Bounded Degree Nonnegative Counting CSP

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Abstract

Constraint satisfaction problems (CSP) encompass an enormous variety of computational problems. In particular, all partition functions from statistical physics, such as spin systems, are special cases of counting CSP (#CSP). We prove a complete complexity classification for every counting problem in #CSP with nonnegative valued constraint functions that is valid when every variable occurs a bounded number of times in all constraints. We show that, depending on the set of constraint functions $\mathcal{F}$, every problem in the complexity class #CSP($\mathcal{F}$) defined by $\mathcal{F}$ is *either* polynomial time computable for all instances without the bounded occurrence restriction, *or* is #P-hard even when restricted to bounded degree input instances. The constant bound in the degree depends on $\mathcal{F}$. The dichotomy criterion on $\mathcal{F}$ is decidable. As a second contribution, we prove a slightly modified but more streamlined decision procedure (from [15]) for tractability. This enables us to fully classify a family of *directed* weighted graph homomorphism problems. This family contains both P-time tractable problems and #P-hard problems. To our best knowledge, this is the first family of such problems explicitly classified that are not *acyclic*, thereby the Lovász-goodness criterion of Dyer-Goldberg-Paterson [24] cannot be applied.

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1 Introduction

Constraint Satisfaction Problems (CSPs) have been a subject of immense interest due to their wide applicability and intrinsic elegance. In particular, counting CSPs, or #CSPs, have been an active subject in computational counting complexity [18, 19, 10, 9, 22, 7, 15, 13], including their approximate solutions [28, 30, 20, 42, 41]. Roughly speaking, an (unweighted) constraint satisfaction problem deals with the following scenario, where there is a set of variables, each taking values over some finite domain D, and a set of constraints, each applied on an (ordered) subsequence of these variables. The #CSP problem on an instance asks how many assignments there are of these variables that satisfy all of the given constraints.

Applications of CSP problems are wide-ranging and varied. They range from within computer science to physical sciences such as physics, chemistry, engineering, even music [52, 1, 38, 47]. Within computer science, belief propagation has been a popular research topic in AI, which are ultimately based on some forms of partition function evaluations [5, 39, 40, 46, 29, 48, 51]. The term partition function, which we define formally later, arises from statistical physics, where one can see special cases of (weighted) counting CSPs in the form of spin systems such as the Ising and Potts models, e.g. [25]. As is the case in physical sciences as well as in applications within computer science, the instances of counting CSP problems that occur in practice are often with the additional restriction that variables occur a bounded number of times.

To define (unweighted) #CSP problems formally, let D be a finite domain set, Γ be a set of constraint relations Θ_i, where each Θ_i is a relation on D of arity $r_i = r(\Theta_i) \geq 1$. An instance of #CSP(Γ) is then defined by a set X of n variables over D, and a list of constraints Θ from Γ, and for each constraint Θ in the list a sequence of $r(\Theta)$ variables from X that the constraint is applied to. This defines an n-ary relation R in D^n on the input variables where an assignment $(x_1, \ldots, x_n) \in D^n$ is in R iff all constraints are satisfied. For any fixed Γ, the counting CSP problem #CSP(Γ) consists of all input instances using constraint relations from Γ. The computational problem is to compute the size of R given an arbitrary input instance, where the (worst case) computational complexity is measured in terms of size n of the set of variables and the size of the list of constraints. For a finite (fixed) Γ, this can be simplified to just n, up to a polynomial factor. A complexity dichotomy theorem can classify, depending on Γ, the problem #CSP(Γ) as either computable in polynomial time (P-time), or #P-complete, with no intermediate cases. Typically, the set Γ is a fixed finite set, which defines the #CSP problem – this Γ is the name of the problem. However, in most dichotomy theorems one can allow infinite sets, where in the P-time computable case we assume the specification of the constraints in the instances counts toward the input size, and in the #P-complete case there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that #CSP(Γ_0) is #P-hard.

For example, if we let $D = \{0, 1\}$ and $\Gamma = \{\mathrm{OR}_k | k \geq 1\} \cup \{\neq_2\}$, where OR_k is the k-ary OR function, and $\neq_2$ the binary disequality function, then the problem #CSP(Γ) is equivalent to #SAT, the counting Boolean satisfiability problem.

This formulation can be generalized to the weighted setting. In the most general case, the constraint functions can take real or complex values. In this paper we only consider #CSP defined by nonnegatively weighted constraint functions. This means that we replace the constraint language Γ by a set of constraint functions $\mathcal{F}$, where each $f_i \in \mathcal{F}$ has some arity $r_i \geq 1$ and maps D^{r_i} to nonnegative algebraic reals, denoted as $\mathbb{R}_+$.[1] Any given instance I defines a function $F_I : D^n \to \mathbb{R}_+$, such that on each assignment of variables, F_I takes the value the product over the constraint functions in I evaluated on the assignment. The solution to this instance I of #CSP($\mathcal{F}$) is then

$$Z_{\mathcal{F}}(I) = \sum_{(x_1, \ldots, x_n) \in D^n} F_I(x_1, \ldots, x_n). \tag{1}$$

This sum-of-products expression in (1) is called the partition function for an instance of #CSP, with the terminology coming from statistical physics [3]. When all functions in $\mathcal{F}$ are 0-1 valued, then the product is also 0-1 valued and is equivalent to the logical AND, and the partition function counts the number of satisfying assignments. Thus this $Z_{\mathcal{F}}(I)$ generalizes the unweighted case when $\mathcal{F}$ is a set of constraint relations Γ.

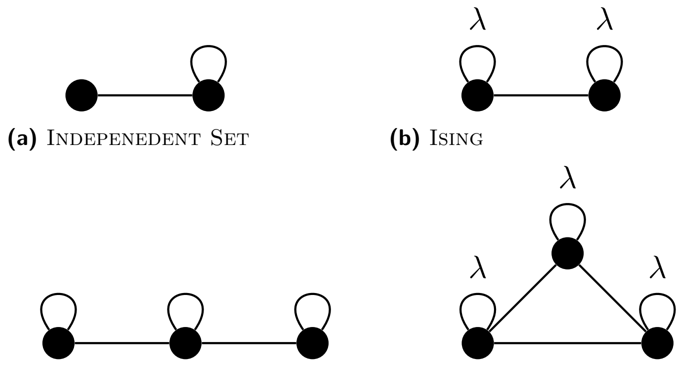
As a special case of #CSP, a q-state spin system is a problem on a domain $[q]$ with the constraint language having only a single binary constraint defined by the $q \times q$ interaction matrix A. An instance to this problem is a graph $G = (V, E)$, where the vertices (sites) are considered to be variables (spins) and the edges (bonds) correspond to the interactions between these vertices. The famous Ising model with parameter λ has domain size $q = 2$,

[1] Restricting to algebraic numbers is standard in this research area because we wish to state our results in the Turing machine model for strict bit complexity. See [14].

and is defined by its interaction matrix $A^{\lambda}_{\text{Ising}} = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}$ (see Figure 1b). The Potts model (Figure 1d) and Widom-Rowlinson model (Figure 1c) on 3 states are defined by the following interaction matrices respectively,

$$A^{\lambda}_{3\text{Potts}} = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} \quad \text{and} \quad A_{\text{WR}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Familiar problems in computer science can also be expressed in this model; e.g., independent set (IS) is defined by $A_{\text{IS}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (Figure 1a).



(a) INDEPENEDENT SET **(b)** ISING **(c)** WR with 3 states **(d)** Potts with 3 states

Figure 1 The graphs corresponding to some well known spin systems.

Bulatov [9] proved a sweeping complexity dichotomy for unweighted #CSPs in, which used deep results from universal algebra. His dichotomy theorem states that #CSP(Γ) is solvable in polynomial time if Γ satisfies a condition called congruence singularity; it is #P-complete otherwise. Dyer and Richerby [22] gave another proof of this dichotomy using a new P-time tractability criterion, which they proved to be equivalent to congruence singularity.

A nonnegative matrix is block-rank-1 if it becomes a block-diagonal matrix after a permutation of its rows and a permutation of its columns separately, such that all blocks are rank 1 except for possibly one all-zero block. (Here the blocks in the block-diagonal form of the matrix need not be square matrices.) For example, the following matrix (where blank entries are 0's)

$$\begin{bmatrix} A_{0,0} & & A_{0,2} & & & & & \\ A_{1,0} & & A_{1,2} & & & & & \\ & & & & A_{2,4} & & A_{2,6} & \\ & & & & A_{3,4} & & A_{3,6} & \\ & A_{4,1} & & A_{4,3} & & & & \\ & A_{5,1} & & A_{5,3} & & & & \\ & & & & & A_{6,5} & & A_{6,7} \\ & & & & & A_{7,5} & & A_{7,7} \end{bmatrix} \tag{2}$$

is block-rank-1 if each nonzero rectangle of the form $\begin{bmatrix} A_{i,j} & A_{i,j'} \\ A_{i',j} & A_{i',j'} \end{bmatrix}$ has rank 1.

For unweighted #CSP, the Dyer-Richerby condition in [22] for polynomial time tractability in the dichotomy theorem is *Strong Balance*. Let $d = |D|$ be the domain size. We say a constraint language Γ is *Strongly Balanced* if every n-ary relation R defined by an instance of #CSP(Γ) satisfies the following condition:

For any $a, b \geq 1$ and $c \geq 0$ with $a+b+c \leq n$, the following $d^a \times d^b$ matrix M is block-rank-1:

$$M(\boldsymbol{u}, \boldsymbol{v}) = \left|\{\boldsymbol{w} \in D^c : \exists \boldsymbol{z} \in D^{n-c-b-a} \text{ s.t. } (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) \in R\}\right|.$$

(If $a+b+c=n$, then the quantified statement "$\exists \boldsymbol{z} \in D^{n-c-b-a}$ such that $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) \in R$" simply means that $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in R$.)

If we are dealing with $\mathcal{F}$ rather than Γ, and if $\mathcal{F}$ is not a set of 0-1 valued functions, then the existential quantified statement "$\exists \boldsymbol{z}$" has no meaning. It turns out that there are several equivalent notions of *Balance*, which when $\mathcal{F}$ is restricted to a set of 0-1 valued functions (i.e. when $\mathcal{F}$ can be identified with a constraint language Γ) are all equivalent to the notion of Strong Balance; see Lemma 9.4 in [15]. These notions of Balance do not use existential quantifiers (see Definition 2 in Section 2). These notions are central to the #CSP dichotomies for #CSP($\mathcal{F}$) for nonnegative valued $\mathcal{F}$.

The study of #CSPs is closely related to that of counting graph homomorphisms [35, 27, 4, 36]. For two graphs G and H, a graph homomorphism from G to H is a mapping $f : V(G) \to V(H)$ that preserves vertex adjacency. In other words, if $e = \{u, v\} \in E(G)$ then $e' = \{f(u), f(v)\} \in E(H)$, for all edges e in G. The question of interest in counting complexity is the number of graph homomorphisms from one graph to another, which can also be represented by a partition function. If we let A be the $m \times m$ adjacency 0-1 matrix of the graph H, then the number of homomorphisms from G to H can be represented as a sum-of-products partition function as follows,

$$Z_A(G) = \sum_{f:V(G)\to[m]} \prod_{\{u,v\}\in E(G)} A_{f(u),f(v)}.$$

Partition functions of graph homomorphism can represent important physical spin systems such as the Ising, Potts, or Widom-Rowlinson models, as well as many other well known problems in computer science.

Counting graph homomorphisms is a special case of #CSP. In fact, the vertex-edge incidence graph of G defines an input to a #CSP problem, where vertices $V(G)$ are variables and edges $E(G)$ are (applications of binary) constraints, and the constraint language consists of a single binary relation represented by the adjacency matrix A defining the graph homomorphism problem $G \mapsto Z_A(G)$. Just as in #CSPs, the counting graph homomorphism function $Z_A(G)$ can be generalized from the 0-1 unweighted case to the weighted case where A is a real or complex matrix. It is symmetric for an undirected graph H, in which case we also only consider undirected G; for directed graph homomorphisms, A need not be symmetric.

The first dichotomy on counting graph homomorphisms was due to Dyer and Greenhill [21] for undirected graphs. They showed that there is a simple criterion such that if A satisfies the criterion then $G \mapsto Z_A(G)$ is computable in P-time, otherwise it is #P-complete. In fact they proved that if A does not satisfy the criterion then the problem of evaluating $Z_A(G)$ remains #P-complete even when restricted to graphs G with bounded degree Δ, for some Δ depending on A. Computing $G \mapsto Z_A(G)$ when restricted to graphs G with bounded degree Δ is called $\text{EVAL}^{(\Delta)}(A)$. The Dyer-Greenhill dichotomy was extended to the nonnegatively weighted case by Bulatov and Grohe in [8]. This dichotomy was then referenced throughout the field, as many other discoveries, including the results on #CSPs, ended up applying it. However, the hardness part of the proof of the Bulatov-Grohe dichotomy theorem required input graphs that have unbounded degrees. When restricted to bounded degree graphs, the worst case complexity of the Bulatov-Grohe dichotomy was left open for 15 years, until it was finally resolved by Govorov, Cai, and Dyer in [26] for graph homomorphisms with nonnegative weights, and continued by Cai and Govorov in [16] for complex weights. Most problems in statistical physics [6, 11, 31] use bounded degree graphs, and also most of the approximation algorithms work on bounded degree graphs [2, 23, 32, 33, 37, 43, 44, 45, 49]. Over the Boolean domain where variables take 0-1 values, it is known that the #CSP dichotomy for complex valued constraint functions holds for input instances where each variable occurs at most three times [17].

It has been an open problem to extend the general domain #CSP dichotomies to include the bounded degree case, i.e. where each variable occurs a bounded number of times. It was open even for the 0-1 unweighted case. For the nonnegative cases, this would be the analogous Govorov-Cai-Dyer extension [26] of the Bulatov-Grohe dichotomy for graph homomorphism, but apply to a much broader class of problems, as graph homomorphism is the special case of $\#\mathrm{CSP}(\mathcal{F})$ where $\mathcal{F}$ consists of a single binary function.

In this paper we prove such a dichotomy for bounded degree nonnegative #CSPs. For any finite domain D, any finite set of nonnegative constraint functions $\mathcal{F}$ on D, and any integer $\Delta \geq 0$, we define $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ to be the #CSP problem, where the input consists of n variables $x_1, \ldots, x_n$ over D and a sequence of constraint functions $f_1, \ldots, f_m \in \mathcal{F}$ each applied to a subsequence of the n variables, where each variable x_i appears no more than Δ times among $f_1, \ldots, f_m$. Note that in general, a function $f \in \mathcal{F}$ may occur multiple times among $f_1, \ldots, f_m$. We take $n + m$ as the input size. We prove that the same dichotomy criterion in [15] applies to the bounded degree case: if the P-time tractability criterion is not satisfied, then $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ remains #P-hard for some $\Delta > 0$. The dichotomy criterion of [15] will be explained in more detail after we introduce some more definitions in Section 2. These notions are further explicated in Theorem 4, and a more technical statement of Theorem 1 is given in Theorem 6.

▶ **Theorem 1.** *For any finite domain D and any nonnegatively weighted constraint functions $\mathcal{F}$ on D, if $\mathcal{F}$ satisfies the tractability criterion in [15], then $\#\mathrm{CSP}(\mathcal{F})$ is P-time computable; otherwise, $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ is #P-hard* [2] *for some $\Delta > 0$.*

For any fixed finite set $\mathcal{F}$ of constraint functions, the arities of $f \in \mathcal{F}$ are bounded. Viewing any instance as a bipartite graph, with the variables $x_1, \ldots, x_n$ on one side and constraints $f_1, \ldots, f_m \in \mathcal{F}$ on the other, with an edge between x_i and f_j if x_i is an input to the function f_j, we can see that the condition for a #CSP instance to be bounded degree corresponds exactly to this bipartite graph having bounded degree.

Our second contribution in this paper is a slightly modified but more streamlined decision procedure (compared to that of [15]) for polynomial time tractability. This enables us to fully classify a family of *directed* weighted graph homomorphism problems. This family contains both P-time tractable problems and #P-hard problems. To our best knowledge, this is the first family of such problems explicitly classified that are not *acyclic*, thereby the Lovász-goodness criterion of Dyer-Goldberg-Paterson [24] cannot be applied.

2 Balance

Several variants of the *Balance* condition have been used in the study of counting constraint satisfaction problems. In addition to the *Strong Balance* condition [22], the following conditions have been introduced in [15]. Recall that $d = |D|$ denotes the domain size.

[2] The problem $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ is also no harder than #P under a polynomial-time Turing reduction for any $\mathcal{F}$. The statement for Theorem 1 does not state #P-complete only for the technical reason that by definition functions in #P take nonnegative integer values while the partition function in (1) need not.

▶ **Definition 2** (Various notions of Balance). *We have the following notions:*

1. *(Balance) We say $\mathcal{F}$ is* Balanced *if for any $n \geq 2$, any $a \geq 1$ and $b \geq 1$ with $a + b \leq n$, and any instance I of #CSP($\mathcal{F}$) which defines an n-ary function $F_I(x_1, \ldots, x_n)$ over D^n, the following $d^a \times d^b$ matrix M_I is block-rank-1: The rows and columns of M_I are indexed by tuples $\boldsymbol{u} \in D^a$ and $\boldsymbol{v} \in D^b$ respectively, and*

$$M_I(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\boldsymbol{w} \in D^{n-a-b}} F_I(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}),$$

for all $\boldsymbol{u} \in D^a, \boldsymbol{v} \in D^b$. If $a + b = n$ then the sum $\sum_{\boldsymbol{w} \in D^{n-a-b}} F_I(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is simply $F_I(\boldsymbol{u}, \boldsymbol{v})$.
2. *(Weak Balance) We say $\mathcal{F}$ is* Weakly Balanced *if the definition for Balance holds for $b = 1$.*
3. *(Primitive Balance) We say $\mathcal{F}$ is* Primitively Balanced *if the definition for Balance holds for $a = b = 1$.*

While these three notions may seem to have varying strengths, all three are in fact equivalent by combining the proof in [15] and [34]. See Theorem 4 below. We need the following definition.

▶ **Definition 3** (Strong Rectangularity). *We say a matrix M is* Rectangular *if after a row permutation and a column permutation it is a block diagonal matrix where all diagonal blocks have no zero entries, except possibly one all zero block. We say a constraint language Γ over D is* Strongly Rectangular *if for any input instance I of #CSP(Γ) which defines an n-ary relation R_I over D^n and for any a and b such that $1 \leq a < b \leq n$, the following $|D|^a \times |D|^{b-a}$ matrix M is rectangular: The rows of M are indexed by $\boldsymbol{u} \in D^a$, the columns of M are indexed by $\boldsymbol{v} \in D^{b-a}$, and*

$$M(\boldsymbol{u}, \boldsymbol{v}) = \left|\{\boldsymbol{w} \in D^{n-b} : (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in R_I\}\right|.$$

▶ **Theorem 4.** *The notions of Balance, Weak Balance, and Primitive Balance are equivalent, and can be taken as the P-time tractability criterion of the dichotomy in [15].*

Proof. For any $\mathcal{F}$ of nonnegative valued constraint functions, Cai, Chen and Lu proved in [15] that (1) if $\mathcal{F}$ is Balanced then it is Weakly Balanced and that the support constraint language of $\mathcal{F}$ satisfies Strong Rectangularity, and (2) the latter two conditions imply that #CSP($\mathcal{F}$) is P-time computable. Here the support constraint language of $\mathcal{F}$ is obtained by taking the support set of each function in $\mathcal{F}$. On the other hand they also proved that if $\mathcal{F}$ is not Balanced then #CSP($\mathcal{F}$) is #P-hard. Thus their dichotomy criterion is that $\mathcal{F}$ is Balanced. They also proved in [15] that Primitive Balance implies Weak Balance. Lin and Wang proved in [34] that Weak Balance implies Balance, thus unifying all three notions. ◀

3 Bounded Degree #CSPs

▶ **Lemma 5.** *Let M be a nonnegative matrix. If M is not block-rank-1 then neither is MM^T.*

Proof. If every two rows of M are either proportional or their nonzero entries are on disjoint subsets of columns, then M would be block-rank-1. Thus there are rows M_i and M_j, such that they are linearly independent, and the subsets of columns where their nonzero entries occur intersect. Being nonnegative, the latter condition implies that they are not orthogonal. So by the Cauchy-Schwarz inequality we have

$$0 < (M_i \cdot M_j)^2 < (M_i \cdot M_i)(M_j \cdot M_j).$$

Letting $A = MM^T$ we find four nonzero elements $A_{i,i}, A_{i,j} = A_{j,i}$, and $A_{j,j}$ satisfying $A_{i,i}A_{j,j} > A_{i,j}^2 > 0$, so A is not block-rank-1. ◀

We can now prove our main result, i.e., if a nonnegative constraint set $\mathcal{F}$ does not satisfy the Balance condition, then $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ is #P-hard for some $\Delta > 0$.

▶ **Theorem 6.** *If $\mathcal{F}$ is Primitively Balanced, then the problem $\#\mathrm{CSP}(\mathcal{F})$ without degree restriction is computable in polynomial time, otherwise $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$ is #P-hard for some $\Delta > 0$.*

Proof. We only need to prove the hardness part. Let $\mathcal{F}$ be a finite set of nonnegatively weighted constraint functions that is not Primitively Balanced. Then for some instance I on n variables, the $|D| \times |D|$ matrix M defined by

$$M(x_1, x_2) = \sum_{(x_3,\ldots,x_n)\in D^{n-2}} F_I(x_1, x_2, x_3, \ldots, x_n)$$

is not block-rank 1. If we let $A = MM^T$, then A is symmetric, nonnegative, and not block-rank 1 by Lemma 5. This A defines a graph homomorphism problem. We know from [26] that the bounded degree nonnegative graph homomorphism problem $\mathrm{EVAL}^{(\Delta)}(A)$ is #P-hard for some $\Delta > 0$, where the constant Δ depends on A. Here we show a reduction $\mathrm{EVAL}^{(\Delta)}(A) \le_P \#\mathrm{CSP}^{(\Delta')}(\mathcal{F})$, for some $\Delta' > 0$, thereby showing that $\#\mathrm{CSP}^{(\Delta')}(\mathcal{F})$ is #P-hard for some $\Delta' > 0$.

To show that, consider graphs G with maximum degree at most Δ as input instances of $\mathrm{EVAL}^{(\Delta)}(A)$. We can compute the value $Z_A(G)$ by expressing it as the partition function $Z_{\mathcal{F}}(I(G))$ for some instance $I(G)$ of polynomial size in $\#\mathrm{CSP}^{(\Delta')}(\mathcal{F})$. We will use the instance I that defines the matrix M as having constant size, as it does not depend on G. We construct $I(G)$, an input to $\#\mathrm{CSP}(\mathcal{F})$, with the additional property that every variable occurs at most Δ' times, such that $Z_{\mathcal{F}}(I(G)) = Z_A(G)$, as follows.

We note that each entry in A is a dot product of two row vectors in M, and every entry of M is a sum over $|D|^{n-2}$ evaluations of F_I.

We will define a (binary) gadget, which is an instance of $\#\mathrm{CSP}(\mathcal{F})$ of bounded size, with two specially labelled variables called x^* and x^{**}. Copies of this gadget will be used in the construction of (global) $\#\mathrm{CSP}(\mathcal{F})$ instances. A (binary) gadget may have other variables, but in the global $\#\mathrm{CSP}(\mathcal{F})$ instances all constraints applied to the variables other than x^* and x^{**} in each copy are from within the gadget. We define $I(G)$ by replacing every edge in G by a copy of this gadget. Formally, the construction is as follows, where the gadget simulates the edge weights in A in the #CSP setting.

1. Define a variable x_v over D for every $v \in V(G)$.
2. For each $e = uv \in E(G)$ we add $2n-3$ variables y_e, $\boldsymbol{z}_e = (z_{e,3}, \ldots, z_{e,n})$, and $\boldsymbol{z}'_e = (z'_{e,3}, \ldots, z'_{e,n})$ over the same domain D. We then apply two copies of I as constraints, one copy over the variables $(x_u, y_e, \boldsymbol{z}_e)$ and another over $(x_v, y_e, \boldsymbol{z}'_e)$. There are no other constraints applied on the variables y_e, $\boldsymbol{z}_e$ and $\boldsymbol{z}'_e$.

Thus one copy of the binary gadget is used to replace each edge $uv \in E(G)$, with the two specially labelled variables x^* and x^{**} identified with x_u and x_v. This gadget defines the following constraint function with input variables x_u and x_v taking values in D,

$$\begin{aligned}
&\sum_{y_e\in D}\left(\sum_{\boldsymbol{z}_e\in D^{n-2}} F_I(x_u, y_e, \boldsymbol{z}_e) \sum_{\boldsymbol{z}'_e\in D^{n-2}} F_I(x_v, y_e, \boldsymbol{z}'_e)\right) \\
&= \sum_{y_e\in D} M_I(x_u, y_e) M_I(x_v, y_e) \\
&= (MM^T)(x_u, x_v) \\
&= A(x_u, x_v).
\end{aligned}$$

Therefore the gadget defines the edge constraint function represented by the matrix A, which is exactly the edge weights in $Z_A(G)$.

Since G has bounded degrees, and I also has constant size, $I(G)$ also has bounded degrees. The variables y_e are "local" to each edge $e \in E(G)$ in the sense that there are no other constraints on them except in the definition of the gadget for this edge e, which has constant size. The same is true for $\boldsymbol{z}_e$ and $\boldsymbol{z}'_e$.

Also, the size of $I(G)$ is linear in the size of G, so this is a polynomial time reduction. That $Z_{\mathcal{F}}(I(G)) = Z_A(G)$ follows from the fact that A is the edge constraint function by our construction. ◀

The constant Δ in Theorem 1 depends on $\mathcal{F}$. In [21], Dyer and Greenhill conjectured that a universal constant $\Delta = 3$ suffices for $\mathrm{EVAL}^{(\Delta)}(A)$ where A is a 0-1 symmetric matrix. This is still open. It is open whether a universal constant Δ, or a constant that only depends on the domain size $|D|$, may suffice for even the 0-1 case, for both $\mathrm{EVAL}^{(\Delta)}(A)$ and $\#\mathrm{CSP}^{(\Delta)}(\mathcal{F})$. In [17] it is known that the constant 3 suffices for the Boolean domain. Xia [50] proved that a universal Δ *does not* exist for $\mathrm{EVAL}^{(\Delta)}(A)$ for complex symmetric matrices A, assuming #P does not collapse to P.

4 Effective Dichotomy and a Family of Directed GH

The condition of Balance (Definition 2) in the dichotomy refers to all instances I of $\#\mathrm{CSP}(\mathcal{F})$, which is an infinitary statement. Thus it is not immediate that the tractability condition in Theorem 6 is decidable. However the condition is the same as the one in [15] for the unbounded degree case, and in that paper a decision procedure is given. Here we give a slight modification of the same decision procedure for the dichotomy in Theorem 6. This form is more symmetric and allows us to apply the procedure more effectively.

▶ **Theorem 7.** *The polynomial-time tractability condition of balance in Theorem 6 can be tested by the following two conditions. Measured in the size of D and $\mathcal{F}$, this shows that the decision problem for testing balance is in* NP.

(A) *There is a Mal'tsev polymorphism $\varphi : D^3 \to D$ for (the support of) every function in $\mathcal{F}$. This means that φ preserves all relations defined as the support of some function in $\mathcal{F}$ (this is called a polymorphism), and satisfies $\varphi(a, a, b) = \varphi(b, a, a) = b$ for all $a, b \in D$ (Mal'tsev property). The existence of such a mapping φ is equivalent to Strong Rectangularity.*

(B) *For all $\alpha \neq \beta, \kappa \neq \lambda \in D$ there is a bijection $\pi : D^6 \to D^6$ satisfying the following three properties:*

1. $\pi((\alpha, \alpha, \alpha, \beta, \beta, \beta)) = (\alpha, \alpha, \alpha, \beta, \beta, \beta)$.
2. $\pi((\kappa, \lambda, \kappa, \lambda, \kappa, \lambda)) = (\lambda, \kappa, \lambda, \kappa, \lambda, \kappa)$.
3. *Any function $f \in \mathcal{F}$ with arity r is invariant under π, that is, for any sequence $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r)$ of length r of 6-tuples where $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,6}) \in D^6$ for $1 \leq i \leq r$, the following holds:*

$$\prod_{j \in [6]} f(y_{1,j}, \ldots, y_{r,j}) = \prod_{j \in [6]} f(\pi(\boldsymbol{y}_1)_j, \ldots, \pi(\boldsymbol{y}_r)_j). \tag{3}$$

The only difference in the statement of this decision criterion compared to the one stated in [15] is a more symmetric expression for condition *B2* of (B). We omit the proof here as it closely follows the proof in [15][3] However, this more symmetric form makes the criterion

[3] Alternatively, one can apply a perm $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}$ which is an automorphism of the relational structure $(\mathfrak{D}, \mathfrak{F})$, the 6^{th} power of $\#\mathrm{CSP}(\mathcal{F})$, to derive this form of the decision criterion from

more easily applicable. We demonstrate this by proving new tractable and intractable cases of directed graph homomorphisms that were previously unknown. The tractability or intractability of these problems were decidable in principle by previous methods; however, the decision procedure of the previous method was in practice too complicated to be useful.

Dyer, Goldberg and Paterson in [24] proved a decidable complexity dichotomy for (unweighted) directed graph homomorphisms that is restricted to directed *acyclic* graphs. Their polynomial-time tractability criterion is an interesting condition of being *layered* and *Lovász-good* for directed acyclic graphs. They state in [24] that "An interesting feature of the dichotomy, which is absent from previously-known dichotomy results, is that there is a rich supply of tractable graphs H with complex structure". Going beyond directed *acyclic* graphs, as it is done with Theorem 6 and 7, is expected to yield even more polynomial-time tractable problems. However, up until now we don't have any interesting concrete examples. (Part of the reason is probably that testing for the tractability criterion, while decidable, is not a simple matter; see below.) The dichotomy theorem in this paper applies more generally without the *acyclicity* restriction. We now give a family of non-acyclic directed graph homomorphism problems that we can completely classify using our tractability criterion in Theorem 7. To our best knowledge, this is the first such explicit family that can be classified, going beyond the Lovász-goodness criterion [24].

To start, if we take all nonzero $A_{i,j} = 1$ in equation (2), we get a binary relation that defines a polynomial-time tractable problem. This represents an adjacency matrix of a directed graph H illustrated in Figure 2. In fact, we can give an infinite family of tractable #CSP based on a weighted binary constraint function given in equation (4). These problems were considered in [12], where it was shown that, while the complexity of the #CSP defined by these relations is provably decidable, the decision criterion was yet too complicated, therefore for what values of $A_{i,j}$ in equation (2) the problem it defines is tractable for #CSP was not resolved.

We will show that a nonnegative binary constraint function given in the form of equation (2) with positive entries $A_{i,j}$ defines a tractable #CSP iff the constraint function is a positive multiple of the function in equation (4)

$$A = \begin{bmatrix} 1 & & v & & & & & \\ u & & uv & & & & & \\ & & & & y & & yv & \\ & & & & yu & & yuv & \\ & x & & xv & & & & \\ & xu & & xuv & & & & \\ & & & & & z & & zv \\ & & & & & zu & & zuv \end{bmatrix} \tag{4}$$

for some positive reals u, v, x, y, z, with the condition that $z = xy$.

The #CSP problem it defines is on a domain of size 8 and has a constraint function set $\mathcal{F}$ consisting of a single binary (but not symmetric) constraint function given by the matrix in (4). The directed graph defined by the support of the function is not acyclic.

We first prove the relation in (4) for any positive u, v, x, y and $z = xy$ defines a tractable problem. After that we prove the reverse direction.

To apply Theorem 7, we treat $D = \{0, \ldots, 7\}$ as a vector space $(\mathrm{GF}[2])^3$ of size 8, represented by three bit strings $\{0,1\}^3$. Then we can take the Mal'tsev polymorphism $\varphi : D^3 \to D$ where $\varphi(x, y, z) = x - y + z$. (Here $-$ is the same as $+$ in $\mathrm{GF}[2^3]$.) It is Mal'tsev because $\varphi(a, a, b) = \varphi(b, a, a) = b$ for all $a, b \in D$. The directed edge relation given by the matrix (2) (by setting all 16 nonzero entries $A_{i,j} = 1$) is given symbolically as follows:

$$A_{i,j} = 1 \text{ where } i = i_1 i_2 i_3, j = j_1 j_2 j_3 \in D \iff j_1 = i_2 \text{ and } j_3 = i_1. \tag{5}$$

the form proved in [15]. For detailed definitions of $(\mathfrak{D}, \mathfrak{F})$ and automorphism of relational structures see page 2190 in [15].

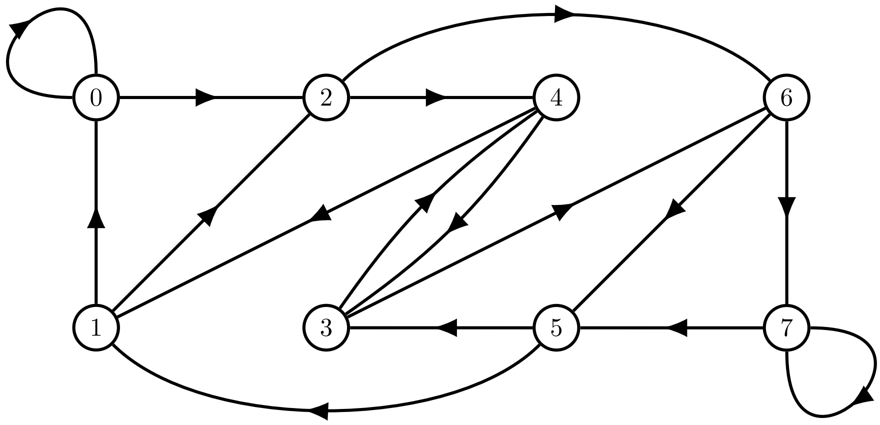


Figure 2 A tractable binary relation represented by a directed graph. The adjacency matrix is given in equation (4).

One can easily check that φ is a polymorphism, i.e. for any (x,x'), (y,y') and (z,z'), if $A_{x,x'} = A_{y,y'} = A_{z,z'} = 1$ then $A_{\varphi(x,y,z),\varphi(x',y',z')} = 1$.

For the second requirement (B), there are $|D|^6! = 262144! > 10^{1306590}$ bijections from D^6 to D^6, so it is infeasible to enumerate them. However the following map π works.

Let $M : \{0,1\}^6 \to \{0,1\}^6$ be the bijection that swaps (010101) and (101010) and acts as the identity on the rest. In particular, M preserves the Hamming weight. Let M_i be the ith output bit of M, and let $x_1, x_2, \ldots, x_6 \in D$ where each x_i consists of three bits $x_i = a_i b_i c_i \in \{0,1\}^3$.

We write $\boldsymbol{x} = (x_1, \ldots, x_6) = (a_1b_1c_1, \ldots, a_6b_6c_6) \in D^6$. We will also represent $\boldsymbol{x}$ bitwise using $\boldsymbol{a} = (a_1, \ldots, a_6) \in \{0,1\}^6$, $\boldsymbol{b} = (b_1, \ldots, b_6) \in \{0,1\}^6$, and $\boldsymbol{c} = (c_1, \ldots, c_6) \in \{0,1\}^6$, and write $\boldsymbol{x} = \boldsymbol{abc}$.

Then we define

$$\begin{aligned}\pi(\boldsymbol{x}) &= \pi(x_1, x_2, \ldots, x_6) = \pi(a_1b_1c_1, a_2b_2c_2, \ldots, a_6b_6c_6) \\ &= (M_1(\boldsymbol{a})M_1(\boldsymbol{b})M_1(\boldsymbol{c}),\ M_2(\boldsymbol{a})M_2(\boldsymbol{b})M_2(\boldsymbol{c}),\ \ldots,\ M_6(\boldsymbol{a})M_6(\boldsymbol{b})M_6(\boldsymbol{c}))\end{aligned}$$

▷ Claim 8. The mapping $\pi : D^6 \to D^6$ satisfies properties B1, B2, and B3 of (B) in Theorem 7 for all $\alpha \neq \beta, \kappa \neq \lambda \in D$. [4]

Proof Sketch. Property B1 holds by construction, because M fixes pointwise 0^6, 1^6, $0^3 1^3$, $1^3 0^3$. It satisfies property B2 because in addition it swaps (010101) and (101010).

For property B3, equation (3) is expressed as (more details are given below)

$$u^{\sum c_i} v^{\sum d_i} x^{\sum a_i} y^{\sum b_i} \left(\frac{z}{xy}\right)^{\sum a_i b_i} = u^{\sum M_i(\boldsymbol{c})} v^{\sum M_i(\boldsymbol{d})} x^{\sum M_i(\boldsymbol{a})} y^{\sum M_i(\boldsymbol{b})} \left(\frac{z}{xy}\right)^{\sum M_i(\boldsymbol{a}) M_i(\boldsymbol{b})} \tag{6}$$

[4] In Theorem 7 the mapping π may depend on $\alpha \neq \beta, \kappa \neq \lambda \in D$, but the π in Claim 8 is in fact the same for all $\alpha, \beta, \kappa, \lambda$.

where all sums (as well as those below) range from $i = 1$ to 6. Because M preserves Hamming weight, we get $\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} M_i(\boldsymbol{a})$, and similarly for $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, and since $z = xy$, this equation holds. ◀

We now investigate *all* possible tractable cases of A in (2) with positive entries $A_{i,j}$. By the necessary condition of Balance applied to the binary function A itself, all relevant four 2×2 blocks must be of rank 1, in order to be tractable, i.e., it takes the form in (4) with some positive u, v, x, y and z, up to a global positive factor. We prove that, up to a global positive factor, a nonnegative matrix A in (2) with the given support structure defines a tractable partition function $Z_{\mathcal{F}}(\cdot)$ where $\mathcal{F} = \{A\}$ iff A has the form in (4) for some positive reals u, v, x, y and $z = xy$; otherwise $Z_{\mathcal{F}}(\cdot)$ is #P-hard.

This #CSP problem has $\mathcal{F}$ consisting of a single binary (nonsymmetric) constraint function defined by the matrix A. By its support structure and the Mal'tsev polymorphism we already satisfied condition (A) of Theorem 7. So, the problem is tractable if and only if for all $\alpha \neq \beta, \kappa \neq \lambda \in D$, there is a bijection $\pi : D^6 \to D^6$ that satisfies the following three properties:

1. $\pi((\alpha, \alpha, \alpha, \beta, \beta, \beta)) = (\alpha, \alpha, \alpha, \beta, \beta, \beta)$.
2. $\pi((\kappa, \lambda, \kappa, \lambda, \kappa, \lambda)) = (\lambda, \kappa, \lambda, \kappa, \lambda, \kappa)$.
3. For the binary function represented by the matrix A, and any 6-tuples $\boldsymbol{x}, \boldsymbol{y} \in D^6$, where $\boldsymbol{x} = (x_1, \dots, x_6)$ and $\boldsymbol{y} = (y_1, \dots, y_6)$, we have the following invariance under π,

$$\prod_{i \in [6]} A_{x_i, y_i} = \prod_{i \in [6]} A_{\pi(\boldsymbol{x})_i, \pi(\boldsymbol{y})_i}. \tag{7}$$

Suppose there is a bijection $\pi : D^6 \to D^6$ that satisfies these properties. Let $\pi(x_1, \dots, x_6) = (\pi_1(x_1, \dots, x_6), \dots, \pi_6(x_1, \dots, x_6))$, where $\pi_i : D^6 \to D$, and $\pi_i(x_1, \dots, x_6)$ is the ith output entry in D of π. Denote the three bits of $\pi_i(x_1, \dots, x_6)$ as $f_i(x_1, \dots, x_6)$, $g_i(x_1, \dots, x_6)$, and $h_i(x_1, \dots, x_6)$. To satisfy property 3, we need each π_i to preserve the edge relation 5, i.e., preserve the support set. Since π is a bijection, if we verify that a nonzero LHS of (7) implies a nonzero RHS of (7), we will also have proved that it maps a zero LHS to a zero RHS; thus it preserves the support set. So, consider arbitrary

$$\boldsymbol{x} = (x_1, \dots x_6) = (a_1 b_1 c_1, \dots, a_6 b_6 c_6) \in D^6, \;\; \boldsymbol{y} = (y_1, \dots, y_6) = (b_1 d_1 a_1, \dots, b_6 d_6 a_6) \in D^6.$$

This is a generic pair of tuples such that $A_{x_i, y_i} \neq 0$, for $1 \leq i \leq 6$. We need $A_{\pi_i(\boldsymbol{x}), \pi_i(\boldsymbol{y})} \neq 0$ for each i. As before we will also represent $\boldsymbol{x}$ bitwise using $\boldsymbol{a} = (a_1, \dots, a_6) \in \{0,1\}^6$, $\boldsymbol{b} = (b_1, \dots, b_6) \in \{0,1\}^6$, $\boldsymbol{c} = (c_1, \dots, c_6) \in \{0,1\}^6$ and $\boldsymbol{d} = (d_1, \dots, d_6) \in \{0,1\}^6$, and write $\boldsymbol{x} = \boldsymbol{abc}$ and $\boldsymbol{y} = \boldsymbol{bda}$.

Therefore, by the edge relation, we have $f_i(\boldsymbol{abc}) = h_i(\boldsymbol{bda})$. Hence f_i is independent of the third part of the input $\boldsymbol{c}$. Also, $g_i(\boldsymbol{abc}) = f_i(\boldsymbol{bda})$, so f_i is also independent of the second part of the input, and therefore is in fact a function on the first part of the input only. Thus there is a function $f_i' : \{0,1\}^6 \to \{0,1\}$, such that $f_i(\boldsymbol{abc}) = f_i'(\boldsymbol{a})$. Then, from $f_i'(\boldsymbol{a}) = h_i(\boldsymbol{bda})$, we know that h_i is actually a function of its third part of the input only. From $g_i(\boldsymbol{abc}) = f_i'(\boldsymbol{b})$, we know that g_i is a function of its second part of the input only. Thus, there are functions $g_i', h_i' : \{0,1\}^6 \to \{0,1\}$, such that $g_i(\boldsymbol{abc}) = g_i'(\boldsymbol{b})$ and $h_i(\boldsymbol{abc}) = h_i'(\boldsymbol{c})$. Putting these together, we see $f_i'(\boldsymbol{a}) = g_i'(\boldsymbol{a}) = h_i'(\boldsymbol{a})$. Since $\boldsymbol{a} \in \{0,1\}^6$ is arbitrary, we get $f_i' = g_i' = h_i'$. We now rename these as $M_i := f_i' = g_i' = h_i'$. In other words, $\pi : D^6 \to D^6$ has the form $\pi = (\pi_1, \pi_2, \dots, \pi_6)$ where $\pi_i(\boldsymbol{abc}) = M_i(\boldsymbol{a}) M_i(\boldsymbol{b}) M_i(\boldsymbol{c})$. We will name the mapping $M = (M_1, M_2, \dots, M_6) : \{0,1\}^6 \to \{0,1\}^6$ with M_i being its ith bit output. Since π is a bijection, so must be M.

Now we pick $\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \kappa = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in D$. (We write them as column vectors to visually aid the readers.) Then clearly $\alpha \neq \beta, \kappa \neq \lambda$. We have $(\alpha, \alpha, \alpha, \beta, \beta, \beta) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. For any π defined by a bijection M as above, it satisfies property 1. above iff M pointwise fixes 000000, 000111 and 111000. We also have $(\kappa, \lambda, \kappa, \lambda, \kappa, \lambda) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$, and $(\lambda, \kappa, \lambda, \kappa, \lambda, \kappa) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. Hence π satisfies property 2. above iff M fixes 111111 and swaps 010101 with 101010. Below we assume M is a bijection that satisfies these properties.

It is easy to verify that for any bijection $M : \{0,1\}^6 \to \{0,1\}^6$, the mapping π defined above preserves the support (defined by nonzero values of the LHS in (7)). Since M is a bijection, in the following we only need to verify that (7) holds for any nonzero LHS of (7) (as any zero LHS automatically has a zero RHS).

Now we show that equation (7) in property 3 is the same as (6).

To see that, take any nonzero of the LHS in (7) with $\boldsymbol{x} = \boldsymbol{abc}$ and $\boldsymbol{y} = \boldsymbol{bda}$, then the LHS is evaluated as

$$\begin{aligned} & \prod_{i \in [6]} (u^{c_i} v^{d_i})^{(1-a_i)(1-b_i)} (y u^{c_i} v^{d_i})^{(1-a_i)b_i} (x u^{c_i} v^{d_i})^{a_i(1-b_i)} (z u^{c_i} v^{d_i})^{a_i b_i} \\ = & \prod_{i \in [6]} u^{c_i} v^{d_i} x^{a_i} y^{b_i} \left(\frac{z}{xy}\right)^{a_i b_i} \\ = & \; u^{\sum c_i} v^{\sum d_i} x^{\sum a_i} y^{\sum b_i} \left(\frac{z}{xy}\right)^{\sum a_i b_i} \end{aligned}$$

The expression for the RHS is nearly identical, with $M_i(\boldsymbol{a})$ substituting a_i, and so on.

Now it is clear that if $z = xy$, then the partition function $Z_{\mathcal{F}}(\cdot)$ is tractable, witnessed by any π defined by a bijection on $\{0,1\}^6$ that preserves Hamming weight, pointwise fixes 000000, 111111, 000111, 111000, and swaps 010101 and 101010.

Next, assume $z \neq xy$. We can multiply both sides of (6) over all 2^6 possible $\boldsymbol{c}$ and all 2^6 possible possible $\boldsymbol{d}$, and using the fact that M is a bijection, to get

$$\begin{aligned} & u^{6 \cdot 2^{11}} v^{6 \cdot 2^{11}} x^{2^{12} \sum a_i} y^{2^{12} \sum b_i} \left(\frac{z}{xy}\right)^{2^{12} \sum a_i b_i} = \\ & \quad u^{6 \cdot 2^{11}} v^{6 \cdot 2^{11}} x^{2^{12} \sum M_i(\boldsymbol{a})} y^{2^{12} \sum M_i(\boldsymbol{b})} \left(\frac{z}{xy}\right)^{2^{12} \sum M_i(\boldsymbol{a}) M_i(\boldsymbol{b})}, \end{aligned}$$

which is equivalent to

$$x^{\sum a_i} y^{\sum b_i} \left(\frac{z}{xy}\right)^{\sum a_i b_i} = x^{\sum M_i(\boldsymbol{a})} y^{\sum M_i(\boldsymbol{b})} \left(\frac{z}{xy}\right)^{\sum M_i(\boldsymbol{a}) M_i(\boldsymbol{b})}. \tag{8}$$

Then multiplying over all 2^6 possible 6-tuples $\boldsymbol{b}$,

$$x^{2^6 \sum a_i} y^{6 \cdot 2^{11}} \left(\frac{z}{xy}\right)^{2^5 \sum a_i} = x^{2^6 \sum M_i(\boldsymbol{a})} y^{6 \cdot 2^{11}} \left(\frac{z}{xy}\right)^{2^5 \sum M_i(\boldsymbol{a})},$$

we get $\left(\frac{xz}{y}\right)^{\sum a_i} = \left(\frac{xz}{y}\right)^{\sum M_i(\boldsymbol{a})}$.

Suppose $xz \neq y$, it follows that $\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} M_i(\boldsymbol{a})$ for any $\boldsymbol{a}$, i.e., M preserves Hamming weight. Then it follows from (6) that

$$\left(\frac{z}{xy}\right)^{\sum a_i b_i} = \left(\frac{z}{xy}\right)^{\sum M_i(\boldsymbol{a}) M_i(\boldsymbol{b})}.$$

But if we take $\boldsymbol{a} = 000111$ and $\boldsymbol{b} = 010101$, we have $M(\boldsymbol{a}) = \boldsymbol{a}$ and $M(\boldsymbol{b}) = 101010$. Then $\sum_{i=1}^{6} a_i b_i = 2$, but $\sum_{i=1}^{6} M_i(\boldsymbol{a})M_i(\boldsymbol{b}) = 1$. This is a contradiction to (6), since $z \neq xy$. Hence we conclude that the partition function $Z_{\mathcal{F}}(\cdot)$ is #P-hard.

Next, suppose that $xz = y$, then (8) is simplified to

$$x^{\sum a_i} y^{\sum b_i} x^{-2\sum a_i b_i} = x^{\sum M_i(\boldsymbol{a})} y^{\sum M_i(\boldsymbol{b})} x^{-2\sum M_i(\boldsymbol{a})M_i(\boldsymbol{b})}. \tag{9}$$

Multiplying over all possible $\boldsymbol{a}$, this becomes

$$x^{2^5} y^{2^6 \sum b_i} x^{-2\cdot 2^5 \sum b_i} = x^{2^5} y^{2^6 \sum M_i(\boldsymbol{b})} x^{-2\cdot 2^5 \sum M_i(\boldsymbol{b})}.$$

This simplifies to

$$\left(\frac{y}{x}\right)^{\sum b_i} = \left(\frac{y}{x}\right)^{\sum M_i(\boldsymbol{b})}.$$

If $x \neq y$, M preserves weight and we are done by the same argument as for when $xz \neq y$.

Otherwise if $x = y$, (9) becomes

$$x^{\sum a_i + \sum b_i - 2\sum a_i b_i} = x^{\sum M_i(\boldsymbol{a}) + \sum M_i(\boldsymbol{b}) - 2\sum M_i(\boldsymbol{a})M_i(\boldsymbol{b})}.$$

This is equivalent to

$$x^{\frac{1}{2} - \frac{1}{2}\sum (2a_i - 1)(2b_i - 1)} = x^{\frac{1}{2} - \frac{1}{2}\sum (2M_i(\boldsymbol{a}) - 1)(2M_i(\boldsymbol{b}) - 1)},$$

which can be written as

$$x^{\sum a_i' b_i'} = x^{\sum M_i(\boldsymbol{a})' M_i(\boldsymbol{b})'}$$

where $a_i' = 2a_i - 1 \in \{-1, 1\}$, and similarly for b_i', and $M_i(\boldsymbol{a})'$, $M_i(\boldsymbol{b})'$. We can fix the same $\boldsymbol{a}$ and $\boldsymbol{b}$ as above, and this gives $\boldsymbol{a}' = (-1, -1, -1, 1, 1, 1)$ and $\boldsymbol{b}' = (-1, 1, -1, 1, -1, 1)$. Then we get $2 = \sum_{i=1}^{6} a_i' b_i' \neq \sum_{i=1}^{6} M_i(\boldsymbol{a})' M_i(\boldsymbol{b})' = -2$. Thus we must have $x = 1$, and then $y = 1$ and $z = 1$, which contradicts $z \neq xy$. We have proved that if $z \neq xy$ the partition function $Z_{\mathcal{F}}(\cdot)$ is #P-hard.

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