates of (a,b)-parking functions that complement the work of Kung and Yan [11], where the explicit formulas for the first and second factorial moments and a general form for the higher factorial moments of sums of (a,b)-parking functions were given.

This paper is organized as follows. Section 2 illustrates the notion of $\mathbf{u}$-*parking function multi-shuffle* that decomposes a $\mathbf{u}$-parking function into smaller components (Definition 3). This construction offers an explicit characterization of multiple coordinates of $\mathbf{u}$-parking functions (Theorems 5 and 7). Theorem 7 enumerates $\mathbf{u}$-parking functions in connection with Gončarov polynomials, but we also provide an alternative description of $\mathbf{u}$-parking functions in Proposition 8. Section 3 uses the multi-shuffle construction introduced in Section 2 to investigate various properties of a parking function chosen uniformly at random from $\mathrm{PF}(cm+b, b, m)$. When the parking preferences $\pi_1, \dots, \pi_l$ are exactly b spots apart, a simplified characterization of $(cm+b, b)$-parking functions is given in Section 3.1. Building upon Theorem 7 and Proposition 10, we compute asymptotics of all moments of multiple coordinates in Theorem 14 in the generic situation $c > 0$. The asymptotic mixed moments in the generic situation $c > 0$ are contrasted with that of the special situation $c = 0$ in Section 3.2. We then focus on the boundary behavior of a single coordinate in Section 3.3. We find that in the generic situation $c > 0$ on the right end it approximates a Borel distribution with parameter $b/(b+c)$ while on the left end it deviates from the constant value in a rescaled Poisson fashion (Corollaries 19 and 20). This asymptotic tendency differs from that in the special situation $c = 0$, where the boundary behavior of a single coordinate on the left and right ends both approach Borel(1) (Corollaries 19 and 21).

Notations

Let $\mathbb{N}$ be the set of positive integers. For $m, n \in \mathbb{N}$, we write $[m,n]$ for the set of integers $\{m, \dots, n\}$ and $[n] = [1,n]$. For vectors $\mathbf{a}, \mathbf{b} \in [n]^m$, denote by $\mathbf{a} \leq_C \mathbf{b}$ if $a_i \leq b_i$ for all $i \in [m]$; this is the component-wise partial order on $[n]^m$. In a similar fashion, denote by $\mathbf{a} <_C \mathbf{b}$ if $a_i \leq b_i$ for all $i \in [m]$ and there is at least one $j \in [m]$ such that $a_j < b_j$. For $\mathbf{b} \in [n]^m$, we write $[\mathbf{b}]$ for the set of $\mathbf{a} \in [n]^m$ with $\mathbf{a} \leq_C \mathbf{b}$.

2 u-parking function multi-shuffle

In this section we explore the properties of generic $\mathbf{u}$-parking functions through a $\mathbf{u}$-parking function multi-shuffle construction. We will write our results in terms of parking coordinates $\pi_1, \ldots, \pi_l$ for explicitness, where $1 \leq l \leq m$ is any integer. But due to permutation symmetry, they may be interpreted for any coordinates. Temporarily fix $\pi_{l+1}, \ldots, \pi_m$. Let

$$A_{\pi_{l+1},\ldots,\pi_m} = \{\mathbf{v} = (v_1, \ldots, v_l) : (v_1, \ldots, v_l, \pi_{l+1}, \ldots, \pi_m) \in \mathrm{PF}(\mathbf{u})\},$$

where $\mathbf{v}$ is in non-decreasing order. Following terminology in [1], we will call such $\mathbf{v}$'s $\mathbf{u}$-parking completions for $\pi_{l+1}, \ldots, \pi_m$.

▶ **Proposition 1.** *Take $1 \leq l \leq m$ any integer. Suppose that $\mathbf{v}$ is in non-decreasing order and is a maximal $\mathbf{u}$-parking completion for the fixed $\pi_{l+1}, \ldots, \pi_m$. Then for all $1 \leq i \leq l$, we have $v_i = u_{k_i}$ with $1 \leq k_1 < \cdots < k_l \leq m$, and $v_i \neq \pi_{l+1}, \ldots, \pi_m$.*

To identify the maximal $\mathbf{v}$ in $A_{\pi_{l+1},\ldots,\pi_m}$, we arrange π_i for $l+1 \leq i \leq m$ in non-decreasing order, denoted by $\pi_{(l+1)} \leq \cdots \leq \pi_{(m)}$. Set $n_l = 0$. We find the minimum index n_i in order, starting with n_{l+1}, such that $n_i > n_{i-1}$ and $u_{n_i} \geq \pi_{(i)}$ for each $l+1 \leq i \leq m$. If such u_{n_i}'s cannot be located, then $A_{\pi_{l+1},\ldots,\pi_m}$ is empty. Otherwise excluding these u_{n_i}'s from $\mathbf{u}$ gives the optimal $\mathbf{v}$. From the parking scheme, if $\mathbf{v} \in A_{\pi_{l+1},\ldots,\pi_m}$, then $\mathbf{w} \in A_{\pi_{l+1},\ldots,\pi_m}$ for all $\mathbf{w} \leq_C \mathbf{v}$, where $\leq_C$ is the component-wise partial order. This implies that if $A_{\pi_{l+1},\ldots,\pi_m}$ is non-empty, then there is a unique maximal $\mathbf{u}$-parking completion $\mathbf{v} \in [u_m]^l$ with $v_i = u_{k_i}$ for all $1 \leq i \leq l$, where $1 \leq k_1 < \cdots < k_l \leq m$, and $A_{\pi_{l+1},\ldots,\pi_m} = [\mathbf{v}]$. Therefore given the last $m - l$ parking preferences, it is sufficient to identify the largest feasible first l preferences (if they exist).

▶ **Example 2.** Take $\mathbf{u} = (u_1, u_2, u_3, u_4) = (2, 3, 5, 8)$, $\pi_3 = 6$, and $\pi_4 = 2$. Then $A_{\pi_3,\pi_4} = [\mathbf{v}] = [(u_2, u_3)] = [(3, 5)]$. See illustration below.

$$\begin{array}{ccccc} \pi_{(3)} & 2 & \leq & 2 & u_1 \\ v_1 & & & 3 & u_2 \\ v_2 & & & 5 & u_3 \\ \pi_{(4)} & 6 & \leq & 8 & u_4 \end{array}$$

We will now introduce an original combinatorial construction which we term a parking function multi-shuffle to generic $\mathbf{u}$-parking functions.

▶ **Definition 3** ($\mathbf{u}$-parking function multi-shuffle)**.** *Take $1 \leq l \leq m$ any integer. Let $\mathbf{v} = (v_1, \ldots, v_l) \in [u_m]^l$ be in increasing order with $v_i = u_{k_i}$ for all $1 \leq i \leq l$, where $1 \leq k_1 < \cdots < k_l \leq m$. Say that $\pi_{l+1}, \ldots, \pi_m$ is a* $\mathbf{u}$-parking function multi-shuffle *of $l+1$ $\mathbf{u}$-parking functions $\boldsymbol{\alpha}_1 \in PF(u_1, \ldots, u_{k_1-1}), \boldsymbol{\alpha}_2 \in PF(u_{k_1+1} - u_{k_1}, \ldots, u_{k_2-1} - u_{k_1}), \ldots, \boldsymbol{\alpha}_l \in PF(u_{k_{l-1}+1} - u_{k_{l-1}}, \ldots, u_{k_l-1} - u_{k_{l-1}})$, and $\boldsymbol{\alpha}_{l+1} \in PF(u_{k_l+1} - u_{k_l}, \ldots, u_m - u_{k_l})$ if $\pi_{l+1}, \ldots, \pi_m$ is any permutation of the union of the $l+1$ words $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 + (u_{k_1}, \ldots, u_{k_1}), \ldots, \boldsymbol{\alpha}_{l+1} + (u_{k_l}, \ldots, u_{k_l})$. (If $k_{j-1} = k_j - 1$ for some j, we take the corresponding $\boldsymbol{\alpha}_j$ as empty.) We will denote this by $(\pi_{l+1}, \ldots, \pi_m) \in MS(\mathbf{v})$.*

▶ **Example 4.** Take $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = (3, 4, 5, 6, 7, 8, 9, 10)$ and $\mathbf{v} = (u_4, u_6) = (6, 8)$. Take $\boldsymbol{\alpha}_1 = (2, 1, 2) \in \mathrm{PF}(u_1, u_2, u_3) = \mathrm{PF}(3, 4, 5)$, $\boldsymbol{\alpha}_2 = (1) \in \mathrm{PF}(u_5 - u_4) = \mathrm{PF}(1)$, and $\boldsymbol{\alpha}_3 = (2, 1) \in \mathrm{PF}(u_7 - u_6, u_8 - u_6) = \mathrm{PF}(1, 2)$. Then $(2, \overline{7}, 2, \underline{9}, \underline{10}, 1) \in \mathrm{MS}(6, 8)$ is a multi-shuffle of the three words $(2, 1, 2)$, (7), and $(10, 9)$.

Take $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) = (2, 4, 5, 6, 8, 12, 15)$ and $\mathbf{v} = (u_2, u_5) = (4, 8)$. Take $\boldsymbol{\alpha}_1 = (1) \in \mathrm{PF}(u_1) = \mathrm{PF}(2)$, $\boldsymbol{\alpha}_2 = (1, 2) \in \mathrm{PF}(u_3 - u_2, u_4 - u_2) = \mathrm{PF}(1, 2)$, and $\boldsymbol{\alpha}_3 = (5, 3) \in \mathrm{PF}(u_6 - u_5, u_7 - u_5) = \mathrm{PF}(4, 7)$. Then $(\underline{11}, \underline{13}, \overline{5}, 1, \overline{6}) \in \mathrm{MS}(4, 8)$ is a multi-shuffle of the three words (1), $(5, 6)$, and $(13, 11)$.

The **u**-parking function multi-shuffle allows for an explicit characterization of multiple coordinates of **u**-parking functions. It connects the identification of the maximal element in $A_{\pi_{l+1},\dots,\pi_m}$ to the decomposition of $\pi_{l+1},\dots,\pi_m$ into a multi-shuffle.

▶ **Theorem 5.** *Take $1 \le l \le m$ any integer. Let $\mathbf{v} = (v_1,\dots,v_l) \in [u_m]^l$ be in increasing order with $v_i = u_{k_i}$ for all $1 \le i \le l$, where $1 \le k_1 < \cdots < k_l \le m$. Then $A_{\pi_{l+1},\dots,\pi_m} = [\mathbf{v}]$ if and only if $(\pi_{l+1},\dots,\pi_m) \in MS(\mathbf{v})$.*

▶ **Example 6** (Continued from Example 4)**.** Take $\mathbf{u} = (3,4,5,6,7,8,9,10)$, then $A_{2,7,2,9,10,1} = [(6,8)]$ is equivalent to $(2,7,2,9,10,1) \in \mathrm{MS}(6,8)$. Take $\mathbf{u} = (2,4,5,6,8,12,15)$, then $A_{11,13,5,1,6} = [(4,8)]$ is equivalent to $(11,13,5,1,6) \in \mathrm{MS}(4,8)$. See illustration below.

$$\begin{array}{cccccc}
\pi_{(3)} & 1 & \le & 3 & u_1 \\
\pi_{(4)} & 2 & \le & 4 & u_2 \\
\pi_{(5)} & 2 & \le & 5 & u_3 \\
v_1 & & & 6 & u_4 \\
\pi_{(6)} & 7 & \le & 7 & u_5 \\
v_2 & & & 8 & u_6 \\
\pi_{(7)} & 9 & \le & 9 & u_7 \\
\pi_{(8)} & 10 & \le & 10 & u_8
\end{array}
\qquad\qquad
\begin{array}{cccccc}
\pi_{(3)} & 1 & \le & 2 & u_1 \\
v_1 & & & 4 & u_2 \\
\pi_{(4)} & 5 & \le & 5 & u_3 \\
\pi_{(5)} & 6 & \le & 6 & u_4 \\
v_2 & & & 8 & u_5 \\
\pi_{(6)} & 11 & \le & 12 & u_6 \\
\pi_{(7)} & 13 & \le & 15 & u_7
\end{array}$$

Of relevance to our investigation, we also utilize the Gončarov polynomials. Let $(a_0, a_1, \dots)$ be a sequence of numbers. The Gončarov polynomials $g_m(x; a_0, a_1, \dots, a_{m-1})$ for $m = 0, 1, \dots$ are the basis of solutions to the Gončarov interpolation problem in numerical analysis. They are defined by the biorthogonality relation:

$$\epsilon(a_i) D^i g_m(x; a_0, a_1, \dots, a_{m-1}) = m! \delta_{im},$$

where $\epsilon(a_i)$ is evaluation at a_i, D is the differentiation operator, and δ_{im} is the Kronecker delta. The Gončarov polynomials satisfy many nice algebraic and analytic properties, making them very useful in analysis and combinatorics. Specifically, we list two properties of Gončarov polynomials below:

1. Determinant formula.

$$g_m(x; a_0, a_1, \dots, a_{m-1}) = m! \begin{vmatrix}
1 & a_0 & \frac{a_0^2}{2!} & \frac{a_0^3}{3!} & \cdots & \frac{a_0^{m-1}}{(m-1)!} & \frac{a_0^m}{m!} \\
0 & 1 & a_1 & \frac{a_1^2}{2!} & \cdots & \frac{a_1^{m-2}}{(m-2)!} & \frac{a_1^{m-1}}{(m-1)!} \\
0 & 0 & 1 & a_2 & \cdots & \frac{a_2^{m-3}}{(m-3)!} & \frac{a_2^{m-2}}{(m-2)!} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & a_{m-1} \\
1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{m-1}}{(m-1)!} & \frac{x^m}{m!}
\end{vmatrix}.$$

2. Shift invariance.

$$g_m(x+y; a_0+y, a_1+y, \dots, a_{m-1}+y) = g_m(x; a_0, a_1, \dots, a_{m-1}).$$

We note that the number of **u**-parking functions of length m is $|\mathrm{PF}(\mathbf{u})| = (-1)^m g_m(0; u_1, \dots, u_m)$. For a full discussion of the connection between Gončarov polynomials and **u**-parking functions, we refer to Kung and Yan [12].

▶ **Theorem 7.** *Take $1 \le l \le m$ any integer. Let $\mathbf{w} = (w_1, \dots, w_l) \in [u_m]^l$ be in non-decreasing order. The number of parking functions $\boldsymbol{\pi} \in PF(\mathbf{u})$ with $\pi_1 = w_1, \dots, \pi_l = w_l$ is*

$$(-1)^{m-l} \sum_{\mathbf{s} \in S_l(\mathbf{w})} \binom{m-l}{\mathbf{s}} \prod_{i=1}^{l+1} g_{s_i}(u_{s_1+\cdots+s_{i-1}+i-1}; u_{s_1+\cdots+s_{i-1}+i}, \dots, u_{s_1+\cdots+s_i+i-1}),$$

where $s_0 = u_0 = 0$, $g_{s_i}(\cdot)$ are the Gončarov polynomials, and

$$S_l(\mathbf{w}) = \left\{ \mathbf{s} = (s_1, \dots, s_{l+1}) \in \mathbb{N}^{l+1} \mid {}^{u_{s_1+\cdots+s_i+i} \ge w_i \ \forall i \in [l]}_{s_1+\cdots+s_{l+1} = m-l} \right\}.$$

Note that this quantity stays constant if all $w_i \le u_i$ and decreases as each w_i increases past u_i as there are fewer resulting summands.

For the special case $l = 0$ and $\mathbf{w} = ()$ (where no parking preferences are specified), we recover the total number of $\mathbf{u}$-parking functions $|\operatorname{PF}(\mathbf{u})| = (-1)^m g_m(0; u_1, \dots, u_m)$. We describe an alternative characterization of this number in the following.

▶ **Proposition 8.** *The number of $\mathbf{u}$-parking functions $|\,PF(\mathbf{u})|$ satisfies*

$$|\,PF(\mathbf{u})| = \sum_{\mathbf{s} \in C(m)} \binom{m}{\mathbf{s}} \prod_{i=1}^{u_m-m+1} (s_i+1)^{s_i-1},$$

where $C(m)$ consists of compositions of m: $\mathbf{s} = (s_1, \dots, s_{u_m-m+1}) \models m$ with $\sum_{i=1}^{u_m-m+1} s_i = m$, subject to $s_1 + \cdots + s_{u_i-i+1} \ge i$ for all $1 \le i \le m$.

▶ **Example 9.** Take $\mathbf{u} = (2,5)$. Then $(\pi_1, \pi_2) \in \operatorname{PF}(\mathbf{u})$ satisfies

$$(\pi_1,\pi_2) \in A := \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),$$
$$(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(4,1),(4,2),(5,1),(5,2)\}.$$

From Proposition 8,

$$\begin{aligned}|A| = &\binom{2}{2,0,0,0} 3^1 1^{-1} 1^{-1} 1^{-1} + \binom{2}{0,2,0,0} 1^{-1} 3^1 1^{-1} 1^{-1} + \binom{2}{1,1,0,0} 2^0 2^0 1^{-1} 1^{-1} \\ &+ \binom{2}{1,0,1,0} 2^0 1^{-1} 2^0 1^{-1} + \binom{2}{1,0,0,1} 2^0 1^{-1} 1^{-1} 2^0 \\ &+ \binom{2}{0,1,1,0} 1^{-1} 2^0 2^0 1^{-1} + \binom{2}{0,1,0,1} 1^{-1} 2^0 1^{-1} 2^0 \\ = &\ 3+3+2+2+2+2+2 = 16.\end{aligned}$$

3 Properties of random (a, b)-parking functions

In general, there are no nice closed-form expressions for Gončarov polynomials related to $\mathbf{u}$-parking functions, but such expressions exist for a specific class of $\mathbf{u}$-parking functions. When the entries of the vector $\mathbf{u}$ form an arithmetic progression: $u_i = a + (i-1)b$ for some positive integers a and b, we get *Abel polynomials*:

$$g_m(x; a, a+b, \dots, a+(m-1)b) = (x-a)(x-a-mb)^{m-1}. \tag{3.1}$$

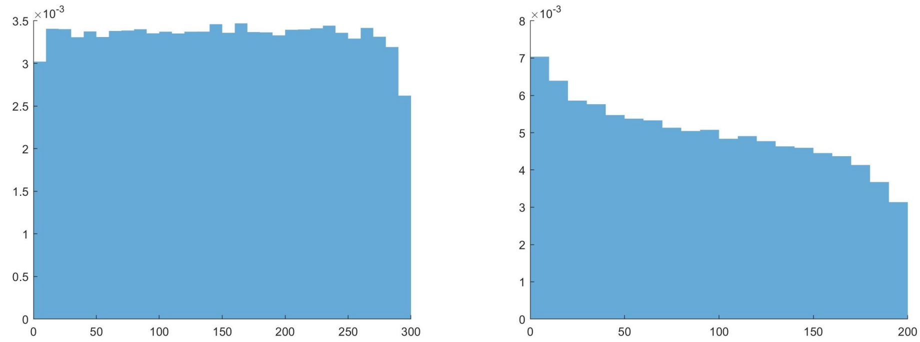

Figure 1 The distribution of π_1 (the first parking coordinate) in $100,000$ samples of $(cm+b,b)$-parking functions chosen uniformly at random, where $m=100$ and $b=2$. In the left plot $c=1$ and in the right plot $c=0$.

We call these $\mathbf{u}$-parking functions (a,b)-parking functions, and denote the set of (a,b)-parking functions of length m by $\mathrm{PF}(a,b,m)$. Using (3.1), the number of (a,b)-parking functions of length m is $|\,\mathrm{PF}(a,b,m)|=(-1)^m g_m(0;a,a+b,\dots,a+(m-1)b)=a(a+mb)^{m-1}$.

In this section we use the multi-shuffle construction introduced in Section 2 to investigate various properties of a parking function chosen uniformly at random from $\mathrm{PF}(a,b,m)$. As stated in the introduction, taking $b=1$, $c=n/m-1$, and $a=cm+b$, an (a,b)-parking function of length m depicts the scenario of parking m cars in n spots sequentially along a one-way street. Therefore among all possible a and b, of particular interest to us is when $a=cm+b$ for some $c\geq 0$. We will write our results in terms of coordinates $\pi_1,\dots,\pi_l$ of parking functions, where $1\leq l\leq m$ is any integer. However, the parking coordinates satisfy permutation symmetry, so the statements in this section may be interpreted for any coordinates.

Before proceeding with the calculations, we outline an effective method for generating a random (a,b)-parking function of length m. The algorithm is suggested by Stanley's generalization [16] of Pollak's circle argument for parking functions [6]. To select $\pi\in\mathrm{PF}(a,b,m)$ uniformly at random:

1. Pick an element $\pi\in(\mathbb{Z}/(a+mb)\mathbb{Z})^m$, where the equivalence class representatives are taken in $1,\dots,a+mb$.
2. For $k\in\{0,\dots,a+mb-1\}$, record k if $\pi+k(1,\dots,1)\in\mathrm{PF}(a,b,m)$ (modulo $a+mb$), where $(1,\dots,1)$ is a vector of length m. There should be exactly a such k's.
3. Pick one k from (2) uniformly at random. Then $\pi+k(1,\dots,1)$ is an (a,b)-parking function of length m taken uniformly at random.

Figure 1 shows a histogram of the values of π_1 based on $100,000$ random samples of $\mathrm{PF}(cm+b,b,m)$ for $m=100$ and $b=2$. The left plot is for $c=1$ and the right plot is for $c=0$. A closed formula for the distribution of π_1 as well as its asymptotic approximation will be provided in Section 3.3.

3.1 Mixed moments of multiple coordinates

In this subsection we study the asymptotics of the generic mixed moments of (a,b)-parking functions of length m when $a=cm+b$ for some $c>0$ via a tree function approach. Building upon Theorem 7 and Proposition 8, we first count the number of (a,b)-parking functions of length m where the specified parking preferences of the first l cars are exactly b spots apart. This calculation will be central in deriving an asymptotic formula for the mixed moments.

▶ **Proposition 10.** *Take $1 \le l \le m$ any integer. Let $0 \le k \le m-l$. The number of parking functions $\boldsymbol{\pi} \in PF(a,b,m)$ with $\pi_1 = a+kb, \dots, \pi_l = a+(k+l-1)b$ is*

$$a \sum_{s=0}^{m-l-k} \binom{m-l}{s} (a+(m-l-s)b)^{m-s-l-1} b^s l(s+l)^{s-1}.$$

The following technical lemma will also be needed in the derivation of asymptotic mixed moments.

▶ **Lemma 11.** *Take $l \ge 1$ any integer and m large. Take $b \ge 1$ any integer and $a = cm+b$ for some $c > 0$. Set $u_i = a+(i-1)b$ for $i \ge 1$ and $u_0 = 0$. For $1 \le i \le l$, take $p_i \ge 1$ any integer and $S_i \sim m$ with $S_1 < \cdots < S_l$. Let $\sum_{j=u_s+1}^{u_{s+1}} j^{p_i} = f_i(s)$ for $s \ge 0$. Then*

$$\sum_{\substack{\#\{i:\, s_i \le S_k\} \ge k \\ \forall k \in [l]}} \prod_{i=1}^{l} f_i(s_i) = \frac{(cm+(S_l+1)b)^{\sum_{i=1}^l p_i + l}}{\prod_{i=1}^l (p_i+1)} \left(1 + \frac{1}{(b+c)m}\left(\frac{\sum_{i=1}^l p_i + l}{2}\right) + O\left(\frac{1}{m^2}\right)\right).$$

We are now ready to establish an asymptotic result for the mixed moments of two coordinates. The key idea is to break apart the parking preferences of the first two cars into blocks and utilize properties of the tree function $F(z) = \sum_{s=0}^{\infty} (s+1)^{s-1} \frac{z^s}{s!}$ in the asymptotic investigation.

▶ **Theorem 12.** *Take $p, q \ge 1$ any integer and m large. Take $b \ge 1$ any integer and $a = cm+b$ for some $c > 0$. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a,b,m)$, we have*

$$\mathbb{E}(\pi_1^p) = \frac{((b+c)m)^p}{p+1}\left(1 + \frac{1}{c(b+c)m}\left(\frac{c(p+1)}{2} - b^2 p\right) + O\left(\frac{1}{m^2}\right)\right),$$

and

$$\mathbb{E}(\pi_1^p \pi_2^q) = \frac{((b+c)m)^{p+q}}{(p+1)(q+1)}\left(1 + \frac{1}{c(b+c)m}\left(\frac{c(p+q+2)}{2} - b^2(p+q)\right) + O\left(\frac{1}{m^2}\right)\right).$$

▶ **Proposition 13.** *Take m large. Take $b \ge 1$ any integer and $a = cm+b$ for some $c > 0$. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a,b,m)$, we have*

$$\mathbb{V}\mathrm{ar}(\pi_1) \sim \frac{((b+c)m)^2}{12} - \frac{b^2(b+c)m}{6c}, \qquad \mathbb{C}\mathrm{ov}(\pi_1, \pi_2) \sim -\frac{b^2(b+c)^2}{4c^2}.$$

Extending the asymptotic expansion approach in the proof of Theorem 12, we have the following more general result.

▶ **Theorem 14.** *Take $l \ge 1$ any integer and m large. Take $b \ge 1$ any integer and $a = cm+b$ for some $c > 0$. For $1 \le i \le l$, take $p_i \ge 1$ any integer. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a,b,m)$, we have*

$$\mathbb{E}\left(\prod_{i=1}^{l} \pi_i^{p_i}\right) = \frac{((b+c)m)^{\sum_{i=1}^l p_i}}{\prod_{i=1}^l (p_i+1)}\left(1 + \frac{1}{c(b+c)m}\left(\frac{c\left(\sum_{i=1}^l p_i + l\right)}{2} - b^2 \sum_{i=1}^{l} p_i\right) + O\left(\frac{1}{m^2}\right)\right).$$

3.2 The special situation $c = 0$

In this subsection we study the asymptotics of the generic mixed moments of (b, b)-parking functions of length m via Abel's multinomial theorem. Indeed, the asymptotic moment calculations in Section 3.1 could as well be approached via Abel's multinomial theorem. Unlike the tree function method which fails for the case $c = 0$ due to divergence, Abel's multinomial theorem applies broadly for $c \geq 0$. However calculation-wise it is in general more cumbersome to apply Abel's multinomial theorem as compared with the tree function method, so we only use this alternative approach when $c = 0$ and so $a = b$.

▶ **Theorem 15** (Abel's multinomial theorem, derived from Pitman [14] and Riordan [15]). *Let*

$$A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) = \sum_{\mathbf{s} \models n} \binom{n}{\mathbf{s}} \prod_{j=1}^{m} (x_j + s_j)^{s_j + p_j},$$

where $\mathbf{s} = (s_1, \ldots, s_m)$ *and* $\sum_{i=1}^m s_i = n$. *Then*

$$A_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m; p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m) \\ = A_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_m; p_1, \ldots, p_j, \ldots, p_i, \ldots, p_m).$$

$$A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) \\ = \sum_{i=1}^{m} A_{n-1}(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_m; p_1, \ldots, p_{i-1}, p_i + 1, p_{i+1}, \ldots, p_m).$$

$$A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) = \sum_{s=0}^{n} \binom{n}{s} s!(x_1 + s) A_{n-s}(x_1 + s, x_2, \ldots, x_m; p_1 - 1, p_2, \ldots, p_m).$$

Moreover, the following special instances hold via the basic recurrences listed above:

$$A_n(x_1, \ldots, x_m; -1, \ldots, -1) = (x_1 \cdots x_m)^{-1}(x_1 + \cdots + x_m)(x_1 + \cdots + x_m + n)^{n-1}.$$

$$A_n(x_1, \ldots, x_m; -1, \ldots, -1, 0) = (x_1 \cdots x_m)^{-1} x_m (x_1 + \cdots + x_m + n)^n.$$

Take $l \geq 1$ any integer and m large. Take $b \geq 1$ any integer and $a = cm + b$ for some $c \geq 0$. For $1 \leq i \leq l$, take $p_i \geq 1$ any integer. In computing $\mathbb{E}(\prod_{i=1}^l \pi_i^{p_i})$, we recognize from Lemma 11 that asymptotically

$$\begin{aligned}
&\sum \left(\prod_{i=1}^{l} \pi_i^{p_i}\right) \{\boldsymbol{\pi} \in \mathrm{PF}(a, b, m) : \pi_i \text{ specified } \forall i \in [l]\} \\
&= \frac{(cm+b) b^{m + \sum_{i=1}^l p_i - 1}}{\prod_{i=1}^l (p_i + 1)} \sum_{s_1=0}^{m-l} \cdots \sum_{s_l=0}^{m-l-s_1-\cdots-s_{l-1}} \binom{m-l}{s_1, \ldots, s_l, m-l-s_1-\cdots-s_l} \cdot \\
&\cdot (m - l - s_1 - \cdots - s_l + 1 + \frac{cm}{b})^{m-l-1-s_1-\cdots-s_l} \prod_{i=1}^{l} (s_i+1)^{s_i-1} \cdot \\
&\cdot (m - s_l + \frac{cm}{b})^{\sum_{i=1}^l p_i + l} \left(1 + \frac{1}{(b+c)m - s_l b} \left(\frac{\sum_{i=1}^l p_i + l}{2}\right) + O\left(m^{-2}\right)\right).
\end{aligned} \tag{3.2}$$

Using Abel's multinomials, the leading order terms in (3.2) may be represented as

$$\frac{(cm+b)b^{m+\sum_{i=1}^{l} p_i-1}}{\prod_{i=1}^{l}(p_i+1)}\left(A_{m-l}\left(1+\frac{cm}{b},\underbrace{1,\ldots,1}_{l\text{ 1's}};\sum_{i=1}^{l}p_i+l-1,\underbrace{-1,\ldots,-1}_{l\text{ -1's}}\right)\right.$$
$$+(l-1)\left(\sum_{i=1}^{l}p_i+l\right)A_{m-l}\left(1+\frac{cm}{b},\underbrace{1,\ldots,1}_{l\text{ 1's}};\sum_{i=1}^{l}p_i+l-2,0,\underbrace{-1,\ldots,-1}_{l-1\text{ -1's}}\right)$$
$$\left.+\frac{1}{2b}\left(\sum_{i=1}^{l}p_i+l\right)A_{m-l}\left(1+\frac{cm}{b},\underbrace{1,\ldots,1}_{l\text{ 1's}};\sum_{i=1}^{l}p_i+l-2,\underbrace{-1,\ldots,-1}_{l\text{ -1's}}\right)\right).$$

This is a general formula that works for any a, b, and l. When $c=0$ and so $a=b$, taking $l=1,2$, we have

$$\mathbb{E}(\pi_1)\sim b\left(\frac{m}{2}-\frac{\sqrt{2\pi}}{4}m^{1/2}+\frac{7}{6}\right)+\frac{1}{2}.$$

$$\mathbb{E}(\pi_1^2)\sim b^2\left(\frac{m^2}{3}-\frac{\sqrt{2\pi}}{4}m^{3/2}+\frac{4}{3}m\right)+\frac{b}{2}m.$$

$$\mathbb{E}(\pi_1\pi_2)\sim b^2\left(\frac{m^2}{4}-\frac{\sqrt{2\pi}}{4}m^{3/2}+\frac{3}{2}m\right)+\frac{b}{2}m.$$

These asymptotic results are in sharp contrast with the case $a=cm+b$ where $c>0$. As $c\to 0$, the correction terms blow up, contributing to the different asymptotic orders between the generic situation $a\gtrsim b$ (corresponding to $c>0$) and the special situation $a=b$ (corresponding to $c=0$). Paralleling Proposition 13, the following asymptotics are immediate.

▶ **Proposition 16.** *Take m large. Take $b\geq 1$ any integer. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(b,b,m)$, we have*

$$\mathbb{V}\mathrm{ar}(\pi_1)\sim\frac{b^2}{12}m^2+\frac{b^2(4-3\pi)}{24}m,\qquad \mathbb{C}\mathrm{ov}(\pi_1,\pi_2)\sim\frac{b^2(8-3\pi)}{24}m.$$

For $\boldsymbol{\pi}=(\pi_1,\ldots,\pi_m)\in\mathrm{PF}(a,b,m)$, the (a,b)-displacement $\mathrm{disp}^{(a,b)}(\boldsymbol{\pi})$ is defined as

$$\mathrm{disp}^{(a,b)}(\boldsymbol{\pi})=b\binom{m}{2}+am-(\pi_1+\cdots+\pi_m).$$

Figure 2 shows a histogram of the displacement based on $100,000$ random samples of $\mathrm{PF}(cm+b,b,m)$ for $m=100$ and $b=2$. The left plot ($c=1$) approximates a normal distribution and the right plot ($c=0$) approximates an Airy distribution. The displacement definition is in connection with the displacement enumerator of (a,b)-parking functions. Note that the set of (a,b)-parking functions of length m is in bijection with the set of length-a sequences of rooted b-forests on m vertices, and there are related formulations for the (a,b)-inversion and inversion enumerator of length-a sequences of rooted b-forests. See Yan [19] for more details.

▶ **Theorem 17.** *Take m large. Take $b\geq 1$ any integer and $a=cm+b$ for some $c>0$. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a,b,m)$, we have*

$$\mathbb{E}(disp^{(a,b)}(\boldsymbol{\pi}))\sim\frac{cm^2}{2}+\left(\frac{b^2}{2c}+\frac{b}{2}-\frac{1}{2}\right)m,\qquad \mathbb{V}\mathrm{ar}(disp^{(a,b)}(\boldsymbol{\pi}))\sim\frac{(b+c)^2}{12}m^3.$$

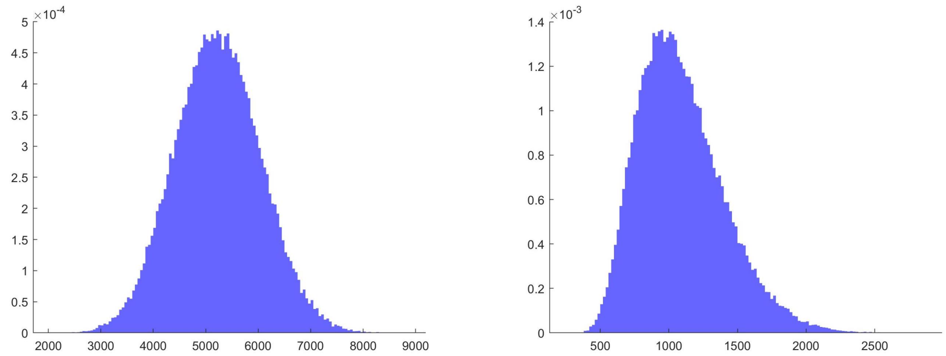

Figure 2 The distribution of displacement in 100,000 samples of $(cm+b, b)$-parking functions chosen uniformly at random, where $m = 100$ and $b = 2$. In the left plot $c = 1$ and in the right plot $c = 0$.

On the other hand, when $c = 0$ and so $a = b$,

$$\mathbb{E}(disp^{(a,b)}(\boldsymbol{\pi})) \sim b\frac{\sqrt{2\pi}}{4}m^{3/2} - \left(\frac{2b}{3}+\frac{1}{2}\right)m, \qquad \mathbb{V}\mathrm{ar}(disp^{(a,b)}(\boldsymbol{\pi})) \sim \frac{b^2(10-3\pi)}{24}m^3.$$

3.3 Boundary behavior of a single coordinate

In this subsection we examine the boundary behavior of a single coordinate of (a, b)-parking functions of length m when $a = cm + b$ for some $c \geq 0$. As in the case of the distribution of multiple coordinates, the asymptotic tendency in the generic situation $c > 0$ and the special situation $c = 0$ are strikingly different. Calculational techniques (tree function, Abel's multinomial theorem) employed in Sections 3.1 and 3.2 will be used in our investigation, with details omitted.

▶ **Proposition 18.** *Let $1 \leq j \leq a+(m-1)b$. The number of parking functions $\boldsymbol{\pi} \in PF(a, b, m)$ with $\pi_1 = j$ is*

$$ab \sum_{s\colon a+sb\geq j} \binom{m-1}{s}(a+sb)^{s-1}((m-s)b)^{m-2-s}.$$

Note that this quantity stays constant for $j \leq a$ and decreases as j increases past a as there are fewer resulting summands.

Recall from Proposition 1 that if $A_{\pi_2,\dots,\pi_m} = [v]$ is non-empty, then $v = a + sb$ for some $0 \leq s \leq m-1$. Let X be a random variable satisfying the Borel distribution with parameter μ $(0 \leq \mu \leq 1)$, that is, with pmf given by, for $j = 1, 2, \dots,$

$$\mathbb{P}_\mu(X = j) = \frac{e^{-\mu j}(\mu j)^{j-1}}{j!}.$$

Denote by $\mathbb{Q}_\mu(j) = \mathbb{P}_\mu(X \geq j)$. We refer to Stanley [18] for some nice properties of this discrete distribution.

▶ **Corollary 19.** *Fix j and take m large relative to j. Take $b \geq 1$ any integer and $a = cm + b$ for some $c \geq 0$. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a, b, m)$, we have*

$$\mathbb{P}(\pi_1 = a + (m-1)b - j) \sim \frac{1 - \mathbb{Q}_{b/(b+c)}(\lfloor j/b \rfloor + 2)}{(b+c)m},$$

where $\mathbb{Q}_{b/(b+c)}(l) = \mathbb{P}_{b/(b+c)}(X \geq l)$ is the tail distribution function of Borel-$b/(b+c)$.

▶ **Corollary 20.** *Fix j and take m large relative to j. Take $b \geq 1$ any integer and $a = cm + b$ for some $c > 0$. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(a, b, m)$, we have*

$$\mathbb{P}(\pi_1 = 1) = \cdots = \mathbb{P}(\pi_1 = a) \sim \frac{1}{(b+c)m},$$

and

$$\mathbb{P}(\pi_1 = a + j) \sim e^{\frac{cm}{be} - \frac{b}{b+c}} \left(\frac{b}{b+c}\right)^{m-1} \frac{1}{cm^2} \left(\mathbb{P}(Y \geq \lceil j/b \rceil) - 1\right) + \frac{1}{(b+c)m},$$

where Y is a Poisson$((cm)/(be))$ random variable.

▶ **Corollary 21.** *Fix j and take m large relative to j. Take $b \geq 1$ any integer. For parking function $\boldsymbol{\pi}$ chosen uniformly at random from $PF(b, b, m)$, we have*

$$\mathbb{P}(\pi_1 = 1) = \cdots = \mathbb{P}(\pi_1 = b) \sim \frac{2}{bm},$$

and

$$\mathbb{P}(\pi_1 = b + j) \sim \frac{1 + \mathbb{Q}_1(\lceil j/b \rceil + 1)}{bm},$$

where $\mathbb{Q}_1(l) = \mathbb{P}_1(X \geq l)$ is the tail distribution function of Borel-1.

4 Final remarks

This paper is a part of an ongoing research direction to answer what random parking functions and their generalizations look like. We consider a generalization of parking functions known as $\mathbf{u}$-parking functions, which are defined for an arbitrarily fixed positive integer-valued vector $\mathbf{u}$. Since $\mathbf{u}$-parking functions include ordinary parking functions, the results in this paper can be specialized to recover other recent results about ordinary parking functions. As an application, we study a specific class of $\mathbf{u}$-parking functions when $\mathbf{u} = (a, a+b, \ldots, a+(m-1)b)$ is a vector of length m. Such $\mathbf{u}$-parking functions are commonly referred to as (a, b)-parking functions and have generated much interest because of their association with particularly nice Gončarov polynomials. We identify a striking contrast between the generic situation $a \gtrsim b$ and the special situation $a = b$ in various asymptotic probabilistic quantities.

Our asymptotic investigations in this work have concentrated mostly on the law of multiple coordinates and displacement of uniformly random (a, b)-parking functions, but there are many other structural properties of parking functions worth studying. For example, descent patterns, cycle counts, and equality processes, among others. The asymptotic distribution of some of these quantities have been explored in [2, 5, 13] for ordinary parking functions, but more generalized models are yet to be examined. Such models include $\mathbf{u}$-parking functions studied in this paper, as well as parking functions on mappings, on graphs, and the effect of group action on parking functions. These topics have amazing connections to many areas in math and computer science. Particularly for the field of analytic combinatorics, it would be interesting to see if there is a generic generating function approach to these random combinatorial structures. The multi-dimensional tree function method utilized in this paper is only one small step on a long journey.

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