Higher-Order Causal Theories Are Models of BV-Logic

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Abstract

The Caus[−] construction takes a compact closed category of basic processes and yields a *-autonomous category of higher-order processes obeying certain signalling/causality constraints, as dictated by the type system in the resulting category. This paper looks at instances where the base category C satisfies additional properties yielding an affine-linear structure on Caus[$\mathcal{C}$] and a substantially richer internal logic. While the original construction only gave multiplicative linear logic, here we additionally obtain additives and a non-commutative, self-dual sequential product yielding a model of Guglielmi's BV logic. Furthermore, we obtain a natural interpretation for the sequential product as "A can signal to B, but not vice-versa", which sits as expected between the non-signalling tensor and the fully-signalling (i.e. unconstrained) par. Fixing matrices of positive numbers for $\mathcal{C}$ recovers the BV category structure of probabilistic coherence spaces identified by Blute, Panangaden, and Slavnov, restricted to normalised maps. On the other hand, fixing the category of completely positive maps gives an entirely new model of BV consisting of higher order quantum channels, encompassing recent work in the study of quantum and indefinite causal structures.

2012 ACM Subject Classification Theory of computation → Quantum computation theory; Theory of computation → Linear logic; Theory of computation → Categorical semantics

Keywords and phrases Causality, linear logic, categorical logic, probabilistic coherence spaces, quantum channels

Digital Object Identifier 10.4230/LIPIcs.MFCS.2022.80

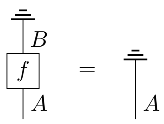
Related Version *Full Version*: `https://arxiv.org/abs/2205.11219` [26]

Funding *Will Simmons*: was supported by Cambridge Quantum (Quantinuum Ltd).
Aleks Kissinger: acknowledges support from Grant No. 61466 from the John Templeton Foundation as part of the QISS project. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

Acknowledgements The authors would like to thank Alessio Guglielmi for posing the question of modelling BV-logic within Caus[$\mathcal{C}$], as well as Chris Barrett, Lutz Straßburger, and members of the quantum group at University of Oxford for useful discussions.

1 Introduction

The causality condition [8] is a simple equation one can impose on a family of processes that states essentially that "discarding the output of f is the same as discarding its input." Pictorially:[1]



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[1] We will use *string diagram* notation for monoidal categories throughout, depicting morphisms as boxes, composition as "plugging" boxes from bottom-to-top and ⊗ as placing boxes side-by-side. See e.g. [25].

47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022).
Editors: Stefan Szeider, Robert Ganian, and Alexandra Silva; Article No. 80; pp. 80:1–80:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

for some fixed family of "discarding" processes $\bar{\top}_A$. While this seems to be a simple equation, and reasonable to impose on most sane collections of physical processes, it deserves the somewhat lofty name of *causality* because it really seems to capture the essence of influences moving in a single direction: forward in time. In particular, it guarantees that distant agents cannot send messages to one another without their existing a forward-directed path in the diagram of their processes. Without referring explicitly to spacetime, this serves as an abstract stand-in for what a relativity theorist would also call causality, namely the limitation that influences cannot propagate faster than the speed of light [17].

In concrete categories of interest, namely those of probabilistic and quantum maps, equation (1) simply imposes that processes should preserve normalised states. Iterating this, we might wish to consider second-order processes that preserve processes that preserve normalised states, and so on. This turns out to provide a rich landscape for modelling causal relationships between events, enabling for example chains [6, 13] (or directed acyclic graphs [19]) of events in a definite causal order, probabilistic mixtures of causal orderings, or even exotic indefinite causal structures [1, 9, 22].

There are several routes to constructing a framework for characterising and composing such higher-order processes, including describing acyclic graphs of interactions between discrete first-order channels [4, 7] or building an infinite hierarchy of types recursively out of constructors describing a kind of causal relationship [2].

The Caus[−] construction of Kissinger and Uijlen [19] fits the latter style through a category where morphisms map normalised states of the source type to normalised states of the target type. For example, Caus[CP*] describes higher-order generalisations of CPTP-maps, containing quantum combs [7], process matrices [22], and bipartite second-order maps [18] which include the quantum switch [9], which have collectively been critical in the formulation and study of indefinite causal structures. The typing of a morphism is a judgement about externally-observable properties of normalisation and information flow as opposed to any assumed internal structure or spacetime geometry. The type constructors model the operators of Multiplicative Linear Logic (MLL), i.e. it forms a $*$-autonomous category. MLL features two logical connectives, tensor and par, which serve as two extremes in the Caus[−] construction. Tensor yields a *non-signalling* composition of processes, where causal influences are not allowed to pass from one side to the other, whereas par gives a *fully-signalling* composition, i.e. one that imposes no signalling constraints.

In between this lies the *one-way signalling* processes, where causal influences can flow from one agent (say, Alice in the past) to another (say, Bob in the future). While special cases of such processes were treated in an *ad hoc* way in [19], here we will show that one-way signalling can in fact be treated as a fully-fledged connective in its own right, yielding a substantially richer logical structure.

BV-logic [11] adds a third logical connective that is non-commutative to capture sequentiality in a similar way to Retoré's pomset logic [24]. It admits a categorical characterisation via BV-categories, and has previously been applied to the study of (probabilistic) coherence spaces [3] and a certain graph-based model of quantum causal structures called discrete quantum causal dynamics [4].

In this paper, we will adapt the Caus[−] construction by modifying some of the assumptions on the base category, requiring it to be *additive precausal*. The extra structure allows us to consider new ways of combining processes and types to describe operational constructions such as binary tests, probability distributions, and one-way signalling processes. These correspond to extending the logical structure of Caus[$\mathcal{C}$] with the additive connectives (sometimes called "with" and "plus") of linear logic, as well as the sequential connective of BV. We show that the main classical and quantum examples of precausal categories are furthermore additive precausal, enabling Caus[$\mathcal{C}$] to model higher-order theories and classical and quantum causal structures, respectively. We also note the existence of non-standard models, such as higher-order affine (a.k.a. quasi-probabilistic) processes.

During the development of this paper, independent work by Hoffreumon and Oreshkov [14] investigated the same spaces of higher-order one-way signalling processes specifically in quantum theory and identified the same self-duality along with some additional results on decompositions into intersections and unions of spaces. Additionally, Cavalcanti et al. [5] obtained the same characterisation of non-signalling spaces as the affine closure of separable processes for multi-partite first-order channels in any generalised probabilistic theory with local tomography.

We refer the reader to the arXiv version of this paper [26] for full proofs of all the results presented here.

2 Higher-Order Causal Structure

We use "process theory" style terminology (see e.g. [10]) for monoidal categories throughout. Namely, we use the terms *type* or *system* interchangeably to refer to objects of a symmetric monoidal category, *process* to refer to a generic morphism $f : A \to B$, *state* to refer to a morphism $\rho : I \to A$ from the tensor unit, *effect* to refer to a morphism $\pi : A \to I$ to the tensor unit, and *scalar* to refer to a morphism $\lambda : I \to I$. In categories admitting higher-order structure, we think of first-order types as state spaces and first-order processes as maps that transform first-order states to first-order states. Higher-order processes are transformations where the input or output system is itself a map. The main example used in causality literature is a quantum 2-comb which is a process taking a channel over first-order types as an input and transforms it into a new channel, typically by composing with some pre- and post-processing which may share some memory channel. One can extend this to an infinite hierarchy of higher-order processes where an $(n+1)$-comb transforms an n-comb into a 1-comb (a channel) [7]. A higher-order process theory is one which describes processes in such an infinite hierarchy uniformly.

Higher-order theories are commonly achieved by providing a mechanism to encode transformations into states of some function type. In category theory, this corresponds to an internal hom in monoidal closed categories, i.e. a bifunctor $\multimap : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ such that $\mathcal{C}(C \otimes A, B) \simeq \mathcal{C}(C, A \multimap B)$ and this is natural in all arguments. Wilson and Chiribella [27] demonstrate that adding basic manipulations (morphisms that capture sequential and parallel composition of such encoded functions) is sufficient to permit the inductive generation of comb types. However, with higher-order theories, there are more ways of composing processes that need not be obtainable in this way.

To solve this, we may further ask that $\mathcal{C}$ be a *compact closed category*, i.e. we require every object A has a "cup" state $\eta_A : I \to A^* \otimes A$ and a "cap" effect satisfying the so-called "yanking equations": $(\mathrm{id}_A \otimes \eta_A) \fatsemi (\epsilon_A \otimes \mathrm{id}_A) = \mathrm{id}_A$ and $(\eta_A \otimes \mathrm{id}_{A^*}) \fatsemi (\mathrm{id}_{A^*} \otimes \epsilon_A) = \mathrm{id}_{A^*}$. These enable us to convert inputs to outputs at will, which in turn lets us "wire up" processes in arbitrary ways to each other. Namely, we can treat all higher order processes as states, and use "caps" to connect them to each other in arbitrary ways. See Figure 1 for an example and [19, Section 2.1] for more details.

The causality condition in equation (1) specialises to states as the requirement that $\rho \fatsemi \bar{\top}_A = \mathrm{id}_I =: 1$ for any state ρ, i.e. it imposes that states be normalised with respect to the discarding effect. Discarding is unique in the sense that it is the only effect normalised for all states. However, for the higher-order analogue of state spaces, there might be more such effects, motivating the following definition.

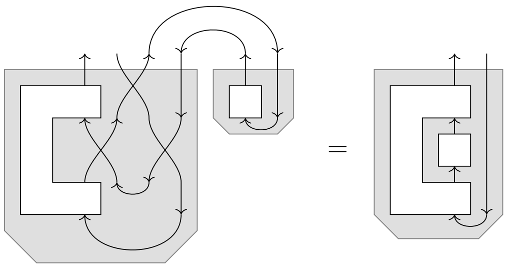


Figure 1 Example of embedding higher-order processes and composition via compact closure.

▶ **Definition 1.** *For $c \subseteq \mathcal{C}(I, A)$, the dual set is $c^* := \{\pi \in \mathcal{C}(A, I) \mid \forall \rho \in c . \rho \,⨟\, \pi = \mathrm{id}_I\}$.*

▶ Remark 2. In this paper, we will freely mix between treating c/c^* as a set of states in $\mathcal{C}(I, A)/\mathcal{C}(I, A^*)$ and a set of effects in $\mathcal{C}(A^*, I)/\mathcal{C}(A, I)$, depending on the context. They are equivalent through the transpose isomorphism induced by compact closure.

From this we can recover the space of first-order/*causal* states for a system A as $\{\bar{\bar{\top}}_A\}^*$. For example, in quantum theory we take $\bar{\bar{\top}}_A \in \mathcal{C}(A, I)$ to represent the partial trace over the system A, so this set describes the space of trace-1 density matrices. First-order/causal processes from A to B are those that map every state in $\{\bar{\bar{\top}}_A\}^*$ to a state in $\{\bar{\bar{\top}}_B\}^*$ (equivalent to mapping the effect $\bar{\bar{\top}}_B$ to $\bar{\bar{\top}}_A$ when our category has enough states/well-pointedness), which we can encode as $\left\{\{\bar{\bar{\top}}_A\}^* \otimes \bar{\bar{\top}}_B\right\}^* \subseteq \mathcal{C}(I, A^* \otimes B)$. These concepts are used to define a precausal category, which specifies one possible set of conditions to enable discussions of higher-order causal structures.

▶ **Definition 3.** *A compact closed category $\mathcal{C}$ is a precausal category if:*

PC1. *$\mathcal{C}$ has a discarding process $\bar{\bar{\top}}_A \in \mathcal{C}(A, I)$ for every system A, compatible with the monoidal structure as below;*

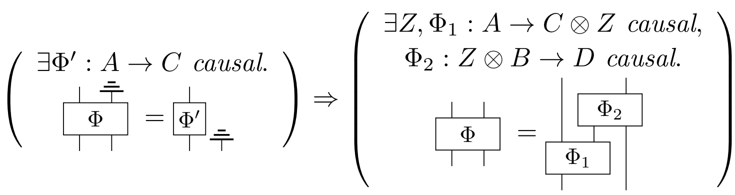
$$\bar{\bar{\top}}_{A \otimes B} = \bar{\bar{\top}}_A \otimes \bar{\bar{\top}}_B \tag{2}$$

$$\bar{\bar{\top}}_I = \mathrm{id}_I \tag{3}$$

PC2. *The dimension $d_A := \bar{\bar{\bot}}_A \,⨟\, \bar{\bar{\top}}_A$ is invertible for all non-zero A;*

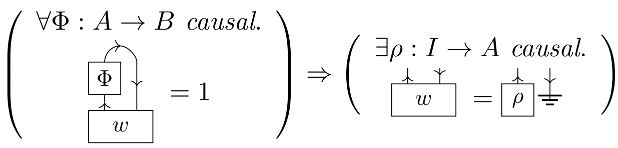
PC3. *$\mathcal{C}$ has enough causal states: $\forall f, g : A \to B. \left(\forall \rho \in \{\bar{\bar{\top}}_A\}^* . \rho \,⨟\, f = \rho \,⨟\, g\right) \Rightarrow f = g$;*

PC4. *Causal one-way signalling processes on first-order types factorise: for any causal $\Phi : A \otimes B \to C \otimes D$,*



(4)

PC5. *For all $w : I \to A \otimes B^*$:*



(5)

Note that in [19], PC4 and PC5 are rolled into a single axiom (C4), which is then proven equivalent to PC4 and PC5.

▶ **Example 4.** Mat[$\mathbb{R}^+$] is a precausal category, whose objects are natural numbers and whose morphisms $M : m \to n$ are $n \times m$ matrices. Then, $\otimes$ is given by tensor product (a.k.a. Kronecker product) of matrices and consequently the tensor unit is the natural number 1. Hence states are column vectors, discarding maps $\bar{\top}_m : m \to 1$ are given by row vectors of all 1's, and the causality condition for states $\rho \fatsemi \bar{\top} = \mathrm{id}_1$ imposes the condition that the entries of ρ sum to 1. The conditions PC1, PC2, and PC3 are easily checked, whereas PC4 and PC5 follow from the product rule for conditional probability distributions (see [19]).

▶ **Example 5.** CP is a precausal category, whose objects are algebras $\mathcal{L}(H)$ of linear operators from a finite-dimensional Hilbert space to itself, and whose morphisms are completely positive maps (CP-maps). $\otimes$ is given by tensor product and consequently the tensor unit is the 1D algebra $\mathcal{L}(\mathbb{C}) \cong \mathbb{C}$. States are CP-maps $\mathbb{C} \to \mathcal{L}(H)$, which correspond to positive semidefinite operators $\rho \in \mathcal{L}(H)$. Discarding is given by the trace, hence causal states are the trace-1 positive semidefinite operators, a.k.a. quantum (mixed) states. Again the conditions PC1, PC2, and PC3 are easily checked, whereas PC4 and PC5 follow from the essential uniqueness of purification for CP-maps (see [19]).

Given a precausal category $\mathcal{C}$, we can refine to a category Caus[$\mathcal{C}$] which equips each object with a nice set of states that should be considered "normalised" (or "causal") and restricts to the morphisms that preserve them. First, we'll define what it means for a set of states to be suitably nice.

▶ **Definition 6.** *A set $c \subseteq \mathcal{C}(I, A)$ is closed if $c = c^{**}$ and flat if either there exist invertible scalars λ, μ such that $\lambda \cdot \underline{\bot}_A \in c$ and $\mu \cdot \bar{\top}_A \in c^*$ or A is a zero system.*

▶ **Definition 7.** *Given a precausal category $\mathcal{C}$, the category* Caus[$\mathcal{C}$] *has as objects pairs $\mathbf{A} = (A, c_\mathbf{A} \subseteq \mathcal{C}(I, A))$ where $c_\mathbf{A}$ is closed and flat. A morphism $f : \mathbf{A} \to \mathbf{B}$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\forall \rho \in c_\mathbf{A}.\rho \fatsemi f \in c_\mathbf{B}$.*

Caus[$\mathcal{C}$] inherits a monoidal structure and monoidal closure from $\mathcal{C}$, from which one can show that it is $*$-autonomous [19]. This is in fact a full subcategory of a particular double gluing construction (specifically the tight orthogonality subcategory of the double glueing construction using the $\{1\}$-focussed orthogonality [16]). Aside from the flatness restriction, this is therefore a relatively well-known means of constructing models of linear logic.

First-order states	$\mathbf{A^1} :=$	$\left(A, \{\bar{\top}_A\}^*\right)$
Dual	$\mathbf{A}^* :=$	$(A^*, c^*_\mathbf{A})$
Tensor product	$\mathbf{A} \otimes \mathbf{B} :=$	$(A \otimes B, \{\rho_A \otimes \rho_B \mid \rho_A \in c_\mathbf{A}, \rho_B \in c_\mathbf{B}\}^{**})$
Par	$\mathbf{A} \parr \mathbf{B} :=$	$(\mathbf{A}^* \otimes \mathbf{B}^*)^*$
Internal hom	$\mathbf{A} \multimap \mathbf{B} :=$	$\mathbf{A}^* \parr \mathbf{B}$
Monoidal unit	$\mathbf{I} :=$	$(I, \{1\})$

The intuition between the two monoidal products is that $\mathbf{A} \otimes \mathbf{B}$ is the closure of the space of local processes and hence we can compose the $\mathbf{A}$ and $\mathbf{B}$ components however we choose, whereas $\mathbf{A} \parr \mathbf{B}$ is the space of bipartite processes that are normalised in local contexts so we generally cannot compose the components, just act locally on each side. Both share the same unit $\mathbf{I} = \mathbf{I}^*$ and there is a canonical inclusion $\mathbf{A} \otimes \mathbf{B} \hookrightarrow \mathbf{A} \parr \mathbf{B}$ making Caus[$\mathcal{C}$] into an isomix category.

From these operators, we can build objects capturing the set of processes compatible with some common causal structures. For example, $(\mathbf{A^1} \multimap \mathbf{B^1}) \parr (\mathbf{C^1} \multimap \mathbf{D^1})$ includes all causal bipartite first-order processes, $(\mathbf{A^1} \multimap \mathbf{B^1}) \otimes (\mathbf{C^1} \multimap \mathbf{D^1})$ is the subset of those that are non-signalling [19, Theorem 6.2], and $(\mathbf{A^1} \multimap \mathbf{B^1}) \multimap (\mathbf{C^1} \multimap \mathbf{D^1})$ includes all 2-combs that map causal channels $\mathbf{A^1} \multimap \mathbf{B^1}$ to causal channels $\mathbf{C^1} \multimap \mathbf{D^1}$.

3 Additive Precausal Categories

When investigating higher order causal theories, it is useful to strengthen the definition of a precausal category in a handful of ways. For the remainder of this paper, we will adapt the definition of Caus[$\mathcal{C}$] to be built from an additive precausal category $\mathcal{C}$.

▶ **Definition 8.** *Let $\mathcal{C}$ be a compact closed category with products (and hence biproducts and additive enrichment [15]). $\mathcal{C}$ is an additive precausal category if:*

APC1. *$\mathcal{C}$ has a discarding process $\bar{\bar{\top}}_A \in \mathcal{C}(A, I)$ for every system A, compatible with the monoidal and biproduct structures as below;*

$$\bar{\bar{\top}}_{A \otimes B} = \bar{\bar{\top}}_A \otimes \bar{\bar{\top}}_B \tag{6}$$

$$\bar{\bar{\top}}_I = \mathrm{id}_I \tag{7}$$

$$\bar{\bar{\top}}_{A \oplus B} = \left[\bar{\bar{\top}}_A, \bar{\bar{\top}}_B\right] \tag{8}$$

APC2. *The dimension $d_A := \underline{\underline{\bot}}_A \,;\, \bar{\bar{\top}}_A$ is invertible for all non-zero A;*

APC3. *Each object $A \in \mathrm{Ob}(\mathcal{C})$ has a finite causal basis: some $\{\rho_i\}_i \subseteq \left\{\bar{\bar{\top}}_A\right\}^*$ such that $\forall B. \forall f, g \in \mathcal{C}(A, B).(\forall i.\ \rho_i \,;\, f = \rho_i \,;\, g) \Rightarrow f = g$.*

APC4. *Addition of scalars is cancellative ($\forall x, y, z.\ x + z = y + z \Rightarrow x = y$), totally pre-ordered ($\forall x, y. \exists z.\ x + z = y \vee x = y + z$), and all non-zero scalars have a multiplicative inverse.*

APC5. *All effects have a complement with respect to discarding: for any $\pi \in \mathcal{C}(A, I)$, there exists some $\pi' \in \mathcal{C}(A, I)$ and scalar λ such that $\pi + \pi' = \lambda \cdot \bar{\bar{\top}}_A$.*

The first 3 axioms relate closely to the corresponding ones in Definition 3, whereas the last two are quite different in flavour, and are in some sense more elementary, as their proofs don't rely on any particularly deep facts about our main classical and quantum examples (see Examples 13 and 15 below).

The axioms APC1 and APC2 above are essentially identical to the corresponding axioms in Definition 3 of precausal categories, with the additional requirement that discarding be compatible with biproducts as well as tensor products.

APC3 is a strengthening of condition PC3. Rather than requiring us to check a pair of processes agree on *all* causal states to be equal, we only require agreement on some fixed finite set of states. In other other words, each system has a set of states that behaves like a basis spanning all of the others, for the purposes of distinguishing maps. In the quantum foundations literature, this is sometimes called a *fiducial set of states*.

APC4 says that the semiring of scalars $\mathcal{C}(I, I)$ behaves somewhat like the set of non-negative real numbers $\mathbb{R}^+$. While the scalars need not be a field (indeed our main example $\mathbb{R}^+$ is not), any field satisfies this axiom as well. We do however exclude categories of non-determinstic processes such as Rel, since the scalars are boolean values with addition given by the (non-cancellative) operation of disjunction.

Note that, with the help of bases (APC3), we can promote additive cancellativity of scalars to additive cancellativity for all processes.

▶ **Proposition 9.** *In an additive precausal category:*

$$\forall f, g, h \in \mathcal{C}(A, B).\ f + h = g + h \Rightarrow f = g \tag{APC5a}$$

This condition allows us to define the free *subtractive closure* Sub($\mathcal{C}$) which extends $\mathcal{C}$ with all negatives and prove that there exists a faithful embedding $[-] : \mathcal{C} \to \mathrm{Sub}(\mathcal{C})$. More details on the explicit construction of Sub($\mathcal{C}$), its properties, and the embedding functor are given in the full version of this paper [26].

The utility of freely introducing negatives is summarised in the following proposition.

▶ **Proposition 10.** *For a precausal category $\mathcal{C}$, the scalars $K := \mathrm{Sub}(\mathcal{C})(I, I)$ are a field, and hence $\mathrm{Sub}(\mathcal{C})$ is enriched over K-vector spaces.*

In particular, we can therefore treat processes in $\mathcal{C}(A, B)$ as being embedded in an ambient vector space $\mathrm{Sub}(\mathcal{C})(A, B)$. So, while we might not be able to make all linear algebraic constructions directly in $\mathcal{C}$, we can do so in $\mathrm{Sub}(\mathcal{C})$. An important example of this is the ability to extend any independent set of states to a basis of states in $\mathcal{C}$ as well as its corresponding dual basis of effects in $\mathrm{Sub}(\mathcal{C})$.

▶ **Lemma 11.** *Given any set of morphisms in $\mathcal{C}(A, B)$ that are linearly independent in $\mathrm{Sub}(\mathcal{C})$, they can be extended to a basis in $\mathcal{C}$ with a dual basis in $\mathrm{Sub}(\mathcal{C})$.*

While the dual basis in $\mathrm{Sub}(\mathcal{C})$ may not be physically meaningful (e.g. in the classical case it will contain vectors with negative probabilities), it will be a useful mathematical tool for working with the morphisms in $\mathcal{C}$.

APC5 allows us to interpret effects (up to some renormalisation) as testing some predicate. To see how this works, first assume for simplicity that $\lambda = \mathrm{id}_I$. For a type A, we can think of $\pi : A \to I$ as some predicate over A, and π' as its negation. For some causal state ρ, we can think of the composition $p_1 := \rho \,⨾\, \pi$ as the probability that π is true for ρ and $p_2 := \rho \,⨾\, \pi'$ as the probability that π is false. The fact that $\pi + \pi' = \overline{\overline{\top}}$ lets us conclude that those probabilities sum to 1:

$$p_1 + p_2 = \rho \,⨾\, \pi + \rho \,⨾\, \pi' = \rho \,⨾\, (\pi + \pi') = \rho \,⨾\, \overline{\overline{\top}} = \mathrm{id}_I$$

If $\lambda \neq \mathrm{id}_I$, the previous reasoning holds after re-normalising, i.e. replacing π and π' with $\lambda^{-1} \cdot \pi$ and $\lambda^{-1} \cdot \pi'$.

Thanks to compact closure, we can promote APC5 to a property about all processes. Namely, any process $f : A \to B$ has a complement f' where, up to re-normalisation, $f + f'$ gives the uniform noise process.

▶ **Proposition 12.** *For any $f : A \to B$ in an additive precausal category, there exists $f' : A \to B$ and a scalar λ such that:*

$$f + f' = \lambda \cdot \overline{\overline{\top}}_A \,⨾\, \underline{\underline{\bot}}_B \tag{APC5a}$$

▶ **Example 13.** $\mathrm{Mat}[\mathbb{R}^+]$ defined as in Example 4 is also an additive precausal category, where $\oplus$ is given by the direct sum of matrices. The standard basis of unit vectors gives a basis for APC3, the semiring of scalars $\mathrm{Mat}[\mathbb{R}^+](I, I) \cong \mathbb{R}^+$ satisfies APC4, and for APC5, we just need to choose a suitably large λ such that $\pi' := \lambda \cdot \overline{\overline{\top}}_A - \pi$ contains all positive numbers.

▶ **Example 14.** In addition to $\mathbb{R}^+$, we can construct an additive precausal category $\mathrm{Mat}[K]$ for any field of characteristic 0. In particular, $\mathrm{Mat}[\mathbb{R}]$ is an additive precausal category that is identical to $\mathrm{Mat}[\mathbb{R}^+]$ but without any positivity constraint, describing affine or "quasi-probabilistic" maps where negative probabilities are permitted. In this case, the subtractive closure gives an equivalent category to $\mathrm{Mat}[\mathbb{R}]$ itself.

▶ **Example 15.** The quantum example is very nearly the category CP, as defined in Example 5, but CP doesn't have biproducts. If we freely add biproducts, we obtain a category CP^* whose objects are all finite-dimensional C*-algebras (or equivalently, algebras of the form $\mathcal{L}(H_1) \oplus \ldots \oplus \mathcal{L}(H_k)$) and completely positive maps. Discarding is again given by the trace operator, so APC1 and APC2 are straightforward to verify. For APC3, we can fix a (non-orthogonal) basis of states for each type. As in the classical case, the scalars are $\mathbb{R}^+$, so APC4 is immediate and since $\bar{\bar{\top}}_A$ is an interior point in the cone of positive effects, π' can defined as $\lambda \cdot \bar{\bar{\top}}_A - \pi$ for suitably large λ.

Given an additive precausal category, we can apply the Caus construction in the same way as before. However, it may now be easier to devise interesting closed sets or interpret the impact of the closure operator since it just corresponds to taking affine combinations of states. For this to make sense, we should say precisely what we mean to take affine combinations of states in $\mathcal{C}$.

▶ **Definition 16.** *For a set of states $c \subseteq \mathcal{C}(I, A)$, we define sets $\mathsf{aff}(c) \subseteq \mathrm{Sub}(\mathcal{C})(I, A)$ and $\mathsf{aff}^+(c) \subseteq \mathcal{C}(I, A)$ as follows, for $K := \mathrm{Sub}(\mathcal{C})(I, I)$:*

$$\mathsf{aff}(c) := \left\{ \rho \,\middle|\, \exists \{\rho_i\}_i \subseteq \mathcal{C}(I, A), \{\lambda_i\}_i \subseteq K.\ \sum_i \lambda_i = \mathrm{id}_I, \rho = \sum_i \lambda_i \cdot [\rho_i] \right\}$$

$$\mathsf{aff}^+(c) := \{\rho \,|\, \exists \rho' \in \mathsf{aff}(c).\ \rho' = [\rho]\}$$

If we identify the set $\mathcal{C}(I, A)$ with its image under $[-]$, we can think if $\mathsf{aff}^+(c)$ as the intersection of the affine closure of c with the set $\mathcal{C}(I, A) \subseteq \mathrm{Sub}(\mathcal{C})(I, A)$ of "positive" states embedded in the subtractive closure. In the classical and quantum case, $\mathsf{aff}^+(-)$ arises from taking all of the affine combinations of elements of c, then intersecting the resulting set with the positive cone of (unnormalised) probability distributions or quantum states, respectively.

▶ **Theorem 17.** *Given any flat set $c \subseteq \mathcal{C}(I, A)$ for a non-zero A, $c^{**} = \mathsf{aff}^+(c)$.*

This characterises the tensor space $c_{\mathbf{A}\otimes\mathbf{B}} = \{\rho_A \otimes \rho_B | \rho_A \in c_{\mathbf{A}}, \rho_B \in c_{\mathbf{B}}\}^{**}$ in $\mathrm{Caus}[\mathcal{C}]$ as the affine closure of the separable states, from which we can prove the following property.

▶ **Theorem 18.** *If $\mathbf{A} = (A, c_{\mathbf{A}})$ with $c_{\mathbf{A}} = \{\mu \cdot \bar{\bar{\bot}}_A\}$ for any non-zero A, then every $h \in c_{\mathbf{A}\otimes\mathbf{B}}$ is a product morphism of the form $\mu \cdot \bar{\bar{\bot}}_A \otimes g$ for some $g \in c_{\mathbf{B}}$.*

This captures what [27] refers to as the principle of "no interaction with trivial degrees of freedom". In particular, it recovers the precausal category axiom PC5 by showing that every state of $(\mathbf{A^1} \multimap \mathbf{B^1})^* = (\mathbf{A^1}^* \parr \mathbf{B^1})^* = \mathbf{A^1} \otimes \mathbf{B^1}^*$ decomposes into a product of a state of $\mathbf{A^1}$ (i.e. a causal state) and $\bar{\bar{\bot}}_{B^*}$.

It should be noted that we have completely dropped condition PC4 on our underlying category of basic processes. In Section 5 we will recover a slightly weaker version of this condition by the equivalence of one-way signalling and the affine closure of semi-localisability (Theorem 30). Fortunately, we can still reuse the same proof to show that $\mathrm{Caus}[\mathcal{C}]$ is $*$-autonomous for additive precausal $\mathcal{C}$ since it doesn't rely on PC4.

4 Additive Types

We may also lean on the relation to the double glueing construction to add type constructors corresponding to the additives of linear logic. In categorical models of linear logic, additive conjunction of types is captured by cartesian product and additive disjunction by coproduct, satisfying a De Morgan duality with one another [20]. In terms of resources, these represent a classical choice: an instance of $A \times B$ is a single resource unit that we can choose to be used either as an instance of A or an instance of B, whereas an instance of $A + B$ is a single resource unit which is fixed on creation as either a unit of A or a unit of B.

The concept of classical choice is often built into operational theories in the form of probability distributions, classical outcomes of tests, and conditional tests. This is typically deemed essential for any experimentalist who is bound to classical data to interact with and make inference from an experiment. On the other hand, the Caus construction concerns the underlying systems present in the physical theory without consideration of any classical agent. We can recover finite-outcome random variables by incorporating binary classical choice through new additive type constructors, i.e. finding constructions for products and coproducts in $\mathrm{Caus}[\mathcal{C}]$.

▶ **Definition 19.** *Given types* $\mathbf{A} = (A, c_{\mathbf{A}}), \mathbf{B} = (B, c_{\mathbf{B}})$ *in* $\mathrm{Caus}[\mathcal{C}]$*, we define* $\mathbf{A} \times \mathbf{B} := (A \oplus B, c_{\mathbf{A}\times\mathbf{B}})$ *and* $\mathbf{A} \oplus \mathbf{B} := (A \oplus B, c_{\mathbf{A}\oplus\mathbf{B}})$ *where:*

$$\begin{aligned} c_{\mathbf{A}\times\mathbf{B}} &= (\{p_A \,;\, \pi_A | \pi_A \in c^*_{\mathbf{A}} \subseteq \mathcal{C}(A, I)\} \cup \{p_B \,;\, \pi_B | \pi_B \in c^*_{\mathbf{B}} \subseteq \mathcal{C}(B, I)\})^* \\ &= \{\langle \rho_A, \rho_B \rangle | \rho_A \in c_{\mathbf{A}}, \rho_B \in c_{\mathbf{B}}\} \end{aligned} \tag{9}$$

$$\begin{aligned} c_{\mathbf{A}\oplus\mathbf{B}} &= (\{\rho_A \,;\, \iota_A | \rho_A \in c_{\mathbf{A}}\} \cup \{\rho_B \,;\, \iota_B | \rho_B \in c_{\mathbf{B}}\})^{**} \\ &= \{[\pi_A, \pi_B] \,| \pi_A \in c^*_{\mathbf{A}} \subseteq \mathcal{C}(A, I), \pi_B \in c^*_{\mathbf{B}} \subseteq \mathcal{C}(B, I)\}^* \end{aligned} \tag{10}$$

▶ **Lemma 20.** *The alternative definitions of* $c_{\mathbf{A}\times\mathbf{B}}$ *and* $c_{\mathbf{A}\oplus\mathbf{B}}$ *are equivalent; that is, Equations 9 and 10 hold.*

▶ **Corollary 21.** *The operators* $\times$ *and* $\oplus$ *are De Morgan duals under* $(-)^*$*.*

▶ **Proposition 22.** $\mathbf{A} \times \mathbf{B}$ *is a categorical product in* $\mathrm{Caus}[\mathcal{C}]$*.*

▶ **Proposition 23.** $\mathbf{A} \oplus \mathbf{B}$ *is a categorical coproduct in* $\mathrm{Caus}[\mathcal{C}]$*.*

The zero object 0 has a unique state $0_{I,0} \in \mathcal{C}(I, 0)$ by terminality and a unique effect $0_{0,I} \in \mathcal{C}(0, I)$ by initiality. There are only two candidates for causal sets: the empty set $\emptyset$ and the singleton $\{0_{I,0}\}$. These have roles in our causal category as the additive units.

▶ **Proposition 24.** *The initial object in* $\mathrm{Caus}[\mathcal{C}]$ *is* $\mathbf{0} := (0, \emptyset)$ *and the terminal object is* $\mathbf{1} := (0, \{0_{I,0}\})$*. Furthermore, they are duals of each other and are units for* $\oplus$ *and* $\times$ *respectively.*

▶ Remark 25. The product and coproduct constructions are only partially-defined as they may not always yield flat sets of states when incorporating the initial or terminal. Specifically, given any $\mathbf{A}$ on a non-zero system, we have

$$\begin{array}{llllll} c_{\mathbf{A}\times\mathbf{0}} & = & \emptyset & c^*_{\mathbf{A}\times\mathbf{0}} & \simeq & \mathcal{C}(A, I) \\ c_{\mathbf{A}\oplus\mathbf{1}} & \simeq & \mathcal{C}(I, A) & c^*_{\mathbf{A}\oplus\mathbf{1}} & = & \emptyset \end{array} \tag{11}$$

though it may be possible that a careful weakening of the flatness condition may permit this more generally without sacrificing some of the other results of this paper.

Thinking of first-order types as describing systems with no input (i.e. no choice in how to consume them), both the product and coproduct have interesting interactions with first-order types because of where the classical choice happens. For coproducts, the choice is already fixed in the creation of a state so we expect it to preserve the first-order property. However, products introduce freedom of choice in effects allowing us to view the projectors as inputs to the system dictating whether it should prepare the left or the right state.

▶ **Proposition 26.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are both first-order types, then so is* $\mathbf{A} \oplus \mathbf{B}$*.*

▶ **Proposition 27.** *If* A *and* B *are non-zero, then* $\mathbf{A} \times \mathbf{B}$ *is never a first-order type.*

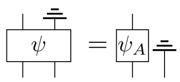
We have both $*$-autonomy and all finite products and coproducts, which is all the evidence required to show that Caus[$\mathcal{C}$] is a model of linear logic with additives. For both the multiplicative and additive structures, we found that the Caus construction took a degenerative categorical structure and generated non-degenerate structures from them. For example, the compact closure of $\mathcal{C}$ means that it is a $*$-autonomous category in which the two monoidal products are the same, but Caus[$\mathcal{C}$] is $*$-autonomous with distinct $\otimes$ and $\parr$. Similarly, the construction for the additives takes biproducts/the zero object and yields distinct products and coproducts/initial and terminal objects. The degenerate exception here is that $\mathbf{I}$ is still the unit for both $\otimes$ and $\parr$.

5 One-Way Signalling Types

When examining multi-partite systems, causal structures can be investigated from the perspective of information signalling (which parties can observe changes to another party's inputs) or decompositions (does the channel admit a decomposition into local channels compatible with some configuration of time- and space-like separations between the parties). In the bipartite case, we can compare these perspectives with the examples of one-way signalling and semi-localisable channels.

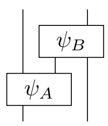
The one-way signalling causal structure refers to a bipartite system (say, between Alice and Bob) in which causal influence may only exist in one direction. There is a standard definition given in causality literature for one-way signalling for quantum channels.

▶ **Definition 28.** *A bipartite process is one-way signalling (Alice to Bob) if discarding Bob's output admits a decomposition into local processes.*



Semi-localisability is an alternative to one-way signalling for expressing compatibility with a time-like separation of parties, giving a constructive example of how the combined channel can be decomposed into local operations.

▶ **Definition 29.** *A bipartite channel between Alice and Bob is semi-localisable (with Alice before Bob) if it decomposes into local processes with a channel from Alice to Bob.*

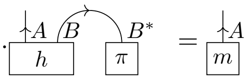


Both of these properties state that the channel is compatible with the setting where Bob is in Alice's future light cone. It is important to note that both are judgements of compatibility with a causal structure rather than inference of any necessary causal relationship between Alice and Bob - completely local processes trivially satisfy these properties but obviously have no causal influence between the parties.

These definitions are specific to first-order channels, but we can consider situations where Alice and Bob have higher-order systems $\mathbf{A}, \mathbf{B} \in \text{Ob}(\text{Caus}[\mathcal{C}])$ representing some more complex interaction with their environments. We will describe these properties as constructions on the causal sets.

The essence of one-way signalling is that Alice's local effect is independent of any input given to Bob by his context:

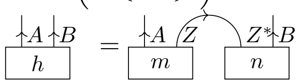
$$c_{\mathbf{A}} < c_{\mathbf{B}} := \left\{ h \in c_{\mathbf{A} \parr \mathbf{B}} \,\middle|\, \exists m \in c_{\mathbf{A}}.\forall \pi \in c_{\mathbf{B}^*}. \right\} \tag{12}$$



When this property holds, we refer to m as the *(left-)residual* of h (with *right-residual* for the corresponding component in the symmetrically-defined $c_{\mathbf{A}} > c_{\mathbf{B}}$). The residual represents the constant local effect Alice observes regardless of inputs provided to Bob. In the case where $\mathbf{A}$ and $\mathbf{B}$ both describe first-order channels, this exactly reduces to Definition 28, since the only causal contexts for a causal channel is to provide an arbitrary state at the input and discard the output.

For semi-localisability, it is not enough to consider a factorisation by any system, but specifically by a first-order system:

$$c_{\mathbf{A}} \lhd c_{\mathbf{B}} := \left\{ h \in c_{\mathbf{A} \parr \mathbf{B}} \,\middle|\, \begin{array}{l} \exists \mathbf{Z} = \left(Z, \left\{\bar{\top}_Z\right\}^*\right), m \in c_{\mathbf{A} \parr \mathbf{Z}}, n \in c_{\mathbf{Z}^* \parr \mathbf{B}}. \end{array} \right\} \tag{13}$$



This is because using a higher-order system (e.g. the dual of a first-order system) may allow information to flow in the opposite direction. Again, we recover the original definition of semi-localisability if we fix $\mathbf{A}$ and $\mathbf{B}$ to types of first-order channels (up to our encoding of channels via the Choi-Jamiołkowski isomorphism).

Intuitively, based on the interpretation of the sequence operator in a BV-category as representing an ordered combination of systems, both of these would be good candidates for sequence operator in Caus[$\mathcal{C}$]. It is also interesting to compare them to sequence operators from other BV-categories. The following construction is inspired by that of the BV-category of probabilistic coherence spaces [3]:

$$c_{\mathbf{A}} \oslash c_{\mathbf{B}} := \left\{ h \in c_{\mathbf{A} \parr \mathbf{B}} \,\middle|\, \begin{array}{l} \exists \mathcal{I}, \{f_i\}_{i \in \mathcal{I}} \subseteq \mathrm{Sub}(\mathcal{C})(I, A \otimes B), \{g_i\}_{i \in \mathcal{I}} \subseteq c_{\mathbf{B}}, f \in c_{\mathbf{A}}. \\ [h] \sim \sum_{i \in \mathcal{I}} f_i \otimes [g_i] \wedge [f] \sim \sum_{i \in \mathcal{I}} f_i \end{array} \right\} \tag{14}$$

Note this definition refers to maps in the subtractive closure Sub($\mathcal{C}$) to make use of the affine-linear structure.

A core result in the field of causal structures is the equivalence of one-way signalling and semi-localisability for first-order channels [8]. In the setting provided by the Caus construction, we find a higher-order generalisation of this, where all three of these definitions coincide up to affine closure, as well as a proof that they are self-dual properties.

▶ **Theorem 30.** $c_{\mathbf{A}} < c_{\mathbf{B}} = (c^*_{\mathbf{A}} < c^*_{\mathbf{B}})^* = (c_{\mathbf{A}} \lhd c_{\mathbf{B}})^{**} = c_{\mathbf{A}} \oslash c_{\mathbf{B}}$

$c_{\mathbf{A}} < c_{\mathbf{B}}$ is closed, since it the dual of another set, and flat, since $c_{\mathbf{A} \otimes \mathbf{B}} \subseteq c_{\mathbf{A}} < c_{\mathbf{B}}$ gives the uniform state and $c^*_{\mathbf{A} \parr \mathbf{B}} \subseteq (c_{\mathbf{A}} < c_{\mathbf{B}})^*$ gives discarding. We can therefore elevate it to a genuine object $\mathbf{A} < \mathbf{B} := (A \otimes B, c_{\mathbf{A}} < c_{\mathbf{B}})$ in Caus[$\mathcal{C}$].

We can go further into the categorical structure induced by this type constructor and show that it adds another monoidal structure with a weak interchange with both $\otimes$ and $\parr$, turning Caus[$\mathcal{C}$] into a model of BV-logic.

▶ **Definition 31.** *Given a symmetric, linearly distributive category $\mathcal{D}$, a weak interchange is an additional monoidal structure $(\mathcal{D}, \oslash, I_\oslash)$ with natural transformations*

$$\begin{array}{llll} w_\otimes & : (R \oslash U) \otimes (T \oslash V) \to (R \otimes T) \oslash (U \otimes V) & w_{I_\otimes} & : I_\otimes \to I_\otimes \oslash I_\otimes \\ w_\parr & : (C \parr E) \oslash (D \parr F) \to (C \oslash D) \parr (E \oslash F) & w_{I_\parr} & : I_\parr \oslash I_\parr \to I_\parr \end{array}$$

which are compatible with the structure isomorphisms of $(\mathcal{D}, \otimes, I_\otimes)$ *and* $(\mathcal{D}, \parr, I_\parr)$ *and the distributive structure (we refer the reader to Blute, Panangaden, and Slavnov [3, Definition 4.1] for the full set of conditions).*

A BV*-category is a symmetric linearly distributive category with a weak interchange and an isomorphism* $m : I_\oslash \to I_\otimes$ *such that* m *is an isomix map and* m^{-1} *is a counit for* $w_{I_\otimes}$*:*

$$(\mathrm{id}_{I_\oslash} \otimes m) \mathbin{⨟} \rho^\otimes_{I_\oslash} = (m \otimes \mathrm{id}_{I_\oslash}) \mathbin{⨟} \lambda^\otimes_{I_\oslash} : I_\oslash \otimes I_\oslash \to I_\oslash$$
$$w_{I_\otimes} \mathbin{⨟} (m^{-1} \oslash \mathrm{id}_{I_\otimes}) \mathbin{⨟} \lambda^\oslash_{I_\otimes} = \mathrm{id}_{I_\otimes} = w_{I_\otimes} \mathbin{⨟} (\mathrm{id}_{I_\otimes} \oslash m^{-1}) \mathbin{⨟} \rho^\oslash_{I_\otimes} : I_\otimes \to I_\otimes$$

▶ **Theorem 32.** $\mathrm{Caus}[\mathcal{C}]$ *is a* BV*-category.*

The equivalence of one-way signalling and (the affine closure of) semi-localisability show that any directed information signalling will always exhibit an equivalent construction where the information transfer is mediated by first-order types. Moreover, first-order types (or scaled versions thereof) are *exactly* those that can carry information in one direction.

▶ **Theorem 33.** $\mathbf{A}^* \parr \mathbf{A} = \mathbf{A}^* < \mathbf{A} \Leftrightarrow |c^*_{\mathbf{A}}| = 1$.

This result presents a characterisation of first-order system types that can be interpreted in any BV-category, which may be a useful lemma in characterising which categories arise as $\mathrm{Caus}[\mathcal{C}]$ for some additive precausal $\mathcal{C}$.

6 Non-Signalling Systems

The one-way signalling structure is not the only interesting causal structure on a bipartite system. For example, the non-signalling causal structure is a symmetric property about the statistical independence of the different parties.

▶ **Definition 34.** *A bipartite process is non-signalling if it satisfies the one-way signalling condition in both directions.*

This is a necessary condition for compatibility with the setting where Alice and Bob are space-like separated. However, this is not a sufficient condition since there exist non-signalling processes, such as a quantum implementation of a PR box, that cannot be factorised into local processes with a shared history [23].

In [19], the tensor product of first-order channel types $(\mathbf{A^1} \multimap \mathbf{B^1}) \otimes (\mathbf{C^1} \multimap \mathbf{D^1})$ was shown to exactly contain those bipartite channels that are non-signalling [19]. This proof relied on some properties that do not generalise beyond first-order channels, such as the ability to apply PC5 to decompose causal contexts for channels into a causal input state and discarding the output. Between Gutoski [12] and Chiribella et al. [9], it was shown that the space of non-signalling channels in quantum theory can be characterised as the affine closure of product channels. By translating this proof into categorical terms, we can generalise the result to hold for arbitrary higher-order systems in $\mathrm{Caus}[\mathcal{C}]$ for any additive precausal $\mathcal{C}$.

▶ **Theorem 35.** $(c_{\mathbf{A}} < c_{\mathbf{B}}) \cap (c_{\mathbf{A}} > c_{\mathbf{B}}) = c_{\mathbf{A} \otimes \mathbf{B}}$.

7 Conclusion

By extending the assumptions on the base category with additive structure, the Caus construction yields a BV-category with additives within which the characterisation of first-order types resembles the causality condition expressed as an equation of types. We also obtain general characterisations of $(-)^{**}$ as affine closure and the tensor product as the space of non-signalling bipartite processes. A number of the proofs are similar to those used to show that probabilistic coherence spaces form a BV-category [3], and the characterisation of non-signalling processes as affine closure of local processes for first-order channels in quantum theory [12]. As BV-logic is known to prove a strict subset of the theorems of pomset logic [21], it remains for future work to investigate whether the Caus construction yields models of the latter or if it is limited to BV-logic.

We recover almost all of the precausal conditions from an additive precausal category. Instead of the equivalence of one-way signalling and semi-localisability for first-order channels (Condition PC4), we obtained equivalence with the affine closure of semi-localisability for arbitrary higher-order systems (Theorem 30). Since existing proofs of PC4 for precausal categories rely on the essential uniqueness of purification, it may be possible to strengthen this result in future work by incorporating notions of purity and purification into this framework.

There are still other interesting process theories that cannot be considered as either the base category or the result of the Caus construction in which it would be interesting to analyse constructions for one-way signalling and their logical role. For example, settings with infinite-dimensional systems are rarely compact closed, real quantum mechanics doesn't have enough causal states (since it is compact closed but doesn't admit local discrimination), and Rel fails Condition PC5. We needn't expect the results of this paper to generalise to other theories since (non-deterministic) coherence spaces form a BV-category using a similar construction to $\mathbf{A} \oslash \mathbf{B}$ for the non-commutative operator [3], but this is distinct from the naive adaptation of $\mathbf{A} < \mathbf{B}$ (instead of a constant residual, every local effect must be a subset of some constant clique) which is not self-dual in this category.

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