Fragmentation Processes Derived from Conditioned Stable Galton-Watson Trees

Gabriel Berzunza Ojeda
Department of Mathematical Sciences, University of Liverpool, United Kingdom

Cecilia Holmgren
Department of Mathematics, Uppsala University, Sweden

Abstract

We study the fragmentation process obtained by deleting randomly chosen edges from a critical Galton-Watson tree $\mathbf{t}_n$ conditioned on having n vertices, whose offspring distribution belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$. This fragmentation process is analogous to that introduced in the works of Aldous, Evans and Pitman (1998), who considered the case of Cayley trees. Our main result establishes that, after rescaling, the fragmentation process of $\mathbf{t}_n$ converges as $n \to \infty$ to the fragmentation process obtained by cutting-down proportional to the length on the skeleton of an α-stable Lévy tree of index $\alpha \in (1, 2]$. We further establish that the latter can be constructed by considering the partitions of the unit interval induced by the normalized α-stable Lévy excursion with a deterministic drift studied by Miermont (2001). In particular, this extends the result of Bertoin (2000) on the fragmentation process of the Brownian CRT.

2012 ACM Subject Classification Mathematics of computing → Probabilistic algorithms

Keywords and phrases Additive coalescent, fragmentation, Galton-Watson trees, spectrally positive stable Lévy processes, stable Lévy tree, Prim's algorithm

Digital Object Identifier 10.4230/LIPIcs.AofA.2022.3

Related Version *Full Version*: `https://arxiv.org/pdf/2010.07880.pdf` [11]

Funding This work was supported by the Knut and Alice Wallenberg Foundation, a grant from the Swedish Research Council and The Swedish Foundations' starting grant from Ragnar Söderbergs Foundation.

1 Introduction and main results

Aldous, Evans and Pitman [3, 19, 28] (see also [13, 24]) considered a fragmentation process of a uniform random tree $\mathbf{t}_n$ on $n \in \mathbb{N}$ labelled vertices (or Cayley tree with n vertices) by deleting the edges of $\mathbf{t}_n$ one by one in uniform random order. More precisely, as time passes, the deletion of edges creates more and more subtrees of $\mathbf{t}_n$ (connected components) such that the evolution of the ranked vector of sizes (number of vertices) of these subtrees (in decreasing order) evolves as a fragmentation process. It turns out that the asymptotic behavior of this fragmentation process, in reverse time, is related to the so-called *standard additive coalescent* [3, 19]. Moreover, this leads to a continuous representation of the standard additive coalescent in terms of the time-reversal of an analogue fragmentation process of the Brownian continuum random tree (Brownian CRT). Evans and Pitman [19, Theorem 2] showed that an additive coalescent is a Feller Markov process with values in the infinite ordered set

$$\mathbb{S} := \Big\{ \mathbf{x} = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0 \ \text{ and } \ \sum_{i=1}^{\infty} x_i < \infty \Big\},$$

33rd International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2022).
Editor: Mark Daniel Ward; Article No. 3; pp. 3:1–3:14
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

endowed with the ℓ^1-norm, $\|\mathbf{x}\|_1 = \sum_{i=1}^{\infty} |x_i|$ for $\mathbf{x} \in \mathbb{S}$, whose evolution is described formally by: given that the current state is $\mathbf{x}$, two terms x_i and x_j, $i < j$, of $\mathbf{x}$ are chosen and merged into a single term $x_i + x_j$ (which implies some reordering of the resulting sequence) at rate equal to $x_i + x_j$.

In this extended abstract, we study the situation where one wants to cut-down critical Galton–Watson trees conditioned on having a fixed number of vertices, but whose offspring distribution belongs to the domain of attraction of a stable law. More precisely, consider a critical offspring distribution $\mu = (\mu(k), k \geq 0)$, i.e., a probability distribution on the nonnegative integers satisfying $\sum_{k=0}^{\infty} k\mu(k) = 1$. In addition, we always implicitly assume that $\mu(0) > 0$ and $\mu(0) + \mu(1) < 1$ to avoid degenerate cases, and that μ is aperiodic[1]. We say that μ belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$ if either the variance of μ is finite, or if $\mu([k, \infty)) = k^{-\alpha} L(k)$ as $k \to \infty$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $L(x) > 0$ for $x \in \mathbb{R}_+$ large enough and $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$ (such function is called slowly varying function). In other terms, if $(Y_i)_{i\geq 1}$ is a sequence of i.i.d. random variables with distribution μ, then there exists a sequence of positive real numbers $(B_n)_{n\geq 1}$ such that

$$B_n \to \infty \quad \text{and} \quad \frac{Y_1 + Y_2 + \cdots + Y_n - n}{B_n} \xrightarrow{d} Y_\alpha, \quad \text{in distribution as } \; n \to \infty \tag{1}$$

where the Laplace exponent of Y_α is given by $\mathbb{E}[\exp(-\lambda Y_\alpha)] = \exp(-\lambda^\alpha)$ whenever $\alpha \in (1, 2)$, and $\mathbb{E}[\exp(-\lambda Y_2)] = \exp(-\lambda^2/2)$ if $\alpha = 2$, for every $\lambda > 0$ ([20, Section XVII.5] guarantees its existence). In particular, for $\alpha = 2$, we have that Y_2 is distributed as a standard Gaussian random variable. The factor B_n is of order $n^{1/\alpha}$ (more precisely, $B_n/n^{1/\alpha}$ is a slowly varying function), and one may take $B_n = \sigma n^{1/2}$ when μ has finite variance σ^2.

We henceforth let $\mathbf{t}_n$ denote a critical Galton-Watson tree whose offspring distribution μ belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$ and refer to it as an α-stable GW-tree, for simplicity. Following Aldous, Evans and Pitman [3, 19], we are interested in the evolution of the ranked vector of sizes (in decreasing order) of the subtrees created by deleting randomly chosen edges from $\mathbf{t}_n$. Indeed, we will consider a continuous-time version of this cutting-down process. Equip each of the edges of $\mathbf{t}_n$ with i.i.d. uniform random variables (or weights) on $[0, 1]$ and independently of the tree $\mathbf{t}_n$. For $u \in [0, 1]$, we then keep the edges of $\mathbf{t}_n$ with weight smaller than u and discard the others. Therefore, one obtains a (fragmentation) forest $\mathbf{f}_n(u)$ conformed by the connected components (or subtrees of $\mathbf{t}_n$) created by the above procedure. In particular, the forest $\mathbf{f}_n(u)$ has the same set of vertices as $\mathbf{t}_n$ but clearly it has a different set of edges. Let $\mathbf{F}_n = (\mathbf{F}_n(u), u \in [0, 1])$ be the process given by

$$\mathbf{F}_n(u) = (F_{n,1}(1 - u), F_{n,2}(1 - u), \dots), \quad \text{for } \; u \in [0, 1],$$

the sequence of sizes (number of vertices) of the connected components of the forest $\mathbf{f}_n(1-u)$, ranked in decreasing order. We have strategically viewed the sequence of sizes of the components of $\mathbf{f}_n(1 - u)$ as an infinite sequence, by completing with an infinite number of zero terms. Plainly as time passes more and more subtrees are created, and thus, the process $\mathbf{F}_n$ evolves as a fragmentation process. Note also that $\mathbf{F}_n(0) = (n, 0, 0, \dots)$ and that $\mathbf{F}_n(1) = (1, 1, \dots, 1, 0, 0, \dots)$ where the first n terms $\mathbf{F}_n(1)$ are ones and the remaining terms are zeros. Since we are interested in studying the asymptotic behaviour of $\mathbf{F}_n$, we consider the (rescaled in time and space) fragmentation process $\mathbf{F}_n^{(\alpha)} = (\mathbf{F}_n^{(\alpha)}(t), t \geq 0)$ given by

$$\mathbf{F}_n^{(\alpha)}(t) = \frac{1}{n}\mathbf{F}_n\left(\frac{B_n}{n} t\right), \;\text{ for }\; 0 \leq t \leq n/B_n, \;\text{ and }\; \mathbf{F}_n^{(\alpha)}(t) = \frac{1}{n}\mathbf{F}_n(1) \;\text{ for }\; t > n/B_n, \tag{2}$$

[1] In the sense that the additive subgroup of the integers $\mathbb{Z}$ spanned by $\{i \geq 0 : \mu(i) \neq 0\}$ is $\mathbb{Z}$.

where $(B_n)_{n\geq 1}$ is a sequence satisfying (1). The process $\mathbf{F}_n^{(\alpha)}$ takes values on the set $\mathbb{S}$. The aim of this extended abstract is to establish a convergence result for the fragmentation process $\mathbf{F}_n^{(\alpha)}$. To state the precise statement (Theorem 1), it will be convenient to first introduce the limiting object.

Bertoin [6] showed that the fragmentation process of the Brownian CRT in [3] can be constructed by considering the partitions of the unit interval induced by a standard Brownian excursion with drift. This latter is sometimes called the *Brownian fragmentation*. In a similar vein, Miermont [25] built other fragmentation processes from Lévy processes with no positive jumps. Specifically, let $X_\alpha^{\mathrm{exc}} = (X_\alpha^{\mathrm{exc}}(s), s \in [0,1])$ be the normalized excursion (with unit length) of an α-stable spectrally positive Lévy process of index $\alpha \in (1,2]$; see e.g., [5]. In particular, X_2^{exc} is the normalized standard Brownian excursion. For every $t \geq 0$, define the processes $Y_\alpha^{(t)} = (Y_\alpha^{(t)}(s), s \in [0,1])$ and $I_\alpha^{(t)} = (I_\alpha^{(t)}(s), s \in [0,1])$ by letting

$$Y_\alpha^{(t)}(s) = X_\alpha^{\mathrm{exc}}(s) - ts \quad \text{and} \quad I_\alpha^{(t)}(s) = \inf_{u\in[0,s]} Y_\alpha^{(t)}(u), \qquad \text{for} \quad s \in [0,1]. \tag{3}$$

For $t \geq 0$, we introduce

$$\mathbf{F}^{(\alpha)}(t) = (F_1^{(\alpha)}(t), F_2^{(\alpha)}(t), \dots) \tag{4}$$

as the random element of $\mathbb{S}$ defined by the ranked sequence (in decreasing order) of the lengths of the intervals components of the complement of the support of the Stieltjes measure $\mathrm{d}(-I_\alpha^{(t)})$; note that $s \mapsto -I_\alpha^{(t)}(s) = \sup_{u\in[0,s]} -Y_\alpha^{(t)}(u)$ is an increasing process. More precisely, the support of $\mathrm{d}(-I_\alpha^{(t)})$ is defined as the set of times when the process $Y_\alpha^{(t)}$ reaches a new infimum. On the other hand, it can be shown that the support of $\mathrm{d}(-I_\alpha^{(t)})$ coincides with the so-called ladder time set of $-Y_\alpha^{(t)}$ which is given by the closure of the set of times when $Y_\alpha^{(t)}$ is equal to its infimum, i.e.,

$$\mathcal{L}^\alpha(t) \coloneqq \overline{\left\{ s \in [0,1] : Y_\alpha^{(t)}(s) = I_\alpha^{(t)}(s) \right\}};$$

see e.g., [5, Proposition 1, Chapter VI] and the discussion after that. Then $\mathbf{F}^{(\alpha)}(t)$ is the lengths of the open intervals in the canonical decomposition of $[0,1] \setminus \mathcal{L}^\alpha(t)$ arranged in the decreasing order. It is well-known that $\mathcal{L}^\alpha(t)$ is a.s. a random closed set with zero Lebesgue measure which implies that $\mathbf{F}^{(\alpha)}(t) \in \mathbb{S}_1$ a.s., where $\mathbb{S}_1 \subset \mathbb{S}$ is the space of the elements of $\mathbb{S}$ with sum 1; see [5, Corollary 5, Chapter VII]. Observe that for every fixed $0 \leq t < t'$, the process $s \mapsto Y_\alpha^{(t)}(s) - Y_\alpha^{(t')}(s) = (t'-t)s$ is monotone increasing which entails that $\mathcal{L}^\alpha(t) \subseteq \mathcal{L}^\alpha(t')$. Then the partition of $[0,1]$ induced by $\mathcal{L}^\alpha(t')$ is finer than that induced by $\mathcal{L}^\alpha(t)$. As a consequence, it has been shown by Miermont [25, Proposition 2] (see also [6, Theorem 1] for $\alpha = 2$) that $\mathbf{F}^{(\alpha)} = (\mathbf{F}^{(\alpha)}(t), t \geq 0)$ is a fragmentation process issued from $\mathbf{F}^{(\alpha)}(0) = (1, 0, 0, \dots)$. A precise description of its transition kernel is given in [25, Definition 4]. From now on, we will refer to $\mathbf{F}^{(\alpha)}$ as the α-stable fragmentation of index $\alpha \in (1,2]$.

We are now able to state our first main result. Let $\mathbb{D}(I, \mathbb{M})$ be the space of càdlàg functions from an interval $I \subseteq \mathbb{R}$ to the separable, complete metric space $(\mathbb{M}, d)$ equipped with the Skorohod topology; (see e.g. [12, Chapter 3] or [21, Chapter VI] for details on this space). We write $\xrightarrow{d}$ to denote convergence in distribution.

▶ **Theorem 1.** *Let* $\mathbf{t}_n$ *be an* α*-stable* GW*-tree of index* $\alpha \in (1,2]$*. Then, we have that*

$$(\mathbf{F}_n^{(\alpha)}(t), t \geq 0) \xrightarrow{d} (\mathbf{F}^{(\alpha)}(t), t \geq 0), \quad \text{as} \ n \to \infty, \quad \text{in the space} \ \mathbb{D}(\mathbb{R}_+, \mathbb{S}).$$

As mentioned earlier, $\mathbf{F}^{(2)}$ is exactly the Brownian fragmentation studied by Bertoin [6], that is to say, it corresponds to the fragmentation process derived from the Brownian CRT of Aldous and Pitman [3]; see also [1]. In view of this, the second goal of this paper is to show that indeed $\mathbf{F}^{(\alpha)}$ is the fragmentation process obtained by cutting-down the "edges" of the α-stable Lévy tree.

The α-stable Lévy tree of index $\alpha \in (1, 2]$ is the continuum random tree analogue of (discrete) α-stable GW-trees. They were introduced by Duquesne and Le Gall [18], and in particular, they also appear as scaling limits of α-stable GW-trees. In brief, the α-stable Lévy tree $\mathcal{T}_\alpha = (\mathcal{T}_\alpha, d_\alpha, \rho_\alpha)$ is a random compact metric space $(\mathcal{T}_\alpha, d_\alpha)$ with one distinguished element $\rho \in \mathcal{T}_\alpha$ called the root such that $(\mathcal{T}_\alpha, d_\alpha)$ is a tree-like space in that for $v, w \in \mathcal{T}_\alpha$, there is a unique non-self-crossing path $[v, w]$ from v to w in $\mathcal{T}_\alpha$, whose length equals $d_\alpha(v, w)$. The leaves $\mathrm{Lf}(\mathcal{T}_\alpha)$ of $\mathcal{T}_\alpha$ are those points that do not belong to the interior of any path leading from one point to another, and the skeleton of the tree is the set $\mathrm{Sk}(\mathcal{T}_\alpha) = \mathcal{T}_\alpha \setminus \mathrm{Lf}(\mathcal{T}_\alpha)$ of non-leaf points. The α-stable Lévy tree $\mathcal{T}_\alpha$ is naturally endowed with a uniform probability measure μ_α (*the mass measure*) that is supported on $\mathrm{Lf}(\mathcal{T}_\alpha)$, and a unique σ-finite measure λ_α (*the length measure*) carried by $\mathrm{Sk}(\mathcal{T}_\alpha)$ that assigns measure $d(v, w)$ to the geodesic path between v and w in $\mathcal{T}_\alpha$.

Following Aldous-Pitman's fragmentation [3] of the Brownian CRT, the analogue of deleting randomly chosen edges in $\mathbf{t}_n$ is to cut the skeleton of $\mathcal{T}_\alpha$ by a Poisson point process of cuts with intensity $\mathrm{d}t \otimes \lambda_\alpha(\mathrm{d}v)$ on $[0, \infty) \times \mathcal{T}_\alpha$. For all $t \geq 0$, define an equivalence relation $\sim_t$ on $\mathcal{T}_\alpha$ by saying that $v \sim_t w$, for $v, w \in \mathcal{T}_\alpha$, if and only if, no atom of the Poisson process that has appeared before time t belongs to the path $[v, w]$. These cuts split the α-stable Lévy tree into a (continuum) forest, that is a countably infinite set of smaller subtrees (connected components) of $\mathcal{T}_\alpha$. Let $\mathcal{T}_{\alpha,1}^{(t)}, \mathcal{T}_{\alpha,2}^{(t)}, \dots$ be the distinct equivalence classes for $\sim_t$ (connected components of $\mathcal{T}_\alpha$), ranked according to the decreasing order of their μ_α-masses. The subtrees $(\mathcal{T}_{\alpha,i}^{(t)}, i \geq 1)$ are nested as t varies, that is, for every $0 \leq t < t'$ and $i \geq 1$, there exits $j \geq 1$ such that $\mathcal{T}_{\alpha,i}^{(t')} \subset \mathcal{T}_{\alpha,j}^{(t)}$. Let $\mathbf{F}_{\mathcal{T}_\alpha} = (\mathbf{F}_{\mathcal{T}_\alpha}(t), t \geq 0)$ be the process given by

$$\mathbf{F}_{\mathcal{T}_\alpha}(t) = (\mu_\alpha(\mathcal{T}_{\alpha,1}^{(t)}), \mu_\alpha(\mathcal{T}_{\alpha,2}^{(t)}), \dots), \quad t \geq 0,$$

where $\mathbf{F}_{\mathcal{T}_\alpha}(0) = (1, 0, 0, \dots)$. Indeed, $\mathbf{F}_{\mathcal{T}_\alpha}$ is a fragmentation process in the sense that $\mathbf{F}_{\mathcal{T}_\alpha}(t')$ is obtained by splitting at random the elements of $\mathbf{F}_{\mathcal{T}_\alpha}(t)$, for $0 \leq t < t'$. We call $\mathbf{F}_{\mathcal{T}_\alpha}$ the fragmentation process of the α-stable Lévy tree. In particular, $\mathbf{F}_{\mathcal{T}_2}$ is the fragmentation process of the Brownian CRT introduced in [3, Section 2.2]. Note that $\mathbf{F}_{\mathcal{T}_\alpha}$ takes values in $\mathbb{S}$, and that [11, Lemma 7] shows that $\mathbf{F}_{\mathcal{T}_\alpha}(t) \in \mathbb{S}_1$ a.s., for every $t \geq 0$. We can now state our second main result.

▶ **Proposition 2.** *We have that* $(\mathbf{F}^{(\alpha)}(t), t \geq 0) \overset{d}{=} (\mathbf{F}_{\mathcal{T}_\alpha}(t), t \geq 0)$*, where* $\overset{d}{=}$ *means equal in distribution (in the sense of finite-dimensional distributions).*

Theorem 3 in [3] shows that the time-reversed fragmentation process of the Brownian CRT, i.e. $(\mathbf{F}_{\mathcal{T}_2}(e^{-t}), t \in \mathbb{R})$, is a version of the standard additive coalescent providing an explicit construction of this last process. In general, Miermont [25, Section 6] has shown that the time-reversed α-stable fragmentation process, i.e. $(\mathbf{F}^{(\alpha)}(e^{-t}), t \in \mathbb{R})$, is an eternal additive coalescent as described by Evans and Pitman [19]. More precisely, it is a mixing of so-called *extremal coalescents* of Aldous and Pitman [4] (see also [7]) which exact law is given in [25, Proposition 3]. Thus, Proposition 2 implies that this eternal additive coalescent can also be constructed from the α-stable Lévy tree by Poisson splitting along its skeleton. On the other hand, Theorem 1 and Proposition 2 clearly generalize Bertoin's work [6] and moreover, complete Miermont's [25] one by identifying the distribution of the α-stable fragmentation with that of the fragmentation process of the α-stable Lévy tree. In particular, Bertoin [8] proved that $\mathbf{F}^{(2)}$ (or equivalently, $\mathbf{F}_{\mathcal{T}_2}$) is a so-called *self-simlar fragmentation process* of index 1/2. However, Miermont [26] has already pointed out that $\mathbf{F}^{(\alpha)}$ (and therefore $\mathbf{F}_{\mathcal{T}_\alpha}$), for $\alpha \in (1, 2)$, is not a self-similar fragmentation due to the existence of points in $\mathcal{T}_\alpha$ with infinite degree.

The proof of Theorem 1 uses some of the ideas developed in [13] where only the case of Cayley tree was treated. However, in our more general framework, there are technical challenges that do not appear in [13], mostly due to the lack of some properties that only the Cayley tree satisfies. Informally, we use the so-called *Prim's algorithm* [29] to obtain a consistent ordering on the vertices of the forest created by deleting randomly chosen edges from $\mathbf{t}_n$ that we refer to as the *Prim order*; see Section 3. This will allow us to precisely encode this forest (and in particular, the sizes of connected components) using a discrete analogue of the process $Y_\alpha^{(t)}$ defined in (3) that we refer to as the *Prim path*. We then show that this (properly rescaled) Prim path indeed converges to its continuous version. Finally, we use a general approach developed in the complete version [11, Section 6] for the convergence of fragmentation processes encoded by functions in $\mathbb{D}([0, 1], \mathbb{R})$ to conclude our proof.

The proof of Proposition 2 follows along the lines of that of Theorem 3 in [3] for the Brownian CRT (see also the proof of Proposition 13 in [4]). Informally, one uses the convergence of rescaled α-stable GW-trees toward the α-stable Lévy tree $\mathcal{T}_\alpha$ in order to approximate the fragmentation process of $\mathcal{T}_\alpha$. The detailed proof of Proposition 2 is given in the complete version [11].

The rest of the manuscript is organized as follows. In Section 2, we discuss some connections with some combinatorial and probabilistic models: additive coalescents, parking schemes, laminations and Bernoulli bond-percolation. Section 3 is devoted to the introduction of Galton-Watson trees as well as the formal definition of the exploration process (the Prim path) associated with the fragmentation forest. Finally, in Section 4 and 5, we provide a fair enough guideline of the proof of Theorem 1.

2 Further remarks

In this section, we highlight some connections with previous works.

Additive coalescents

A Cayley tree of size n can be viewed as a Galton-Watson tree with Poissonian offspring distribution of parameter 1 and conditioned to have n vertices, where the labels are assigned to the vertices uniformly at random. In particular, Aldous, Evans and Pitman fragmentation process [3, 19, 28], say $\mathbf{F}_n^+ = (\mathbf{F}_n^+(t), t \geq 0)$, corresponds precisely to $\mathbf{F}_n^{(\alpha)}$ in (2), with $\alpha = 2$ and $B_n = n^{1/2}$. The fragmentation process $\mathbf{F}_n^+$ leads to a representation of an additive coalescent by an appropriate time reversal, that is, the exponential time-change $t \to e^{-t}$. Specifically, $(\mathbf{F}_n^+(e^{-t}), t \geq -(1/2)\ln n)$ is an additive coalescent starting at time $-(1/2)\ln n$ from the state $(1/n, 1/n, \dots, 1/n, 0, 0, \dots) \in \mathbb{S}$. Evans and Pitman [19] (see also [3, Proposition 2]) showed that this time-reversed version of $\mathbf{F}_n^+$ converges in distribution to the standard additive coalescent, i.e., $(\mathbf{F}_{\mathcal{T}_2}(e^{-t}), t \in \mathbb{R})$.

Aldous and Pitman [4] (see also [19, Construction 5]) also studied the fragmentation process derived by cutting-down *birthday trees*. They are a family of trees that generalizes the Cayley tree in allowing "weights" on the vertices. Aldous and Pitman showed that this fragmentation process, suitable rescaled, converges to the fragmentation process associated to the continuum counterpart of birthday trees, the *inhomogeneous continuum random trees* (ICRT). Moreover, the time-reversed version of the fragmentation process of the ICRT can be viewed as version of an eternal additive coalescent. On the other hand, Bertoin [7] has proved that the fragmentation process of the ICRT can also be constructed by considering the partitions of the unit interval induced by certain bridges with exchangeable increments.

Parking schemes

Chassaing and Louchard [14] provided another representation of the standard additive coalescent as parking schemes related to the *Knuth's parking problem*; see also [15, 24]. Bertoin and Miermont [10] extended the work [14] and relate the Knuth's parking problem for caravans to different versions of eternal additive coalescent. On the other hand, the Knuth's parking problem bear some similarities with the dynamics of an aggregating server studied by Bertoin [7].

Lamination process

In [30], Thévenin has provided a geometric representation of the fragmentation process $\mathbf{F}_{\mathcal{T}_\alpha}$ by introducing a new lamination-valued process. In particular, Theorem 1.1 in [30] combined with Proposition 1 allows us to deduce the exact distribution of the ranked sequence (in decreasing order) of the masses of the faces of this lamination-valued process.

Bernoulli bond-percolation

Bernoulli bond-percolation on finite connected graphs is perhaps the simplest example of a percolation model. In this model, each edge in the connected graph is removed with probability $1-p \in (0,1)$, and it is kept with probability p, independently of the other edges. This induces a partition of the set of vertices of the graph into connected components usually referred to as clusters. It should be intuitively clear that there is a link between Bernoulli bond-percolation on α-stable GW-trees and their associated fragmentation processes. More precisely, let $\mathbf{t}_n$ be an α-stable GW-tree. For $u \in [0,1]$, recall that the continuous-time cutting-down procedure of $\mathbf{t}_n$ described in the introduction results in a random forest of connected components. Indeed, the probability that a given edge of $\mathbf{t}_n$ has not yet been removed at time u is exactly u. Thus, the configuration of the connected components at time u is precisely that resulting from Bernoulli bond-percolation on $\mathbf{t}_n$ with parameter u. A natural problem in this setting is then to investigate the asymptotic behavior of the sizes of the largest clusters for appropriate percolation regimes. In this direction, let $(B_n)_{n\geq 1}$ be a sequence of positive real numbers satisfying (1). An application of Theorem 1 shows that for the percolation parameter $1-(B_n/n)t$ with a fixed $t \geq 0$, the sequence of sizes of the clusters ranked in decreasing order and renormalized by a factor of $1/n$ (i.e. $\mathbf{F}_n^{(\alpha)}(t)$) converges in distribution, as $n \to \infty$, to $\mathbf{F}^{(\alpha)}(t)$. Theorem 2 in [25] allows us to describe explicitly the distribution of $\mathbf{F}^{(\alpha)}$ at fixed times. Let $(p_s(z), z \in \mathbb{R}, s \geq 0)$ be the family of densities of the distribution of a strictly stable spectrally positive Lévy process with index $\alpha \in (1,2]$.

► **Corollary 3.** *For $t > 0$, let $\mathrm{a}_1^{(\alpha)}(t) > \mathrm{a}_2^{(\alpha)}(t) > \cdots$ be the atoms of a Poisson measure on $(0,\infty)$ with intensity $\Lambda_\alpha^{(t)}(\mathrm{d}z) := z^{-1}p_z(-tz)\mathbb{1}_{\{z>0\}}\mathrm{d}z$, ranked in decreasing order. Then*

$$\mathbf{F}^{(\alpha)}(t) \overset{d}{=} \Big((\mathrm{a}_1^{(\alpha)}(t), \mathrm{a}_2^{(\alpha)}(t), \dots) \,\Big|\, \sum_{i=1}^{\infty} \mathrm{a}_i^{(\alpha)}(t) = 1 \Big).$$

Following Bertoin's [9] work about Bernoulli bond-percolation on random trees. The percolation regime $1-(B_n/n)t$ on $\mathbf{t}_n$ corresponds to the so-called supercritical regime. Indeed, the result in Corollary 3 has already been proved by Pitman [28] for Cayley trees.

3 The coding of Galton-Watson trees and their fragmentation

In this section, we formally introduce the family of critical Galton-Watson trees and explain how they can be coded by different functions, namely the so-called Łukasiewicz path and a similar path derived by the Prim's algorithm.

Plane trees

We follow the formalism of Neveu [27]. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, set $\mathbb{N}^0 = \{\varnothing\}$ and consider the set of labels $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$. For $u = (u_1, \dots, u_n) \in \mathbb{U}$, we denote by $|u| = n$ the length (or generation, or height) of u; if $v = (v_1, \dots, v_m) \in \mathbb{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ be the concatenation of u and v. A plane tree is a nonempty, finite subset $\tau \subset \mathbb{U}$ such that: (i) $\varnothing \in \tau$; (ii) if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}$, then $u \in \tau$; (iii) if $u \in \tau$, then there exists an integer $c(u) \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq c(u)$. We will view each vertex u of a tree τ as an individual of a population whose τ is the genealogical tree. The vertex $\varnothing$ is called the root of the tree and for every $u \in \tau$, $c(u)$ is the number of children of u (if $c(u) = 0$, then u is called a leaf, otherwise, u is called an internal vertex). The total progeny (or size) of τ will be denoted by $\zeta(\tau) = \mathrm{Card}(\tau)$ (i.e., the number of vertices of τ). We denote by $\mathbb{T}$ the set of plane trees and for each $n \in \mathbb{N}$, by $\mathbb{T}_n$ the set of plane trees with n vertices, or equivalently $n - 1$ edges.

Galton-Watson trees

Let μ be a probability measure on $\mathbb{Z}_+$ which satisfies $\mu(0) > 0$, expectation $\sum_{k=0}^{\infty} k\mu(k) = 1$ and such that $\mu(0) + \mu(1) < 1$. The law of a critical Galton–Watson tree with offspring distribution μ is the unique probability measure $\mathbb{P}_\mu$ on $\mathbb{T}$ satisfying: (i) $\mathbb{P}_\mu(c(\varnothing) = k) = \mu(k)$ for every $k \geq 0$; (ii) For every $k \geq 1$ such that $\mu(k) > 0$, conditioned on the event $\{c(\varnothing) = k\}$, the subtrees that stem from the children of the root $\{u \in \mathbb{U} : 1u \in \tau\}, \dots, \{u \in \mathbb{U} : ku \in \tau\}$ are independent and distributed as $\mathbb{P}_\mu$. A random tree whose distribution is $\mathbb{P}_\mu$ will be called a Galton–Watson tree with offspring distribution μ. We also denote by $\mathbb{P}_\mu^{(n)}$ the law on $\mathbb{T}_n$ of a Galton-Watson tree with offspring distribution μ conditioned to have n vertices, providing that this conditioning makes sense.

Coding planar trees by discrete paths

We will use two different orderings of the vertices of a tree $\tau \in \mathbb{T}$:

(i) Lexicographical ordering. Given $v, w \in \tau$, we write $v \prec_{\mathrm{lex}} w$ if there exits $z \in \tau$ such that $v = z(v_1, \dots, v_n)$, $w = z(w_1, \dots, w_m)$ and $v_1 < w_1$.

(ii) Prim ordering. Let $\mathbf{edge}(\tau)$ be the set of edges of τ and consider a sequence of distinct and positive weights $\mathbf{w} = (w_e : e \in \mathbf{edge}(\tau))$ (i.e., each edge e of τ is marked with a different and positive weight w_e). Given two distinct vertices $u, v \in \tau$, we write $\{u, v\}$ for the edge connecting u and v in τ. Let us describe the Prim order $\prec_{\mathrm{prim}}$ of the vertices in τ, that is, $\varnothing = u(0) \prec_{\mathrm{prim}} u(1) \prec_{\mathrm{prim}} \cdots \prec_{\mathrm{prim}} u(\zeta(\tau) - 1)$. We will use the notation V_i for the set $\{u(0), \dots, u(i-1)\}$, for $1 \leq i \leq \zeta(\tau)$. First set $u(0) = \varnothing$ and $V_0 = \{u(0)\}$. Suppose that for some $1 \leq i \leq \zeta(\tau) - 1$, the vertices $u(0), \dots, u(i-1)$ have been defined. Consider the weights $\{w_{\{u,v\}} : u \in V_i, v \notin V_i\}$ of edges between a vertex of V_i and another outside of V_i. Since all the weights are distinct, the minimum weight in $\{w_{\{u,v\}} : u \in V_i, v \notin V_i\}$ is reached at an edge $\{\tilde{u}, \tilde{v}\}$ where $\tilde{u} \in V_i$ and $\tilde{v} \notin V_i$. Then set $u(i) = \tilde{v}$. This iterative procedure completely determines the Prim order $\prec_{\text{prim}}$.

For $* \in \{\text{lex}, \text{prim}\}$, we associate to every ordering $\varnothing = u(0) \prec_* u(1) \prec_* \cdots \prec_* u(\zeta(\tau) - 1)$ of the vertices of τ a path $W^* = (W^*(k), 0 \leq k \leq \zeta(\tau))$, by letting $W^*(0) = 0$ and for $0 \leq k \leq \zeta(\tau) - 1$, $W^*(k+1) = W^*(k) + c(u(k)) - 1$, where we recall that $c(u(k))$ denotes the number of children of the vertex $u(k) \in \tau$. Observe that $W^*(k+1) - W^*(k) = c(u(k)) - 1 \geq -1$ for every $0 \leq k \leq \zeta(\tau) - 1$, with equality if and only if $u(k)$ is a leaf of τ. Note also that $W^*(k) \geq 0$, for every $0 \leq k \leq \zeta(\tau) - 1$, but $W^*(\zeta(\tau)) = -1$. We shall think of such a path as the step function on $[0, \zeta(\tau)]$ given $s \mapsto W^*(\lfloor s \rfloor)$. The path W^{lex} is commonly called Łukasiewicz path of τ, and from now on we refer to W^{prim} as the Prim path; see [23] for more details and properties on the Łukasiewicz path.

The procedure just described to obtain the Prim ordering is known as Prim's algorithm (or Prim-Jarník algorithm); see [29]. This algorithm associates to any properly weighted graph its unique minimum spanning tree. In practice, one could also consider that $\mathbf{w}$ is a sequence of i.i.d. positive random variables such that they are all distinct a.s. and independent of the tree.

Define the probability measure $\hat{\mu}$ on $\{-1, 0, 1, \dots\}$ by $\hat{\mu}(k) = \mu(k+1)$ for every $k \geq -1$. Let $X = (X(k), k \geq 0)$ be a random walk which starts at 0 with jump distribution $\hat{\mu}$ and define also the time $\zeta_1 = \inf\{k \geq 0 : X(k) = -1\}$. In the Prim ordering, consider that the weights $\mathbf{w}$ is a sequence of i.i.d. positive random variables such that they are distinct a.s. and independent of the tree.

▶ **Proposition 4.** *For every $* \in \{\text{lex}, \text{prim}\}$, if we sample a plane tree $\mathbf{t}$ according to $\mathbb{P}_\mu$, then W^* is distributed as $(X(0), X(1), \dots, X(\zeta_1))$. In particular, the total progeny of the sample plane tree has the same distribution as ζ_1.*

Proof. The proof for the Łukasiewicz path can be found in [23, Proposition 1.5]. For the Prim path the proof follows from a simple adaptation of that of [23, Proposition 1.5]; see also [13, Lemmas 15 and 16] for an alternative approach. ◀

Fragmentation of a plane tree

Consider $\tau \in \mathbb{T}$ and let $\mathbf{edge}(\tau)$ denote its set of edges. Equip the edges of τ with i.i.d. uniform random variables (or weights) $\mathbf{w} = (w_e : e \in \mathbf{edge}(\tau))$ on $[0,1]$ and independently of the tree τ. In particular, for a vertex $v \in \tau$ with $c(v) \geq 1$ children, we write $(w_{v,k}, 1 \leq k \leq c(v))$ for the weights of the edges connecting v with its children. For $t \in [0,1]$, we then keep the edges of τ with weight smaller than t and discard the others. This gives rise to a forest $\mathbf{f}_\tau(t)$ with the same set of vertices as τ but with set of edges given by $\mathbf{edge}(\mathbf{f}_\tau(t)) = \{e \in \mathbf{edge}(\tau) : w_e \leq t\}$. Furthermore, each vertex $v \in \mathbf{f}_\tau(t)$ has $c_t(v) = \sum_{k=1}^{c(v)} \mathbb{1}_{\{w_{v,k} \leq t\}}$ children if $c(v) \geq 1$; otherwise, $c_t(v) = 0$ whenever $c(v) = 0$. In what follows, we refer to the forest $\mathbf{f}_\tau(t)$ associated to a plane tree τ and uniform weights $\mathbf{w}$ as the *fragmentation forest*. In this manuscript we restrict ourselves to the case uniform i.i.d. weights, but certainly some of the forthcoming results can be extended for more general sequences of weights.

Prim exploration of the fragmentation forest

For a plane tree $\tau \in \mathbb{T}$ and sequence of i.i.d. uniform random weights $\mathbf{w}$ on $[0,1]$, let $\mathbf{f}_\tau(t)$ be the fragmentation forest of τ at time $t \in [0,1]$. Let us now explain how to explore the subtree components of the forest $\mathbf{f}_\tau(t)$ by using the approach outlined in [13, page 532] (see also [2]).

For $t \in [0,1]$, denote by $\mathbf{Neigh}_t(v) \coloneqq \{u \in \mathbf{f}_\tau(t) : \{u,v\} \in \mathbf{edge}(\mathbf{f}_\tau(t))\}$ the set of neighbours of $v \in \mathbf{f}_\tau(t)$. For a set of vertices V of $\mathbf{f}_\tau(t)$, let also $\mathbf{Neigh}_t(V) \coloneqq (\bigcup_{v\in V} \mathbf{Neigh}_t(v)) \setminus V$, the set of neighbours of vertices in V but not in V. We associate to the prim ordering $\varnothing = u(0) \prec_{\text{prim}} u(1) \prec_{\text{prim}} \cdots \prec_{\text{prim}} u(\zeta(\tau)-1)$ of the vertices of τ the following exploration process of $\mathbf{f}_\tau(t)$ (recall that $\mathbf{f}_\tau(t)$ and τ have the same set of vertices). The first visited vertex is $v_t(0) = u(0)$. Suppose that we have explored the vertices $V_k = \{v_t(0), \dots, v_t(k-1)\}$ at some time $1 \leq k \leq \zeta(\tau)$. If $k = \zeta(\tau)$, we have finished the exploration, and otherwise, one has two possibilities:

(i) if $\mathbf{Neigh}_t(V_k) \neq \varnothing$, then $v_t(k)$ is the next vertex according to the order $\prec_{\text{prim}}$ that belongs to $\mathbf{Neigh}_t(V_k)$, or

(ii) if $\mathbf{Neigh}_t(V_k) = \varnothing$, then $v_t(k)$ is the next vertex according to the order $\prec_{\text{prim}}$ that belongs to $\tau \setminus V_k$.

This exploration process results in an order for the vertices of $\mathbf{f}_\tau(t)$ (equivalently, to the vertices of τ) that we denote by $<_{\text{prim}}$ (i.e. $\varnothing = v_t(0) <_{\text{prim}} v_t(1) <_{\text{prim}} \cdots <_{\text{prim}} v_t(\zeta(\tau)-1)$) and call *Prim exploration*. An important feature of the Prim exploration of $\mathbf{f}_\tau(t)$ is that the Prim ordering $<_{\text{prim}}$ of its vertices is preserved for all values of $t \in [0,1]$. More precisely, for $t_1, t_2 \in [0,1]$, $v_{t_1}(k) = v_{t_2}(k)$, for all $0 \leq k \leq \zeta(\tau)-1$. This is a consequence of the algorithm to obtain the Prim ordering of the vertices in τ which associates to any properly weighted graph its unique minimum spanning tree. We henceforth write $\prec_{\text{prim}}$ instead of $<_{\text{prim}}$ and remove the subindex t from our notation, i.e., we write $\varnothing = v(0) \prec_{\text{prim}} v(1) \prec_{\text{prim}} \cdots \prec_{\text{prim}} v(\zeta(\tau)-1)$ for the vertices of $\mathbf{f}_\tau(t)$ in Prim order, which is the same as the Prim ordering of the vertices of the tree τ, $\varnothing = u(0) \prec_{\text{prim}} u(1) \prec_{\text{prim}} \cdots \prec_{\text{prim}} u(\zeta(\tau)-1)$ presented earlier.

Following the presentation of [13, pages 532-533], one can associate to the Prim ordering of the vertices of $\mathbf{f}_\tau(t)$, an *exploration path* $Z_t = (Z_t(k), 0 \leq k \leq \zeta(\tau)+1)$ by letting $Z_t(0) = Z(\zeta(\tau)+1) = 0$, and for $1 \leq k \leq \zeta(\tau)$, $Z_t(k) = \mathrm{Card}(\mathbf{Neigh}_t(V_k))$. Furthermore, let $\mathbf{CC}(\mathbf{f}_\tau(t))$ be the set of connected components of $\mathbf{f}_\tau(t)$. Then [13, Lemma 14] shows that

$$\mathrm{Card}(\{k \in \{1,\dots,\zeta(\tau)\} : Z_t(k) = 0\}) = \mathrm{Card}(\mathbf{CC}(\mathbf{f}_\tau(t))),$$

and that the successive sizes of the connected components ordered by the exploration coincide with the distances between successive 0's in the sequence $Z_t = (Z_t(k), 0 \leq k \leq \zeta(\tau)+1)$.

In this manuscript, and in analogy with the coding paths of τ introduced earlier, we will consider a slight modification of the exploration path Z_t. More precisely, define the Prim path $W_t^{\text{prim}} = (W_t^{\text{prim}}(k), 0 \leq k \leq \zeta(\tau))$ by letting $W_t^{\text{prim}}(0) = 0$, and for $0 \leq k \leq \zeta(\tau)-1$, $W_t^{\text{prim}}(k+1) = W_t^{\text{prim}}(k) + c_t(v_t(k)) - 1$, where $c_t(v)$ denotes the number of children of $v \in \mathbf{f}_\tau(t)$. We shall also think of such a path as the step function on $[0, \zeta(\tau)]$ given by $s \mapsto W_t^{\text{prim}}(\lfloor s \rfloor)$.

▶ **Lemma 5.** *Let $\tau \in \mathbb{T}$ and $\mathbf{w}$ be a sequence of i.i.d. uniform random weights on $[0,1]$ which is also independent of τ. For any time $t \in [0,1]$,*

$$\mathrm{Card}\left(\left\{k \in \{1,\dots,\zeta(\tau)\} : W_t^{\text{prim}}(k) = \min_{0\leq m\leq k} W_t^{\text{prim}}(m)\right\}\right) = \mathrm{Card}(\mathbf{CC}(\mathbf{f}_\tau(t))),$$

Moreover, the successive sizes of the connected components of $\mathbf{f}_\tau(t)$ ordered by the exploration process coincide with the distances between successive new minimums in the sequence $(W_t^{\text{prim}}(k), 0 \leq k \leq \zeta(\tau))$.

Proof. The result is an immediate consequence of the previous discussion. ◀

Indeed, the sizes of the connected components of $\mathbf{f}_\tau(t)$ coincides with the length of the excursions of the walk W_t^{prim} above its minimum.

Following Proposition 4, the Prim path of the fragmentation forest associated to a critical Galton-Watson tree with offspring distribution μ can also be related to a random walk. Recall that $X = (X(k), k \geq 0)$ denotes a random walk that starts at 0 and has jump distribution $\hat{\mu}$ on $\{-1, 0, 1, \dots\}$. Recall also that $\zeta_1 = \inf\{k \geq 0 : X(k) = -1\}$. Denote by $(\xi(k), k \geq 1)$ the increments of X, i.e. $\xi(k) = X(k) - X(k-1)$, for $k \geq 1$. Let $(U_k(j))_{k,j\geq 1}$ be a sequence of i.i.d. uniform random variables on $[0,1]$. For $t \in [0,1]$, define $(\xi_t(k), k \geq 1)$ by letting

$$\xi_t(k) = \sum_{j=1}^{\xi(k)+1} \mathbb{1}_{\{U_k(j) \leq t\}}, \quad \text{for } t \in [0,1], \quad k \geq 1,$$

with the convention $\sum_{j=1}^{0} \mathbb{1}_{\{U_k(j)\leq t\}} = 0$. Hence, $\xi_0(k) = 0$, $\xi_1(k) = \xi(k) + 1$ and for any $k \geq 1$, the mapping $t \mapsto \xi_t(k)$ is non-decreasing. Let $X_t = (X_t(k), k \geq 0)$ be the process defined by

$$X_t(0) = 0 \quad \text{and} \quad X_t(k) = \sum_{i=1}^{k} (\xi_t(i) - 1), \quad \text{for } t \in [0,1], \quad k \geq 1. \tag{5}$$

▶ **Proposition 6.** *Sample a plane tree* $\mathbf{t}$ *according to* $\mathbb{P}_\mu$*, i.e., consider a critical Galton-Watson tree* $\mathbf{t}$ *with offspring* μ*. Let* $\mathbf{w} = (w_e : e \in \mathbf{edge}(\mathbf{t}))$ *be a sequence of i.i.d. uniform random weights on* $[0,1]$ *which is also independent of* $\mathbf{t}$*. Then, the Prim path* W_t^{prim} *satisfies*

$$(W_t^{\mathrm{prim}}(0), W_t^{\mathrm{prim}}(1), \dots, W_t^{\mathrm{prim}}(\zeta(\mathbf{t})))_{t\in[0,1]} \overset{d}{=} (X_t(0), X_t(1), \dots, X_t(\zeta_1))_{t\in[0,1]},$$

where $\overset{d}{=}$ *means equal in distribution (in the sense of finite-dimensional distributions).*

Proof. The proof of can be found in the complete version [11]. ◀

4 Convergence of the exploration processes

Recall that $\mathbb{P}_\mu^{(n)}$ denotes the law of a critical Galton-Watson tree with offspring distribution μ conditioned to have $n \in \mathbb{N}$ vertices. For every $n \in \mathbb{N}$, for which $\mathbb{P}_\mu^{(n)}$ is well-defined, sample a plane tree on $\mathbb{T}_n$, say $\mathbf{t}_n$, according to $\mathbb{P}_\mu^{(n)}$, i.e., $\mathbf{t}_n$ is a critical Galton-Watson tree conditioned to have n vertices. Through this section we assume that μ belongs to the domain of attraction of a stable law of index $\alpha \in (1,2]$, and refer to $\mathbf{t}_n$ as an α-stable GW-tree. We will always let $\mathbf{w} = (w_e : e \in \mathbf{edge}(\mathbf{t}_n))$ be a sequence of i.i.d. uniform random weights on $[0,1]$ which is also independent of $\mathbf{t}_n$. We write $W_n^{\mathrm{lex}} = (W_n^{\mathrm{lex}}(\lfloor nu \rfloor), u \in [0,1])$ for the associated time-scaled Łukasiewicz path of $\mathbf{t}_n$. We also write $W_n^{\mathrm{prim}} = (W_n^{\mathrm{prim}}(\lfloor nu \rfloor), u \in [0,1])$ for the time-scaled Prim path of $\mathbf{t}_n$ with respect to the weights $\mathbf{w}$.

The asymptotic behavior of large α-stable GW-trees is well understood, in particular through scaling limits of their associated Łukasiewicz paths; see, e.g., [17]. In this section, we first show that the Prim path of $\mathbf{t}_n$ has the same asymptotic behavior as its Łukasiewicz path. This will serve as a stepping stone to study the Prim path of the fragmentation forest of $\mathbf{t}_n$ associated to the weights $\mathbf{w}$. Recall that $X_\alpha^{\mathrm{exc}} = (X_\alpha^{\mathrm{exc}}(u), u \in [0,1])$ denotes the α-stable excursion of index α.

▶ **Theorem 7.** *Let* $\mathbf{t}_n$ *be an* α*-stable* GW*-tree, and let* $(B_n)_{n\geq 1}$ *be a sequence of positive real numbers satisfying (1). For* $* \in \{\mathrm{lex}, \mathrm{prim}\}$*, we have that*

$$\left(\frac{1}{B_n} W_n^*(\lfloor nu \rfloor), u \in [0,1]\right) \xrightarrow{d} (X_\alpha^{\mathrm{exc}}(u), u \in [0,1]), \quad \textit{as } n \to \infty, \quad \textit{in the space } \mathbb{D}([0,1], \mathbb{R}).$$

Proof. It follows from [17, Theorem 3.1] and Proposition 4. ◀

For $s \in [0,1]$, let $\mathbf{f}_n(s)$ be the fragmentation forest of $\mathbf{t}_n$ at time s. Denote by $W^{\mathrm{prim}}_{n,s} = (W^{\mathrm{prim}}_{n,s}(\lfloor nu \rfloor), u \in [0,1])$ the time-scaled Prim path of $\mathbf{f}_n(s)$. In particular, $W^{\mathrm{prim}}_{n,1}$ is exactly W^{prim}_n. For fixed $t \geq 0$, consider the sequence $(s_n(t))_{n \geq 1}$ of positive times given by $s_n(t) = 1 - (B_n/n)t$, where $(B_n)_{n\geq 1}$ is a sequence of positive real numbers satisfying (1). Define the process $W^{(t)}_n = (W^{(t)}_n(u), u \in [0,1])$ by letting

$$W^{(t)}_n(u) = \frac{1}{B_n} W^{\mathrm{prim}}_{n,s_n(t)}(\lfloor nu \rfloor), \quad \text{for} \ \ u \in [0,1]. \tag{6}$$

Later, we refer to $W^{(t)}_n$ as the (normalized and time-scaled) Prim path of the fragmentation forest at time $s_n(t)$, i.e., $\mathbf{f}(s_n(t))$. We set $W_n = (W^{(t)}_n, t \geq 0)$. From the previous section, the mapping $t \mapsto W^{(t)}_n(u)$ is non-increasing in t which implies that W_n has càdlàg paths. Thus, we will view $(t,u) \mapsto W^{(t)}_n(u)$ as a random variable taking values in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0,1],\mathbb{R}))$ of $\mathbb{D}([0,1],\mathbb{R})$-valued càdlàg functions on $\mathbb{R}$ equipped with the Skorokhod topology. In other words, for fixed $t \geq 0$, $W^{(t)}_n$ is a random variable in $\mathbb{D}([0,1],\mathbb{R})$.

We introduce the continuous counterpart of the process W_n. For every $t \geq 0$, let $Y^{(t)}_\alpha = (Y^{(t)}_\alpha(u), u \in [0,1])$ be defined by $Y^{(t)}_\alpha(u) = X^{\mathrm{exc}}_\alpha(u) - tu$, for $u \in [0,1]$. In particular, for $t = 0$, $Y^{(0)}_\alpha = X^{\mathrm{exc}}_\alpha$. Then, define the process $Y_\alpha = (Y^{(t)}_\alpha, t \geq 0)$.

The following theorem is the main result of this section.

▶ **Theorem 8.** *We have the convergence*

$$(W^{(t)}_n, t \geq 0) \xrightarrow{d} (Y^{(t)}_\alpha, t \geq 0), \quad \textit{as} \ \ n \to \infty, \ \ \textit{in the space} \ \ \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0,1],\mathbb{R})).$$

Theorem 8 generalizes [13, Theorem 10]. Specifically, in [13], the authors only consider the case when $\mathbf{t}_n$ is a GW-tree with μ being the law of a Poisson random variable of parameter 1 (i.e., $\mathbf{t}_n$ is a Cayley tree) while our setting is clearly more general.

As in most proofs for convergence of stochastic processes, the proof of Theorem 8 consists in two steps: convergence of the finite-dimensional distributions and tightness of the sequence of processes $(W_n)_{n\geq 1}$. To accomplish the above, one uses the random walk connected to the Prim path of the fragmentation forest of the α-stable GW-tree $\mathbf{t}_n$ in Proposition 6. More precisely, for $s \in [0,1]$, let $X_s = (X_s(k), k \geq 0)$ be the stochastic process defined in (5). For $n \in \mathbb{N}$ and $t \geq 0$, define the process $Y^{(t)}_n = (Y^{(t)}_n(u), u \in [0,1])$ by letting

$$Y^{(t)}_n(u) = \frac{1}{B_n} X_{s_n(t)}(\lfloor nu \rfloor), \quad \text{for} \ \ u \in [0,1],$$

and set $Y_n = (Y^{(t)}_n, t \geq 0)$. From Proposition 6, we see that W_n has the same finite-dimensional distribution as Y_n under the conditional probability distribution $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot | \zeta_1 = n)$. Therefore, the proof of Theorem 8 boils down to establishing the convergence for Y_n instead of W_n. Although it is simpler to work with Y_n than with W_n, the proof of the convergence is rather technical and it is given in the complete version [11, Section 5].

5 Proof of Theorem 1

In this section, we prove Theorem 1. The final ingredient is the general approach developed in the complete version for the convergence of fragmentation processes encoded by functions in $\mathbb{D}([0,1],\mathbb{R})$; see [11, Section 6]. Before that, we need to introduce some notation. For an increasing function $h = (h(s), s \in [0,1]) \in \mathbb{D}([0,1],\mathbb{R})$, write $\mathbf{F}(h) := (F_1(h), F_2(h), \dots) \in \mathbb{S}$, for the sequence of the lengths of the intervals components of the complement of the support of the Stieltjes measure $\mathrm{d}h$, arranged in decreasing order; we tacitly understand $\mathbf{F}(h)$ as an infinite sequence, by completing with an infinite number of zero terms.

Proof of Theorem 1. Let $\mathbf{t}_n$ be an α-stable GW-tree of index $\alpha \in (1,2]$. Recall that $(B_n)_{n\geq 1}$ denotes a sequence of positive real numbers satisfying (1). For $t \geq 0$, let $W_n^{(t)}$ be the (normalized and time-scaled) Prim path defined in (6) of the fragmentation forest at time $s_n(t) = 1-(B_n/n)t$, i.e. $\mathbf{f}(s_n(t))$, associated to $\mathbf{t}_n$ and the i.i.d. uniform random weights $\mathbf{w}$. Define the process $I_n^{(t)} = (I_n^{(t)}(u), u \in [0,1])$ by letting

$$I_n^{(t)}(u) = \inf_{s\in[0,u]} W_n^{(t)}(s), \quad \text{for } s \in [0,1].$$

Recall that $\mathbf{F}_n^{(\alpha)} = (\mathbf{F}_n^{(\alpha)}(t), t \geq 0)$ stands for the fragmentation process of $\mathbf{t}_n$ defined in (2). From Lemma 5 and the preceding discussion, it is clear that $\mathbf{F}_n^{(\alpha)}(t) = \mathbf{F}(-I_n^{(t)})$, for $t \geq 0$. Let $Y_\alpha^{(t)}$ and $I_\alpha^{(t)}$ be the processes defined in (3), and recall that the α-stable fragmentation process, $\mathbf{F}^{(\alpha)} = (\mathbf{F}^{(\alpha)}(t), t \geq 0)$, is given by $\mathbf{F}^{(\alpha)}(t) = \mathbf{F}(-I_\alpha^{(t)})$, for $t \geq 0$. Note that for all $t \geq 0$, $W_n^{(t)}(0) = Y_\alpha^{(t)}(0) = 0$. Then, to prove Theorem 1, one only needs to check that the processes $W_n = (W_n^{(t)}, t \geq 0)$ and $Y_\alpha = (Y_\alpha^{(t)}, t \geq 0)$ satisfy the conditions of [11, Lemma 5].

We start by verifying that Y_α fulfills (i), (ii) and (iii) of [11, Lemma 5]. Indeed, (i) has been proven in Theorem 8. Recall that X_α^{exc} can be defined as the Vervaat transform of the so-called stable Lévy bridge; see [16]. Since the stable Lévy bridge has exchangeable increments (see e.g., [22, Chapters 11 and 16]), (ii) follows along the lines of the proof of Lemma 7 (i) in [7] thanks to [16, Theorem 4]. To prove that $Y_\alpha^{(t)}$ fulfills condition (iii) for every $t \geq 0$, recall that the support of the Stieltjes measure $\mathrm{d}(-I_\alpha^{(t)})$ coincides with the ladder time set $\mathcal{L}^\alpha(t)$ of $Y_\alpha^{(t)}$, which is a random closed set with zero Lebesgue measure. The latter follows from [5, Corollary 5, Chapter VII] but alternatively, it can be deduced from [16, Theorem 4] by following the same argument as in [7, Proof of Lemma 7]. Since $\mathbf{F}(-I_\alpha^{(t)})$ is defined as the ranked sequence of the lengths of the open intervals in the canonical decomposition of $[0,1]/\mathcal{L}^\alpha(t)$, condition (iii) follows.

We now check that the sequence $(W_n)_{n\geq 1}$ fulfills [11, (17)]. Note that, for every $t \geq 0$, $\|\mathbf{F}(-I_n^{(t)})\|_1 = 1$. Fix $t_\star, t^\star$ such that $0 \leq t_\star \leq t^\star < \infty$. For every $t \in [t_\star, t^\star]$ and $m \in \mathbb{N}$,

$$\|\mathbf{F}(-I_n^{(t)})\|_1 - \sum_{i=1}^{m} \mathbf{F}_i(-I_n^{(t)}) = \sum_{i>m} \mathbf{F}_i(-I_n^{(t)})$$

reaches its maximum at $t = t_\star$. Then for [11, (17)] to be satisfied, it suffices that for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ and $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \mathbf{F}_i(-I_n^{(t_\star)}) \geq \lim_{r\to\infty} \sum_{i=1}^{r} \mathbf{F}_i(-I_n^{(t_\star)}) - \varepsilon = 1-\varepsilon. \tag{7}$$

This would imply that for any $t \in [t_\star, t^\star]$, we have that $\sum_{i=1}^m \mathbf{F}_i(-I_n^{(t)}) \geq 1-\varepsilon$, which shows that $(W_n)_{n\geq 1}$ satisfies [11, (17)].

Theorem 8 implies that $(W_n^{(t)}, t \in [t_\star, t^\star]) \to (Y_\alpha^{(t)}, t \in [t_\star, t^\star])$, in distribution, as $n \to \infty$, in the space $\mathbb{D}([t_\star, t^\star], \mathbb{D}([0,1], \mathbb{R}))$. By the Skorokhod representation theorem, we can and we will work on a probability space on which this convergence holds almost surely. Since we have proven that the process $Y_\alpha^{(t_\star)}$ fulfills (iii) of [11, Lemma 5], for any $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that $\sum_{i=1}^m \mathbf{F}_i(-I_\alpha^{(t_\star)}) \geq 1-\varepsilon/2$. On the other hand, recall that $Y_\alpha^{(t_\star)}$ fulfills (ii) of [11, Lemma 5]. Then [7, Lemma 4] implies that a.s., $\mathbf{F}(-I_n^{(t_\star)}) \to \mathbf{F}(-I_\alpha^{(t_\star)})$, as $n \to \infty$ in the space $\mathbb{S}$ with the ℓ^1-norm. Hence, a.s. for all n large enough, $\sum_{i=1}^m \mathbf{F}_i(-I_n^{(t_\star)}) \geq 1-\varepsilon$, which proves (7). ◀

References

1. Romain Abraham and Laurent Serlet. Poisson snake and fragmentation. *Electron. J. Probab.*, 7:no. 17, 15, 2002. doi:10.1214/EJP.v7-116.
2. David Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.*, 25(2):812–854, 1997. doi:10.1214/aop/1024404421.
3. David Aldous and Jim Pitman. The standard additive coalescent. *Ann. Probab.*, 26(4):1703–1726, 1998. doi:10.1214/aop/1022855879.
4. David Aldous and Jim Pitman. Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent. *Probab. Theory Related Fields*, 118(4):455–482, 2000. doi:10.1007/PL00008751.
5. Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
6. Jean Bertoin. A fragmentation process connected to Brownian motion. *Probab. Theory Related Fields*, 117(2):289–301, 2000. doi:10.1007/s004400050008.
7. Jean Bertoin. Eternal additive coalescents and certain bridges with exchangeable increments. *Ann. Probab.*, 29(1):344–360, 2001. doi:10.1214/aop/1008956333.
8. Jean Bertoin. Self-similar fragmentations. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(3):319–340, 2002. doi:10.1016/S0246-0203(00)01073-6.
9. Jean Bertoin. Almost giant clusters for percolation on large trees with logarithmic heights. *J. Appl. Probab.*, 50(3):603–611, 2013. doi:10.1239/jap/1378401225.
10. Jean Bertoin and Grégory Miermont. Asymptotics in Knuth's parking problem for caravans. *Random Structures Algorithms*, 29(1):38–55, 2006. doi:10.1002/rsa.20092.
11. Gabriel Berzunza Ojeda and Cecilia Holmgren. Invariance principle for fragmentation processes derived from conditioned stable Galton-Watson trees. *arXiv e-prints*, page arXiv:2010.07880, October 2020. arXiv:2010.07880.
12. Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication. doi:10.1002/9780470316962.
13. Nicolas Broutin and Jean-François Marckert. A new encoding of coalescent processes: applications to the additive and multiplicative cases. *Probab. Theory Related Fields*, 166(1-2):515–552, 2016. doi:10.1007/s00440-015-0665-1.
14. P. Chassaing and G. Louchard. Phase transition for parking blocks, Brownian excursion and coalescence. *Random Structures Algorithms*, 21(1):76–119, 2002. doi:10.1002/rsa.10039.
15. Philippe Chassaing and Svante Janson. A Vervaat-like path transformation for the reflected Brownian bridge conditioned on its local time at 0. *Ann. Probab.*, 29(4):1755–1779, 2001. doi:10.1214/aop/1015345771.
16. L. Chaumont. Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bull. Sci. Math.*, 121(5):377–403, 1997.
17. Thomas Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.*, 31(2):996–1027, 2003. doi:10.1214/aop/1048516543.
18. Thomas Duquesne and Jean-François Le Gall. Random trees, Lévy processes and spatial branching processes. *Astérisque*, 281:vi+147, 2002.
19. Steven N. Evans and Jim Pitman. Construction of Markovian coalescents. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(3):339–383, 1998. doi:10.1016/S0246-0203(98)80015-0.
20. William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.
21. Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003. doi:10.1007/978-3-662-05265-5.
22. Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002. doi:10.1007/978-1-4757-4015-8.

23 Jean-François Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311, 2005. doi:10.1214/154957805100000140.

24 Jean-François Marckert and Minmin Wang. A new combinatorial representation of the additive coalescent. *Random Structures Algorithms*, 54(2):340–370, 2019. doi:10.1002/rsa.20775.

25 Grégory Miermont. Ordered additive coalescent and fragmentations associated to Levy processes with no positive jumps. *Electron. J. Probab.*, 6:no. 14, 33, 2001. doi:10.1214/EJP.v6-87.

26 Grégory Miermont. Self-similar fragmentations derived from the stable tree. II. Splitting at nodes. *Probab. Theory Related Fields*, 131(3):341–375, 2005. doi:10.1007/s00440-004-0373-8.

27 J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(2):199–207, 1986. URL: http://www.numdam.org/item?id=AIHPB_1986__22_2_199_0.

28 Jim Pitman. Coalescent random forests. *J. Combin. Theory Ser. A*, 85(2):165–193, 1999. doi:10.1006/jcta.1998.2919.

29 R. C. Prim. Shortest connection networks and some generalizations. *The Bell System Technical Journal*, 36(6):1389–1401, 1957.

30 Paul Thévenin. A geometric representation of fragmentation processes on stable trees. *The Annals of Probability*, 49(5):2416–2476, 2021. doi:10.1214/21-AOP1512.