

Combinatorial and Algorithmic Aspects of Monadic Stability

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Abstract

Nowhere dense classes of graphs are classes of sparse graphs with rich structural and algorithmic properties, however, they fail to capture even simple classes of dense graphs. Monadically stable classes, originating from model theory, generalize nowhere dense classes and close them under transductions, i.e. transformations defined by colorings and simple first-order interpretations. In this work we aim to extend some combinatorial and algorithmic properties of nowhere dense classes to monadically stable classes of finite graphs. We prove the following results.

- For every monadically stable class \mathcal{C} and fixed integer $s \geq 3$, the Ramsey numbers $R_{\mathcal{C}}(s, t)$ are bounded from above by $\mathcal{O}(t^{s-1-\delta})$ for some $\delta > 0$, improving the bound $R(s, t) \in \mathcal{O}(t^{s-1}/(\log t)^{s-1})$ known for the class of all graphs and the bounds known for k -stable graphs when $s \leq k$.
- For every monadically stable class \mathcal{C} and every integer r , there exists $\delta > 0$ such that every graph $G \in \mathcal{C}$ that contains an r -subdivision of the biclique $K_{t,t}$ as a subgraph also contains K_{t^δ, t^δ} as a subgraph. This generalizes earlier results for nowhere dense graph classes.
- We obtain a stronger regularity lemma for monadically stable classes of graphs.
- Finally, we show that we can compute polynomial kernels for the independent set and dominating set problems in powers of nowhere dense classes. Formerly, only fixed-parameter tractable algorithms were known for these problems on powers of nowhere dense classes.

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1 Introduction

The notion of nowhere density was introduced by Nešetřil and Ossona de Mendez [22] and provides a very robust notion of uniform sparsity of graphs. These classes can be characterized in various ways, leading to a rich combinatorial theory and enabling the design



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of efficient algorithms for problems that are hard in general. For example, the (distance- r) dominating set problem [6] and more generally, the first-order model-checking problem [16], is fixed-parameter tractable on nowhere dense graph classes.

Unfortunately, the techniques developed for nowhere dense classes are not applicable on dense graphs. However, there has been recent success of defining notions of *structural sparsity* by considering first-order transductions of sparse classes. Intuitively, a first-order transduction transforms graphs by first non-deterministically adding colors and then reinterpreting the edge relation using a first-order formula $\varphi(x, y)$, see e.g. [9, 14, 15, 21, 25]. This has led to the notion of *structurally nowhere dense* classes, which are first-order transductions of nowhere dense classes. If \mathcal{C} is a nowhere dense class, then for example the class of all complements of graphs from \mathcal{C} or for an integer d the class of all d th powers of graphs from \mathcal{C} forms a structurally nowhere dense class.

It has been conjectured that structurally nowhere dense classes coincide with *monadically stable* classes (see e.g. [24]), which have long been studied in model theory [4]. This conjecture is based on the results of Adler and Adler [1], showing that every nowhere dense class is monadically stable and in fact for monotone classes (i.e. classes closed under taking subgraphs) the notions of nowhere density and monadic stability coincide. However, monadically stable classes are vastly more general than nowhere dense classes. Further indications for the truth of the conjecture is given by a recent characterization of monadically stable classes in terms of *flip-wideness* [10], which generalizes the notion of uniform quasi-wideness, which characterizes nowhere dense classes.

So far, the main tool for the study of monadically stable classes has been *indiscernible sequences* [10]. In our restricted context we focus on indiscernible sequences of vertices, that is, sequences $(v_i)_{1 \leq i \leq n}$ of vertices such that all (increasing tuples of) vertices from the sequence satisfy the same first-order formulas. For example, an indiscernible sequence of vertices in a graph will induce an independent set or a clique. Furthermore, indiscernible sequences impose strong structural properties on their neighborhoods, which we exploit in our combinatorial and algorithmic study of monadically stable classes.

In the remainder of this section we describe our contribution in detail.

Ramsey numbers in monadically stable classes of graphs. Ramsey's theorem is a foundational result in combinatorics, stating in one of its graph theoretic formulations that in every sufficiently large graph one either finds a large clique or a large independent set. For a graph class \mathcal{C} , the Ramsey number $R_{\mathcal{C}}(s, t)$ denotes the smallest number of vertices that a graph from \mathcal{C} must have such that one can either find a clique of size s or an independent set of size t . If \mathcal{C} is the class of all graphs, we omit the subscript. Focusing on the case that s is fixed and t is growing, folklore bounds give us $R(s, t) \leq \binom{s+t-2}{s-1} \in \mathcal{O}_s(t^{s-1})$ (the subscript s in the asymptotic notation indicates that constant factors depending on s are hidden). For $s = 1, 2$ this bound is trivial and tight. The classical result of Ajtai, Komlós and Szemerédi [2] improves this for $s \geq 3$ to $R(s, t) \in \mathcal{O}_s(t^{s-1})/(\log t)^{s-2}$. We can rephrase this by saying that every K_s -free n -vertex graph with $s \geq 3$ contains an independent set of order $\Omega_s(n^{1/(s-1)} \cdot (\log n)^{(s-2)/(s-1)})$. For every stable class \mathcal{C} there exists an integer k (the ladder index, see Section 2 for formal definitions) such that every K_s -free n -vertex graph $G \in \mathcal{C}$ with $s \geq k - 1$ contains an independent set of order $\Omega((n/k)^{1/(k+1)})$, strongly improving the general bound by removing the dependence on s from the exponent. These bounds are implicit in [20] and made explicit in [27, Observation 6.12]. However, on stable graph classes for $s < k - 1$ the classical bounds of Ajtai et al. are the currently best known. Things look better on structurally nowhere dense classes: For every such class \mathcal{C} , $\epsilon > 0$ and

integer s , it holds that every K_s -free n -vertex graph from \mathcal{C} contains an independent set of size $\Omega_{s,\epsilon,\mathcal{C}}(n^{1-\epsilon})$. This follows from the fact that graphs from structurally nowhere dense classes can be partitioned into $\mathcal{O}_{\mathcal{C},\epsilon,s}(n^\epsilon)$ parts, each of which induces a cograph [9], and we have $R_{\text{co-graphs}}(s, t) = st$. (To be more precise, it is proved in [9] that each of the parts has bounded shrubdepth and it is proved in [26] that every graph of bounded shrubdepth can be decomposed into a bounded number of cographs.)

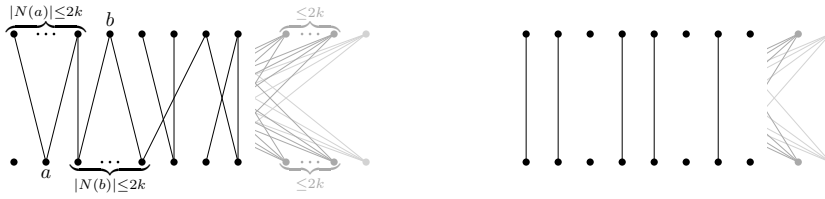
In light of the conjecture that every monadically stable class is a transduction of a nowhere dense class, we conjecture that also for every monadically stable class \mathcal{C} , $\epsilon > 0$ and integer s , it holds that every K_s -free n -vertex graph from \mathcal{C} contains an independent set of size $\Omega_{\mathcal{C},\epsilon,s}(n^{1-\epsilon})$. We were unable to prove this conjecture, however, we were able to improve the bounds for monadically stable classes with ladder index k and $s < k - 1$ by proving the following. If \mathcal{C} is a K_s -free monadically stable class for some fixed $s \geq 3$, then there exists a real $\delta > 0$ such that in every n -vertex graph from \mathcal{C} we can find an independent set of size $\Omega_{\mathcal{C}}(n^{1/(s-1)+\delta})$. This implies that we have $R_{\mathcal{C}}(s, t) \in \mathcal{O}_{\mathcal{C}}(t^{s-1-\delta})$ (Remark 3.2 below). To put this into perspective, Ajtai et al.'s classical result improves the size of an independent set in K_s -free graphs by a factor $\Theta((\log n)^{(s-2)/(s-1)})$ compared to the folklore bounds of $\Omega(n^{1/(s-1)})$ and our result improves it by a factor $\Theta(n^\delta)$ for monadically stable graph classes.

Large bicliques in monadically stable classes of graphs. An r -subdivision of a graph H is a graph obtained by replacing the edges of H by paths of length (exactly) $r + 1$ (containing r inner vertices). A class \mathcal{C} of graphs is nowhere dense if and only if there exists a function f such that for every $r \geq 0$ the r -subdivision of a biclique (complete bipartite graph) $K_{f(r),f(r)}$ with parts of size $f(r)$ is excluded as a subgraph [23]. Hence, every class that is not nowhere dense must contain for some value of r arbitrarily large r -subdivided bicliques. It follows from [11] that every monadically stable class that is not nowhere dense even contains arbitrarily large bicliques as subgraphs. We demonstrate that these two quantities (size of r -subdivided bicliques and size of bicliques) are polynomially bounded in monadically stable classes, that is, we prove that for every monadically stable class \mathcal{C} and every $r > 0$, there exist an integer m and a real $\delta > 0$, such that for every $t \geq m$ it holds that every graph $G \in \mathcal{C}$ that contains an r -subdivision of $K_{t,t}$ as a subgraph, also contains K_{t^δ,t^δ} as a subgraph.

Regularity lemma for monadically stable classes of graphs. Szemerédi's regularity lemma [30] is a celebrated result in extremal graph theory. Let us give the required formal definitions to formulate the lemma. Let G be a graph and let $A, B \subseteq V(G)$ be two disjoint non-empty subsets of vertices. We write $E(A, B)$ for the set of edges with one end in A and the other end in B . We define the *density* of the pair (A, B) as $\text{dens}(A, B) := |E(A, B)|/|A||B|$. For a real $\epsilon > 0$, we call the pair (A, B) ϵ -regular if, for all subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$, we have $|\text{dens}(A', B') - \text{dens}(A, B)| \leq \epsilon$. A partition V_1, \dots, V_k of a set into disjoint parts is called an *equipartition* if $||V_i| - |V_j|| \leq 1$ for $1 \leq i < j \leq k$.

Szemerédi's Regularity Lemma [30] states that for every real $\epsilon > 0$ and integer $m \geq 1$ there exist two integers M and n_0 with the following property. For every graph G with $n \geq n_0$ vertices there exists an equipartition of the vertex set into k parts V_1, \dots, V_k , $m \leq k \leq M$, such that all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ -regular.

Irregular pairs cannot be avoided as witnessed by the example of ladders, see [3]. A *ladder* of order k is the bipartite graph H_k on vertices a_1, \dots, a_k and b_1, \dots, b_k with edges $\{a_i, b_j\}$ for $i \leq j$. For disjoint subsets X, Y of vertices of a graph G , we write $G[X, Y]$ for the subgraph of G semi-induced by X and Y , that is, the subgraph with vertex set $X \cup Y$



■ **Figure 1** A possible connection between two parts for Malliaris and Shelah’s regularity lemma (left) and our regularity lemma for monadically stable classes (right). In this example in both cases, two parts A and B form an independent set and are placed at the top and bottom, the local exceptional vertices ($2k$ or one) of each part are placed on the right in dark gray, and the additional single global exceptional vertex of each part is placed even further right in light gray. The defining statements are applied once from A to B and once from B to A .

and all the edges of G with one endpoint in X and one endpoint in Y . A bipartite graph H is a semi-induced subgraph of G if H is isomorphic to $G[X, Y]$ for some disjoint subsets X and Y of $V(G)$.

Malliaris and Shelah [20] proved that semi-induced ladders are the only reason for the exceptional pairs and proved a stronger regularity lemma (see [18] for an in-depth discussion of regularity lemmas for restricted graph classes). By giving up the requirement of partitioning into a constant number of parts, they proved that for every class \mathcal{C} of graphs that exclude a ladder of order k as semi-induced subgraph there exist an integer n_0 and a real δ , such that for every graph $G \in \mathcal{C}$ of size $n \geq n_0$ there exists an equipartition into disjoint parts of size $\Omega(n^\delta)$ such that after possibly omitting one vertex from each part the following holds.

1. Every part forms a clique or an independent set.
2. For every two parts A and B , either for all vertices $a \in A$, except for at most $2k$, it holds that $|N(a) \cap B| \leq 2k$, or for all $a \in A$, except for at most $2k$, it holds that $|N(a) \cap B| \geq |B| - 2k$.

For every monadically stable class of graphs \mathcal{C} we strengthen this result by replacing the at most $2k$ exceptional vertices by single exceptional vertices. More precisely, we prove that there exist an integer n_0 and a real δ , such that for every graph $G \in \mathcal{C}$ of size $n \geq n_0$ there exists an equipartition into disjoint parts of size $\Omega(n^\delta)$ such that after possibly omitting one vertex from each part the following holds (see Figure 1 for an illustration of both regularity lemmas).

1. Every part forms a clique or an independent set.
2. For every two parts A and B , either for all vertices $a \in A$, except for at most one, it holds that $|N(a) \cap B| \leq 1$, or for all $a \in A$, except for at most one, it holds that $|N(a) \cap B| \geq |B| - 1$. This means, with the exception of one vertex per part, the connection between A and B semi-induces either a subgraph of a matching or a supergraph of a co-matching.

Polynomial kernels for independent and dominating set in powers of nowhere dense graphs. Finally, we study algorithmic applications of structurally nowhere dense classes. A parameterized graph problem is called fixed-parameter tractable with respect to a parameter k if on input G and k it can be solved in time $f(k) \cdot |V(G)|^c$ for some function f and constant c . The independent set and dominating set problem are important algorithmic problems, which are unlikely to be fixed-parameter tractable in general graphs [8]. A kernel is a polynomial time algorithm that reduces an instance (G, k) to an equivalent instance (H, k') such that $|H| + |k'|$ is bounded by a function $g(k)$ only. In case g is a polynomial function, one

speaks of a polynomial kernel. For an integer d and a graph G , the d th power of G is the graph G^d on the same vertex set as G , where two (distinct) vertices are connected if their distance in G is at most d . A class \mathcal{D} is a power of another class \mathcal{C} if there exists d such that $\mathcal{D} = \{G^d : G \in \mathcal{C}\}$. It was recently shown that the independent set problem and dominating set problem are fixed-parameter tractable on powers of nowhere dense graphs [13], however no polynomial kernels were known. We complete the picture by proving that both problems admit polynomial kernels on powers of nowhere dense classes of graphs.

Note that the independent/dominating set problem on the d th power G^d of a graph G naturally corresponds to the following distance- d version of the problem on G . A *distance- d independent set* in a graph G is a set of vertices of pairwise distance greater than d , while a *distance- d dominating set* is a set of vertices such that every vertex of G has distance at most d to a vertex of the dominating set. The distance- d independent set problem and distance- d dominating set problem are fixed-parameter tractable on every nowhere dense class of graphs [6], and in fact admit polynomial kernels [12, 19]. However, note that our result is not implied by these results, as the difficulty in dealing with powers of nowhere dense classes stems from the fact that it is hard to compute for a given graph H a graph G such that $H = G^d$.

2 Preliminaries

We use standard notation from graph theory and model theory and refer to [7] and [17] for extensive background. We write $[m]$ for the set of integers $\{1, \dots, m\}$.

Graphs. Given a graph G , we denote its vertex set by $V(G)$ and its edge set by $E(G)$. For a vertex $u \in V(G)$ we define $N(u) := \{v \mid \{u, v\} \in E(G)\}$ to be the *open neighborhood* of u . A *partition* of $V(G)$ is a collection of pairwise disjoint, non-empty subsets whose union is $V(G)$. An *r -subdivision* of G is a graph obtained by subdividing each edge of G exactly r times. We call a bipartite graph H a *semi-induced subgraph* of G if H can be obtained from G by choosing two disjoint subsets A, B of $V(G)$, taking the subgraph of G induced by $A \cup B$ and removing all edges with both endpoints in A , respectively B . Let H be a bipartite graph with vertices a_1, \dots, a_n and b_1, \dots, b_n . Then H is

- a *biclique* of order n if we have $\{a_i, b_j\} \in E(H)$ for all $i, j \in [n]$,
- a *matching* of order n if we have $\{a_i, b_j\} \in E(H) \iff i = j$ for all $i, j \in [n]$,
- a *co-matching* of order n if we have $\{a_i, b_j\} \in E(H) \iff i \neq j$ for all $i, j \in [n]$, and
- a *ladder* of order n if $\{a_i, b_j\} \in E(H) \iff i \leq j$ for all $i, j \in [n]$.

The *ladder index* of a graph G is the order of the longest ladder, which is contained as a semi-induced subgraph in G .

Monadic stability. A *transduction* (on graphs) is an operation mapping an input graph to a set of output graphs. A transduction is defined by an integer k and two first-order formulas $\varphi(x, y), \nu(x)$ over the language of vertex colored graphs, where φ is a symmetric and irreflexive formula (that is $\models \varphi(x, y) \leftrightarrow \varphi(y, x)$ and $\models \neg \varphi(x, x)$) and is the composition of the following operations. First, the input graph is mapped to k disjoint copies. Edges between the copies are created by pairwise connecting copies of the same vertex. The resulting graph is mapped to the set of all of its possible vertex colorings. At last, every colored graph G^+ is mapped to an uncolored graph H whose vertex set $\{v \in V(G^+) \mid G^+ \models \nu(v)\}$ is defined by ν and whose edge set $\{\{u, v\} \mid G^+ \models \varphi(u, v), u, v \in V(H)\}$ is defined by φ . Note that since φ is symmetric and irreflexive, it defines the edge set of an undirected graph.

This definition lifts to classes of graphs where we say a class \mathcal{C} *transduces* a class \mathcal{D} if there exists a fixed transduction T which produces from \mathcal{C} all graphs from \mathcal{D} , i.e. $\mathcal{D} \subseteq \bigcup_{G \in \mathcal{C}} \mathsf{T}(G)$. We call a class *monadically stable* if it does not transduce the class of all ladders. Generalizing this notion, we call a class *monadically NIP* if it does not transduce the class of all graphs.

Transductions are transitive, i.e., for classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ we have that if \mathcal{C}_1 transduces \mathcal{C}_2 and \mathcal{C}_2 transduces \mathcal{C}_3 , then also \mathcal{C}_1 transduces \mathcal{C}_3 . Combining this observation with the fact that there exists a simple transduction yielding all induced subgraphs of an input graph, we will from now on assume without loss of generality that every monadically stable class is *hereditary*, i.e., closed under taking induced subgraphs.

Indiscernible sequences. Let G be a graph and $\varphi(x_1, \dots, x_m)$ a formula. A sequence $(v_i)_{1 \leq i \leq n}$ of vertices from G is a φ -*indiscernible sequence* of length n , if for every two sequences of indices $i_1 < \dots < i_m$ and $j_1 < \dots < j_m$ from $[n]$ we have $G \models \varphi(v_{i_1}, \dots, v_{i_m})$ if and only if $G \models \varphi(v_{j_1}, \dots, v_{j_m})$. For a set of formulas Δ we call a sequence Δ -*indiscernible* if it is φ -indiscernible for all $\varphi \in \Delta$. In classical model theory, indiscernible sequences are usually defined over arbitrary structures, where they have infinite length and may consist of tuples of elements. However for our purpose we can restrict ourselves to finite sequences of vertices in graphs.

Indiscernible sequences have recently gained attention also in finite model theory and graph theory when Malliaris and Shelah showed that in *stable* structures we can find polynomially large indiscernible sequences [20, Theorem 3.5]. The extraction process was made algorithmic in [19, Theorem 2.8] and we now use the formulation of [10].

► **Theorem 2.1.** *Let \mathcal{C} be a stable class of graphs and let Δ be a finite set of first-order formulas. Assume that we can evaluate, given $\varphi \in \Delta$, an n -vertex graph $G \in \mathcal{C}$ and a tuple \bar{v} in G , whether $G \models \varphi(\bar{v})$ in time $f(\mathcal{C}, \Delta, n)$. Then there is an integer t and an algorithm that finds in any vertex-sequence of size at least m^t in an n -vertex graph $G \in \mathcal{C}$ a Δ -indiscernible sub-sequence of length m . The running time of the algorithm is $\mathcal{O}(|\Delta| \cdot k \cdot n^{1/k} \cdot f(\mathcal{C}, \Delta, n))$, where k is the maximal number of free variables of formulas in Δ .*

We refrain from formally defining the notion of stability and just note that every monadically stable class is also stable, hence, the theorem is applicable in our context.

Let $E(x, y)$ be the formula testing whether two vertices are adjacent in a graph. For $k \in \mathbb{N}$ we define $\Delta_k := \{E(x, y)\} \cup \{\gamma_k^S \mid S \subseteq [k]\}$ where for every $S \subseteq [k]$ we define

$$\gamma_k^S(y_1, \dots, y_k) := \exists z \left(\bigwedge_{i \in S} E(z, y_i) \wedge \bigwedge_{i \notin S} \neg E(z, y_i) \right).$$

Since for every graph G , set $S \subseteq [k]$ and tuple $\bar{v} \in V(G)^k$, we can evaluate $G \models \gamma_k^S(\bar{v})$ in time $\mathcal{O}(k \cdot n)$, we derive the following corollary of Theorem 2.1.

► **Corollary 2.2.** *For every stable class of graphs \mathcal{C} and integer k there exist an integer t and an algorithm that, given an n -vertex graph $G \in \mathcal{C}$ and a sequence $J \subseteq V(G)$ of length m^t , extracts a Δ_k -indiscernible sub-sequence of J of length m in time $\mathcal{O}(2^k \cdot k^2 \cdot n^2)$.*

Note that since the formula $E(x, y)$ is included in Δ_k , every Δ_k -indiscernible sequence is either a clique or an independent set. In [5], Blumensath studied the behaviour of indiscernible sequences in monadically NIP classes of infinite structures. Recently Dreier, Mählmann, Siebertz, and Toruńczyk finitized and extended those results with a special focus on monadically stable classes [10]. Their results are of very general nature and work for

arbitrary signatures and formulas. For our purposes we restrict their results to monadically stable classes of graphs using only the edge relation and distance formulas. Here we have that Δ_k -indiscernible sequences impose a strong structure on their neighborhood. The following follows from [10, Corollary 3.5].

► **Theorem 2.3.** *For every monadically stable class of graphs \mathcal{C} there exist integers n, k such that for every Δ_k -indiscernible sequence I of length at least n in a graph $G \in \mathcal{C}$ and for every vertex $v \in V(G)$ we have that $|N(v) \cap I| \leq 1$ or $|N(v) \cap I| \geq |I| - 1$.*

They also prove strong properties for sequences of disjoint first-order definable sets. In our case we are only interested in disjoint balls of radius r around vertices. The following follows from [10, Theorem 4.2 and proof of Theorem 5.2].

► **Theorem 2.4.** *For every monadically stable class of graphs \mathcal{C} and integer r , there exist integers k and t such that for every $m \in \mathbb{N}$, every graph $G \in \mathcal{C}$, and every set J of size m^t of disjoint r -balls in G the following holds. There exists a subset $I \subseteq J$ of size at least m with the following two properties.*

1. *There exists a set S of at most k sample vertices such that for every $v \in V(G)$ there exist a sample vertex $s(v) \in S$ and an exceptional ball $X(v) \in I$ such that for all $B \in I \setminus \{X(v)\}$ it holds that $N(v) \cap B = N(s(v)) \cap B$.*
2. *There exist a coloring $\text{col} : V(G) \rightarrow [k]$ with k colors and a relation $R \subseteq [k]^2$ such that for every two vertices u, v from different r -balls in I we have that $G \models E(u, v)$ if and only if $(\text{col}(u), \text{col}(v)) \in R$.*

In the above theorem we can assume that for every sample vertex $s \in S$ we have that $N(s)$ intersect with every or none of the r -balls from I . This can be achieved by adding the vertices of S as constants to G and taking a Ψ -indiscernible subsequence I' of I where $\Psi(y) := \{\psi_s(y) = \exists z \in N(s) : \text{dist}_{\leq r}(y, z) \mid s \in S\}$. By Theorem 2.1 the size of I' will still be polynomial.

3 Ramsey numbers in monadically stable graph classes

For a graph class \mathcal{C} , the Ramsey number $R_{\mathcal{C}}(s, t)$ denotes the smallest number of vertices that a graph from \mathcal{C} must have such that one can either find a clique of size s or an independent set of size t . If \mathcal{C} is the class of all graphs, we omit the subscript. As shown by Ajtai, Komlós and Szemerédi [2] we have $R(s, t) \in \mathcal{O}_s(t^{s-1}/(\log t)^{s-2})$. We improve these bounds in monadically stable classes by showing that for every fixed $s \geq 3$ and K_s -free monadically stable class \mathcal{C} of graphs, there exists a constant $\delta > 0$ such that we have $R_{\mathcal{C}}(s, t) \in \mathcal{O}_{\mathcal{C}}(t^{s-1-\delta})$. The main result of this section is the following theorem.

► **Theorem 3.1.** *Let $s \geq 3$ and \mathcal{C} be a K_s -free monadically stable class of graphs. There exist a real $\delta > 0$ and an integer n_0 such that every graph $G \in \mathcal{C}$ with $n \geq n_0$ vertices contains an independent set of size at least $n^{\frac{1}{s-1}+\delta}$.*

The aforementioned bound $R_{\mathcal{C}}(s, t) \in \mathcal{O}_{\mathcal{C}}(t^{s-1-\delta})$ then follows from the following remark.

► **Remark 3.2.** Assume we have a graph class \mathcal{C} , integers $s \geq 3$ and n_0 and a real $0 < \delta \leq 1$ such that every K_s -free graph from \mathcal{C} with $n \geq n_0$ vertices contains an independent set of size at least $n^{\frac{1}{s-1}+\delta}$, that is, $R_{\mathcal{C}}(s, n^{\frac{1}{s-1}+\delta}) \leq n$ for all $n \geq n_0$. By setting $t = n^{\frac{1}{s-1}+\delta}$, we obtain $n = t^{\frac{s-1}{1+(s-1)\delta}}$. Basic arithmetic then yields $\frac{s-1}{1+(s-1)\delta} \leq s-1-\delta$ for $0 < \delta \leq 1$, $s \geq 3$ and therefore $n \leq t^{s-1-\delta}$. This means we have $R_{\mathcal{C}}(s, t) \leq t^{s-1-\delta}$.

The proof of Theorem 3.1 will be by induction on s . The heart of the proof is the base case $s = 3$, where \mathcal{C} is triangle-free.

► **Lemma 3.3.** *For every triangle-free monadically stable class \mathcal{C} of graphs there exist a real $\delta > 0$ and an integer n_0 such that every graph $G \in \mathcal{C}$ with $n \geq n_0$ vertices contains an independent set of size at least $n^{\frac{1}{2}+\delta}$.*

The core of the proof of Lemma 3.3 is to compute indiscernible sequences and reason about their neighborhoods using Theorem 2.3 and Theorem 2.4. For technical reasons we want the additional property that the indiscernible sequences have a “large” neighborhood. We first show that if we cannot find such indiscernible sequences, then the graph contains a large independent set.

Proof of Lemma 3.3. Let us fix a graph $G \in \mathcal{C}$. We greedily build an independent set \mathcal{I} as follows. We look for an independent set I of size $\lceil (n/2)^\delta \rceil$ with $|N(I) \setminus I| \leq n^{\frac{1+\delta}{2}}$, where the value $0 < \delta \leq 1$ depends only on \mathcal{C} and will be specified later. Then we place the vertices of I into the independent set \mathcal{I} and remove I and its neighborhood from the graph, since these vertices are not allowed to be chosen into \mathcal{I} anymore. We repeat this process as long as possible. Let us first show that the statement of the lemma follows if we can repeat this process as long as there are still at least $n/2$ vertices in the graph. In each iteration we remove at most $\lceil (n/2)^\delta \rceil + n^{\frac{1+\delta}{2}}$ vertices from the graph, add $\lceil (n/2)^\delta \rceil$ vertices to the independent set \mathcal{I} and do not terminate unless $n/2$ vertices are removed. We choose n_0 to be large enough such that with $n \geq n_0$ it holds that $4\lceil (n/2)^\delta \rceil + 4n^{\frac{1+\delta}{2}} \leq n^{\frac{1+1.1\delta}{2}}$. The process therefore yields an independent set \mathcal{I} of size at least

$$\lceil (n/2)^\delta \rceil \cdot \frac{n/2}{\lceil (n/2)^\delta \rceil + n^{\frac{1+\delta}{2}}} \geq \frac{n^{1+\delta}}{4\lceil (n/2)^\delta \rceil + 4n^{\frac{1+\delta}{2}}} \geq \frac{n^{1+\delta}}{n^{\frac{1+1.1\delta}{2}}} = n^{\frac{1+0.9\delta}{2}}$$

and the statement of the lemma follows by rescaling δ .

From now on, we can therefore assume that the process could not proceed, while there were still at least $n/2$ vertices in the graph. Thus, we have a subgraph G' of G with at least $n/2$ vertices and every independent set I in G' of size $\lceil (n/2)^\delta \rceil$ has $|N(I) \setminus I| \geq n^{\frac{1+\delta}{2}}$ (later on referred to as assumption (*)).

We search for a Δ_{k_1} -indiscernible sequence I_0 in G' with k_1 chosen as required by Theorem 2.3. By Corollary 2.2, I_0 can be chosen of length at least $(n/2)^{\delta_1}$ for some real δ_1 depending only on \mathcal{C} .

By triangle-freeness, I_0 is not a clique but an independent set. Let G'_1 be the induced subgraph of G' where all vertices adjacent to at least two elements of I_0 have been removed. The radius 1-balls around the elements of I_0 in G'_1 are disjoint and we can apply Theorem 2.4, yielding a real δ_2 and an integer k_2 , both depending only on \mathcal{C} , with the following properties. There exists a subset $I \subseteq I_0$ of size at least $(n/2)^\delta$ for $\delta := \delta_1 \cdot \delta_2$ together with a k_2 -coloring \mathcal{K} of the 1-balls around I in G'_1 and a relation $R \subseteq [k_2]^2$, completely determining the edge relation between vertices from different balls.

We go back to the graph G' where we partition $N(I)$ into two sets $B_1 \cup B_2$, where B_1 contains all vertices that are adjacent to exactly one vertex in I and B_2 contains all vertices that are adjacent to at least two vertices in I . Note that in G' the coloring \mathcal{K} still describes the adjacency between elements from B_1 with different neighbors in I . Since I is a sub-sequence of the Δ_{k_1} -indiscernible sequence I_0 , Theorem 2.3 still applies and all vertices in B_2 are adjacent to at least $|I| - 1$ vertices of I . This means any two vertices $u, v \in B_2$ have a common neighbor in I and since G' is triangle-free, there can be no edge between u and v , and B_2 forms an independent set. We can assume $|B_2| \leq \frac{1}{2}n^{\frac{1+\delta}{2}}$, since otherwise

the statement of the lemma follows. As argued before I is also an independent set. Thus, by assumption (*), $|N(I) \setminus I| \geq n^{\frac{1+\delta}{2}}$. In summary, we so far have $N(I) \setminus I = B_1 \cup B_2$, $|N(I) \setminus I| \geq n^{\frac{1+\delta}{2}}$ and $|B_2| \leq \frac{1}{2}n^{\frac{1+\delta}{2}}$. This means $|B_1| \geq \frac{1}{2}n^{\frac{1+\delta}{2}}$.

We pick the color class $K \in \mathcal{K}$ containing the most vertices of B_1 . It has size at least $|K| \geq |B_1|/k_2 \geq \frac{1}{2k_2}n^{\frac{1+\delta}{2}}$. For $v \in I$ let $K_v = K \cap N(v)$. Since every vertex in B_1 has exactly one neighbor in I , this describes a partition of K . For every $v \in I$, the vertices in K_v have a common neighbor and thus form (by triangle-freeness of G) an independent set. If the relation R does not contain (K, K) then vertices from different sets K_v and K_w are not connected for all $v \neq w \in I$ and therefore K forms an independent set. On the other hand, if the relation R contains (K, K) then vertices from different sets K_v and K_w are connected for all $v \neq w \in I$. Then there can be at most two vertices $v \in I$ with $K_v \neq \emptyset$, since otherwise there would be a triangle. Hence, there exists $v \in I$ with $|K_v| \geq |K|/2$. In either case, we find an independent set of size at least $|K|/2 \geq \frac{1}{4k_2}n^{\frac{1+\delta}{2}}$. The statement of the lemma now follows by rescaling δ . ◀

We now turn to the proof of Theorem 3.1. The proof is by induction with Lemma 3.3 as the base case.

Proof of Theorem 3.1. If $s = 3$, the claim follows from Lemma 3.3. For larger values of s , we prove the claim by induction. The class \mathcal{C}' of K_{s-1} -free induced subgraphs of \mathcal{C} is monadically stable. By induction, there exist $0.1 \geq \delta' > 0$ and n'_0 such that every graph in \mathcal{C}' with $n \geq n'_0$ vertices contains an independent set of size at least $n^{\frac{1}{s-2} + \delta'} \geq n^{\frac{1+\delta'}{s-2}}$. Note that the upper bound of 0.1 for δ' is no restriction: if the statement holds for $\delta' > 0.1$, then it also holds for $\delta' = 0.1$. Let $G \in \mathcal{C}$ be a graph of size $n \geq n_0 \geq n'_0$, where we specify n_0 later. Let $\beta = (1 - \delta'/2)^{\frac{s-2}{s-1}}$. We distinguish two cases.

First, assume there is a vertex v with degree at least n^β . Then the induced subgraph $G[N(v) \setminus \{v\}]$ has size at least n^β and is K_{s-1} -free. Since it is contained in \mathcal{C}' , there is an independent set in $G[N(v) \setminus \{v\}]$ (and therefore also in G) of size at least $n^\beta \frac{1+\delta'}{s-2}$. Since $\delta' \leq 0.1$, we have $(1 - \delta'/2)(1 + \delta') = 1 + \delta'/2 - \delta'^2/2 \geq 1 + \delta'/3$ and therefore:

$$\beta \frac{1 + \delta'}{s - 2} = (1 - \delta'/2) \cdot \frac{s - 2}{s - 1} \cdot \frac{1 + \delta'}{s - 2} = (1 - \delta'/2)(1 + \delta') \cdot \frac{1}{s - 1} \geq \frac{1}{s - 1} + \frac{\delta'/3}{s - 1}.$$

We set $\delta = (\delta'/3)/(s - 1)$. Then G contains an independent set of size at least $n^{\frac{1}{s-1} + \delta}$.

If on the other hand there is no vertex with degree at least n^β then all vertices have degree less than n^β . In this case, we greedily construct an independent set by repeatedly picking a vertex and removing it together with its neighbors from the graph. This yields an independent set of size at least $n/(n^\beta + 1)$. We choose n_0 to be large enough such that with $n \geq n_0$ it holds that $n^\beta + 1 \leq n^{(1-\delta'/3)\frac{s-2}{s-1}}$. Then G contains an independent set of size at least $n/(n^\beta + 1) \geq n^{1-(1-\delta'/3)\frac{s-2}{s-1}} = n^{\frac{1}{s-1} + (\delta'/3)\frac{s-2}{s-1}}$. The claim follows with $\delta = (\delta'/3)\frac{s-2}{s-1}$. ◀

4 Subdivided bicliques

For every $r, s, t \in \mathbb{N}$ denote by $\text{sd}_r(K_{s,t})$ the r -subdivision of the complete bipartite graph with partitions of size s and t . When dealing with subdivisions, we will call the vertices of the original graph *principal vertices* and the newly inserted vertices *subdivision vertices*. In this section we prove the following theorem.

► **Theorem 4.1.** *For every monadically stable class \mathcal{C} and every integer r , there exist an integer n_0 and a real $\delta > 0$, such that for every $n \geq n_0$ it holds that every graph $G \in \mathcal{C}$ that contains $\text{sd}_r(K_{n,n})$ as a subgraph, also contains $K_{\lceil n^\delta \rceil, \lceil n^\delta \rceil}$ as a subgraph.*

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Proof. For the class \mathcal{C} and radius r , Theorem 2.4 yields a sample set size bound $k \in \mathbb{N}$ and an exponent $\frac{1}{\lambda} \in \mathbb{N}$ (referred to as t in Theorem 2.4) bounding the size of the set required to apply Theorem 2.4. Define $t_0 := n$ and $t_{i+1} := \frac{1}{2}t_i^\lambda$. Choose n_0 and δ such that it holds for all $n \geq n_0$ and $0 \leq i \leq r+1$ that $k \cdot n^\delta \leq \frac{1}{2}t_i$.

We work in the induced subgraph H of G that only contains the vertices of the $\text{sd}_r(K_{n,n})$. Let A and B be the sets of principal vertices in the $\text{sd}_r(K_{n,n})$. Assume towards a contradiction, that H contains no $K_{\lceil n^\delta \rceil, \lceil n^\delta \rceil}$ as a subgraph.

▷ **Claim 4.2.** For every $0 \leq i \leq r+1$, H contains an induced subgraph H_i that contains as a subgraph $\text{sd}_r(K_{\lceil t_i \rceil, \lceil t_i \rceil})$, with sets of principal vertices $A_i \subseteq A$ and $B_i \subseteq B$ such that the vertices from A_i have disjoint distance- i neighborhoods.

Proving the above claim will immediately lead to a contradiction, as the distance- $(r+1)$ neighborhoods of any two principal vertices in A_{r+1} overlap in every vertex of B_{r+1} as they are the principal vertex of an r -subdivided biclique. We prove the claim by induction on i .

Proof. For the base case set $H_0 := H$, $A_0 = A$, and $B_0 = B$. For the inductive step we are given H_i such that the vertices of A_i have disjoint distance- i neighborhoods. They therefore form the centers of a set J of disjoint i -balls and we can apply Theorem 2.4. We obtain a subset $I \subseteq J$ of size at least $|A_i|^\lambda$ together with a set S of at most k sample vertices such that for every $v \in H_i$, there exist a sample $s(v) \in S$ and an exceptional ball such that $N(v) \cap B = N(s(v))$ for every ball $B \in I$ which is not the exceptional ball. Let $s^{-1}(s) := \{v \in V(H_i) \mid s(v) = s\}$ be the set of all vertices described by $s \in S$. Let S_I be subset of sample vertices from S whose neighborhood intersects all of the balls from I . Conversely let S_\emptyset be subset of sample vertices from S whose neighborhood intersects none of the balls from I . As argued in Section 2 following Theorem 2.4 we have that $S = S_I \cup S_\emptyset$. We note that one can also assume, without loss of generality, that $|S_\emptyset| \leq 1$.

We now argue that $s^{-1}(s)$ has less than n^δ elements for every $s \in S_I$. Towards a contradiction, assume otherwise and let Q be a subset of $s^{-1}(s)$ of size $\lceil n^\delta \rceil$. Since every vertex from Q can behave exceptionally to at most one ball from I and we have that $|I| \geq 2\lceil n^\delta \rceil$, we find a subset of balls centered at $I' \subseteq I$ of size $\lceil n^\delta \rceil$ in which every vertex from Q has the same neighborhood as s . Since s has at least one neighbor in every ball of I' this yields a biclique of size $\lceil n^\delta \rceil$; a contradiction.

It follows that the set $S_I^{-1} := \bigcup_{s \in S_I} s^{-1}(s)$ of vertices described by sample vertices from S_I has size less than $k \cdot n^\delta$. Next we build the induced subgraph H_{i+1} of H_i by removing the vertices of S_I^{-1} . Let $A'_i \subseteq A_i$ be the center vertices of the balls in I . After the deletion of S_I^{-1} we have that every vertex in $v \in V(H_{i+1})$ is described by a sample vertex from S_\emptyset . Since the neighborhoods of the vertices in S_\emptyset intersect none of the i -balls in I , the neighborhood of v can intersect at most one of the i -balls in I (i.e. the exceptional ball of v). It follows that the vertices from A'_i have disjoint $(i+1)$ -balls.

It remains to show that there exist sets $A_{i+1} \subseteq A'_i$ and $B_{i+1} \subseteq B_i$ forming the principal vertices of a $\text{sd}_r(K_{\lceil t_{i+1} \rceil, \lceil t_{i+1} \rceil})$ subgraph in H_{i+1} . We started with the graph H_i containing as a subgraph $\text{sd}_r(K_{\lceil t_i \rceil, \lceil t_i \rceil})$ with principal vertices A_i and B_i . It follows that H_i contains as a subgraph $\text{sd}_r(K_{\lceil A'_i \rceil, \lceil t_i \rceil})$ with principal vertices A'_i and B_i . It holds that for every $t \geq 2$, every graph obtained by removing a single vertex from $\text{sd}_r(K_{t,t})$ still contains $\text{sd}_r(K_{t-1, t-1})$ as a subgraph. Therefore after we removed S_I^{-1} from H_i to build H_{i+1} we still find in it a set $A_{i+1} \subseteq A'_i$ of size $|A_{i+1}| \geq |A'_i| - |S_I^{-1}| \geq |A_i|^\lambda - |S_I^{-1}| \geq t_i^\lambda - k \cdot n^\delta \geq \frac{1}{2}t_i^\lambda = t_{i+1}$ forming one side of a $\text{sd}_r(K_{\lceil t_{i+1} \rceil, \lceil t_{i+1} \rceil})$ subgraph in H_{i+1} . The other side is formed by a subset of B_i . ◁

This completes the proof of the theorem. ◀

5 Regularity in monadically stable classes of graphs

In this section we prove the following theorem.

► **Theorem 5.1.** *For every monadically stable class of graphs \mathcal{C} , there exist an integer n_0 and a real δ , such that for every graph $G \in \mathcal{C}$ of size $n \geq n_0$ there exists an equipartition into parts of size n^δ , such that after possibly omitting one element from each part, they satisfy the following properties:*

1. *Every part forms a clique or an independent set.*
2. *For every two parts A and B , except for at most one vertex $a_{\text{ex}} \in A$, for all other vertices $a \in A$, either $|N(a) \cap B| \leq 1$ or $|N(a) \cap B| \geq |B| - 1$. This means, with the exception of one vertex per part, the connections between A and B semi-induce either a subgraph of a matching or a supergraph of a co-matching.*

Note that the additional vertex that needs to be omitted from a part A when comparing it to a part B may differ from the vertex that needs to be omitted from A when comparing it to a different part C .

We follow the approach by Malliaris and Shelah, by partitioning the graph into indiscernible sequences. We start by analyzing the interaction between two indiscernible sequences.

► **Lemma 5.2.** *For every monadically stable class of graphs \mathcal{C} , there exist integers n_0 and k such that for every graph $G \in \mathcal{C}$ the following holds. Let A and B be disjoint Δ_k -indiscernible sequences of length at least n_0 in G . Except for at most one vertex $a_{\text{ex}} \in A$, for all other vertices $a \in A$, either $|N(a) \cap B| \leq 1$ or $|N(a) \cap B| \geq |B| - 1$. In particular, after removing at most one vertex from each sequence, the connections between A and B semi-induce either a subgraph of a matching or a supergraph of a co-matching.*

Proof. Choose $n_0 \geq 5$ and k as required by Theorem 2.3.

▷ **Claim 5.3.** The set of vertices of A adjacent to at least two vertices from B has size 0, 1, $|A| - 1$, or $|A|$.

Proof. Assume towards a contradiction that there exist two elements $a_1, a_2 \in A$ adjacent to at least two vertices in B and two vertices $a_3, a_4 \in A$ adjacent to at most one vertex in B . Since a_1, a_2 have at least two neighbors in B , by Theorem 2.3 they both must have at least $|B| - 1$ neighbors in B . Therefore there must be a set $B' \subseteq B$ of size at least $|B| - 2 \geq 3$ containing only vertices adjacent to both a_1 and a_2 . Again by Theorem 2.3 every vertex in B' must have at least $|A| - 1$ neighbors in A . As a result either a_3 or a_4 will have at least two neighbors in B' and therefore in B and we reach a contradiction. ◁

Let $A' \subseteq A$ be the set of vertices adjacent to at least two vertices from B . It has size 0, 1, $|A| - 1$, or $|A|$. Assume $|A'| = 0$. All vertices of A have at most one neighbor in B and we have at most $|A|$ edges between the two sets. If no two vertices have the same neighbor, then we have a subgraph of a matching between A and B and we are done. Assume two vertices of A share a neighbor b in B . By Theorem 2.3, b must have at least $|A| - 1$ neighbors in A . We therefore have at most one edge between A and $B \setminus \{b\}$, a subgraph of a matching as required.

If $|A'| = 1$, only one vertex $a \in A$ has at least two neighbors in B we omit a from A and argue as in the previous case. The two cases where $|A'| \geq |A| - 1$ follow by symmetry. ◀

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Proof of Theorem 5.1. Let n'_0 and k be from Lemma 5.2 and let $\lambda \leq \frac{1}{2}$ be the real given by Corollary 2.2 such that any graph from \mathcal{C} of size m contains a Δ_k -indiscernible sequence of size at least m^λ . Therefore, we can iteratively build a partition \mathcal{G} of Δ_k -indiscernible sequences of size $n^{\frac{\lambda}{2}}$ in G as long as $|V(G) \setminus \bigcup \mathcal{G}| = n - |\bigcup \mathcal{G}| \geq n^{\frac{\lambda}{2}}$. Once we have that $|\bigcup \mathcal{G}| > n - n^{\frac{\lambda}{2}}$, we know that \mathcal{G} contains at least

$$\left\lfloor \frac{n - n^{\frac{\lambda}{2}}}{n^{\frac{\lambda}{2}}} \right\rfloor = \left\lfloor n^{1-\frac{\lambda}{2}} - n^{\frac{1}{2}-\frac{\lambda}{2}} \right\rfloor \geq n^{\frac{1}{2}}$$

many parts. Therefore we can distribute the leftover elements from $V(G) \setminus \bigcup \mathcal{G}$ among the parts of \mathcal{G} , such that every part of \mathcal{G} gets at most one additional vertex. Now \mathcal{G} is the desired equipartition with part size at least n^δ for $\delta := \frac{\lambda}{2}$. By omitting the leftover element property (1) follows from the fact that any Δ_k -indiscernible sequence is either a clique or an independent set and property (2) follows from Lemma 5.2 and by setting $n_0 \geq n_0'^{\frac{1}{\delta}}$. ◀

6 Polynomial kernels in powers of nowhere dense classes

In this section we turn to algorithmic applications of indiscernible sequences in monadically stable classes of graphs. We focus on powers of classes of nowhere dense graphs. Recall that the d th power G^d of a graph G is obtained by pairwise connecting in G vertices that have distance at most d . This notion is lifted to classes of classes by defining $\mathcal{C}^d := \{G^d \mid G \in \mathcal{C}\}$ to be the d th power of a class \mathcal{C} .

Powers of nowhere dense graphs were studied by Fabiański et al. who showed that they admit fixed-parameter tractable algorithms for INDEPENDENT SET and DOMINATING SET parameterized by solution size k [13]. We will strengthen their result by giving polynomial kernels for both problems and prove the following theorem.

► **Theorem 6.1.** *For every integer d and nowhere dense class \mathcal{C} , INDEPENDENT SET and DOMINATING SET parameterized by solution size k admit a polynomial kernel on \mathcal{C}^d .*

To obtain this result we will make use of the following two properties of powers of nowhere dense graphs. For every integer d and nowhere dense class \mathcal{C} , the class \mathcal{C}^d (1) is monadically stable, and (2) excludes a fixed co-matching as a semi-induced subgraph. For every integer d the operation of taking the d th power of a graph is expressible as a transduction. Therefore the first property follows trivially from the monadic stability of nowhere dense classes. The second property was proved by Fabiański et al. in [13, Theorem 11]. Assuming those two properties, we will construct our kernels. In fact, our results apply to all graph classes satisfying these two properties, with powers of nowhere dense classes merely being the most prominent example.

► **Theorem 6.2.** *INDEPENDENT SET parameterized by solution size k admits a polynomial kernel on every monadically stable class of graphs which excludes a co-matching of size ℓ as a semi-induced subgraph.*

Proof. We search for an independent set of size k in a graph G , which we assume to be from a monadically stable class \mathcal{C} that excludes a co-matching of size ℓ . By Theorem 2.3, for every monadically stable class of graphs \mathcal{C} there exist integers n_0 and p such that for every Δ_p -indiscernible sequence I of length at least n_0 in a graph $G \in \mathcal{C}$ and for every vertex $v \in V(G)$ we have that $|N(v) \cap I| \leq 1$ or $|N(v) \cap I| \geq |I| - 1$. We fix such a pair of integers n_0 and p , as given by Theorem 2.3. Now by Corollary 2.2, there exist an integer t and an algorithm that, given an n -vertex graph $G \in \mathcal{C}$, where $n \geq n_0^t$, extracts a Δ_p -indiscernible

sequence $I \subseteq V(G)$ of length n_0 in time $\mathcal{O}(2^p \cdot p^2 \cdot n^2)$. We now have that in any graph $G \in \mathcal{C}$ of size at least n_0^t , we can find in time $\mathcal{O}(2^p \cdot p^2 \cdot n^2)$ a Δ_p -indiscernible sequence I of size at least $n^{1/t}$ such that, by Theorem 2.3, every vertex of $V(G) \setminus I$ is adjacent to at most one vertex from I or all but at most one vertex from I . Recall that since we include the edge relation in Δ_p , every I is either a clique or an independent set.

If $n \leq (k + 1)^t$, then the graph itself is a polynomial kernel, so we can assume without loss of generality that $n > (k + 1)^t$. This means that we can find in any graph $G \in \mathcal{C}$ of size at least $(k + 1)^t$ in time $\mathcal{O}(2^p \cdot p^2 \cdot n^2)$ a Δ_p -indiscernible sequence I of size at least $k + 1$. We will (for later) similarly assume, without loss of generality, that $k + 1 \geq \ell$; since otherwise we can solve the problem in polynomial time, i.e., $\mathcal{O}(n^\ell)$ -time, and return a trivial yes- or no-instance. Our kernelization algorithm proceeds by applying the following reduction rule exhaustively.

If G has at least $(k + 1)^t$ vertices, find a Δ_p -indiscernible sequence I of size at least $k + 1$. If I is an independent set return a trivial yes-instance. Otherwise remove from G all vertices of I except for k many.

Since \mathcal{C} is closed under taking induced subgraphs, we can apply this rule repeatedly without leaving \mathcal{C} . The procedure runs in polynomial time and obviously, upon termination, the resulting graph has size at most $(k + 1)^t$. Since the procedure only removes vertices, it cannot turn no-instances into a yes-instances. It remains to argue that the procedure does not turn yes-instances into no-instances.

If I forms an independent set, then we return a trivial yes-instance, so the only remaining case is I forming a clique. In this case, I can contain at most one vertex from a solution. Assume there exists a vertex outside I that is non-adjacent to only one element of I . By indiscernibility of I , there exists such a “private vertex“ for every element in I which implies a co-matching of size at least $k + 1 \geq \ell$. This is a contradiction to our assumption of excluding a co-matching of size ℓ . Therefore, by Theorem 2.3, every vertex outside I is adjacent to either all or at most one vertex of I .

Assume we have a yes-instance, witnessed by an independent set S of size k . Since I is a clique, I contains at most one vertex from S . If S contains no vertex from I , then after applying the reduction rule we trivially still have a yes-instance. On the other hand, if S contains exactly one vertex from I then the remaining $k - 1$ vertices outside I are not adjacent to all vertices in I and therefore have at most one neighbor each within I . This means in any subset of k vertices of I there will be at least one vertex that is non-adjacent to the remaining $k - 1$ vertices of S , completing it to an independent set of size k and preserving the yes-instance. ◀

► **Theorem 6.3.** *DOMINATING SET parameterized by solution size k admits a polynomial kernel on every monadically stable class of graphs which excludes a co-matching of size ℓ as a semi-induced subgraph.*

Proof. We search for a dominating set of size k in a graph from a monadically stable graph class \mathcal{C} which excludes a co-matching of size ℓ as a semi-induced subgraph. We outline our approach: Given a DOMINATING SET instance (G_0, k) , we will first construct an equivalent instance (G, k) of the RED BLUE DOMINATING SET problem. The input to this problem is a bipartite graph G whose two parts are colored red and blue, respectively, and a number k , and the task is to find k blue vertices that dominate all red vertices. We will ensure that G is still part of another monadically stable class \mathcal{D} that excludes a co-matching. We then

kernelize (G, k) to obtain a kernel (K, k) to RED BLUE DOMINATING SET whose size is polynomial in k . Finally we will transform (K, k) back into a DOMINATING SET instance $(K_0, k + 1)$ by only introducing $\mathcal{O}(k)$ many new vertices.

From Dominating Set to Red Blue Dominating Set. We create the instance (G, k) from (G_0, k) as follows. The vertices of G consist of a red copy A and a blue copy B of $V(G_0)$. For every vertex $v \in V(G_0)$ we connect the red copy of v with each blue copy in $N[v]$. Note that no vertices of the same color are connected. There exists a fixed transduction which produces G from G_0 , so we know that G must be from a monadically stable class \mathcal{D} depending only on \mathcal{C} . It is easy to see that G_0 contains a size k dominating set if and only if there exist k vertices from A which together dominate all of B . Now assume towards a contradiction that G contains a co-matching C of size 3ℓ as a semi-induced subgraph.

We build a semi-induced co-matching C_0 in G_0 by iteratively taking a red vertex a and a blue vertex b from C that are not connected and adding them to C_0 . Since a and b are non-neighbors in different colors, we know that they are not connected and correspond to different vertices in G_0 . If the blue copy of a or the red copy of b is contained in C we remove them together with their co-matching partners from C . In G_0 , a will be connected to the original of every blue vertex in C except for b , and analogously b will be connected to the original of every red vertex in C except for a . In every step, we add one pair to C_0 and remove at most three pairs from C . Therefore by fully exhausting C we find a semi-induced co-matching of size at least ℓ in C , a contradiction. It follows that G contains no co-matching of size 3ℓ .

Kernelization. By Theorem 2.3, for every monadically stable class of graphs \mathcal{D} there exist integers n_0 and p such that for every Δ_p -indiscernible sequence I of length at least n_0 in a graph $G \in \mathcal{D}$ and for every vertex $v \in V(G)$ we have that $|N(v) \cap I| \leq 1$ or $|N(v) \cap I| \geq |I| - 1$. We fix such a pair of integers n_0 and p , as given by Theorem 2.3. Now by Corollary 2.2, there exist an integer t and an algorithm which given an n -vertex graph $G \in \mathcal{D}$, where $n \geq n_0^t$, extracts a Δ_p -indiscernible sequence $I \subseteq V(G)$ of length n_0 in time $\mathcal{O}(2^p \cdot p^2 \cdot n^2)$. We now have that in any graph $G \in \mathcal{D}$ of size at least n_0^t , we can find in time $\mathcal{O}(2^p \cdot p^2 \cdot n^2)$ a Δ_p -indiscernible sequence I of size at least n_0 such that, by Theorem 2.3, every vertex of $V(G) \setminus I$ is adjacent to at most one vertex from I or all but at most one vertex from I .

We assume (similarly to Theorem 6.2), without loss of generality, that $k + 2 \geq 3\ell$, $k > 1$, $|B| > (k + 2)^t$. Thus, if the set of to-be-dominated vertices B contains at least $(k + 2)^t$ vertices, there exists a Δ_p -indiscernible sequence I of size at least $k + 2$ among B such that by Theorem 2.3, all vertices in A are connected to at most one or all but at most one vertex in I . We will kernelize it using the following reduction rules.

1. If A contains two vertices with the same neighborhood in B , remove one of them.
2. If the set of to-be-dominated vertices B contains at least $(k + 2)^t$ vertices, find a Δ_p -indiscernible sequence I of size at least $k + 2$ among B . Remove from G all vertices of I except for $k + 1$ many.

The first reduction rule is obviously efficient and safe. As discussed above, the second rule can be implemented in polynomial time. Let (G', k) be the instance obtained from (G, k) by the second rule, where a Δ_p -indiscernible sequence I is reduced to a sub-sequence I' of size $k + 1$. If (G, k) is a yes-instance, then removing to-be-dominated vertices from B will still yield a yes-instance. To show safety of the second rule, we assume that the reduced

instance (G', k) is a yes-instance and show that (G, k) also is a yes-instance. Since (G', k) is a yes-instance, there exists a set $S \subseteq A$ of size at most k that dominates all of I' . As discussed above, by Theorem 2.3 every vertex in S is connected to at most one, all, or all but one vertex of I . Assume for contradiction there exists a vertex connected to all but one vertex of I . By indiscernibility, one vertex of I having a private non-neighbor would mean that all vertices of I have a private non-neighbor, implying the existence of a co-matching of size $|I| \geq k + 2 \geq 3\ell$, a contradiction. Therefore every vertex in S is either connected to at most one or all vertices of I . Since I' has $k + 1$ vertices and S only k , S must include a vertex connected to at least two elements of I and therefore connected to all of I . Hence, S is also a solution to (G, k) , meaning that (G, k) is a yes-instance

It remains to prove that the kernel (K, k) obtained by exhaustively applying the two rules has size polynomial in k . Both rules reduce the size of the graph and when applied exhaustively, B has size at most $(k + 2)^t$ and A contains no more twins. We use the following theorem which applies to *NIP* classes which contain monadically *NIP* and therefore also monadically stable classes.

► **Theorem 6.4** (Sauer-Shelah Lemma [28, 29, 31]). *Let $G \in \mathcal{D}$ and let $B \subseteq V(G)$. Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a partition of $V(G) \setminus B$ such that two vertices u and v belong to $P \in \mathcal{P}$ whenever $N(u) \cap B = N(v) \cap B$. If \mathcal{D} is a *NIP* class of graphs then there exists a constant d such that $|\mathcal{P}| \leq |B|^d + 1$.*

Since we have no twins in A , every vertex in $a \in A$ is uniquely identified by its neighborhood $N(a) \cap B$ and therefore $|A| \leq |B|^d + 1 \leq (k + 2)^{t \cdot d} + 1$. Thus, (K, k) has polynomial size.

From Red Blue Dominating Set to Dominating Set. Finally we are going to convert (K, k) into a DOMINATING SET instance $(K_0, k + 1)$ by creating a star $K_{1, k+2}$ of size $k + 2$ and connecting its center c to every red (i.e. dominating) vertex from K .

If (K, k) admits a red blue dominating set S of size k , then $S \cup \{c\}$ is a $k + 1$ dominating set for K_0 . Vice versa, the only way to dominate the $k + 2$ leaves of $K_{1, k+2}$ with $k + 1$ vertices is by picking the center c . Hence, if there exists a solution S for $(K_0, k + 1)$, it must contain c . Note that all the red vertices are dominated by c and every blue vertex has no blue neighbor. Therefore, we can replace every blue vertex in S with any of its red neighbors. This yields a set S' containing c and at most k red vertices dominating all blue vertices. These red vertices witness that (K, k) is a yes-instance as well. ◀

Having proved Theorem 6.2 and Theorem 6.3, this concludes the proof of Theorem 6.1.

References

- 1 Hans Adler and Isolde Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *European Journal of Combinatorics*, 36:322–330, 2014.
- 2 Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on ramsey numbers. *Journal of Combinatorial Theory, Series A*, 29(3):354–360, 1980.
- 3 Noga Alon, Richard A Duke, Hanno Lefmann, Vojtěch Rödl, and Raphael Yuster. The algorithmic aspects of the regularity lemma. *Journal of Algorithms*, 16(1):80–109, 1994.
- 4 John T Baldwin and Saharon Shelah. Second-order quantifiers and the complexity of theories. *Notre Dame Journal of Formal Logic*, 26(3):229–303, 1985.
- 5 Achim Blumensath. Simple monadic theories and indiscernibles. *Mathematical Logic Quarterly*, 57(1):65–86, 2011. doi:10.1002/malq.200910121.

- 6 Anuj Dawar and Stephan Kreutzer. Domination problems in nowhere-dense classes. In *FSTTCS*, volume 4 of *LIPICs*, pages 157–168. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2009.
- 7 Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.
- 8 Rod G Downey and Michael R Fellows. Fixed-parameter tractability and completeness ii: On completeness for $w[1]$. *Theoretical Computer Science*, 141(1-2):109–131, 1995.
- 9 Jan Dreier, Jakub Gajarský, Sandra Kiefer, Michał Pilipczuk, and Szymon Toruńczyk. Treelike decompositions for transductions of sparse graphs. *arXiv preprint*, 2022. [arXiv:2201.11082](https://arxiv.org/abs/2201.11082).
- 10 Jan Dreier, Nikolas Mählmann, Sebastian Siebertz, and Szymon Toruńczyk. Indiscernibles and wideness in monadically stable and monadically NIP classes. *arXiv preprint*, 2022. [arXiv:2206.13765](https://arxiv.org/abs/2206.13765).
- 11 Zdeněk Dvořák. Induced subdivisions and bounded expansion. *European Journal of Combinatorics*, 69:143–148, 2018.
- 12 Kord Eickmeyer, Archontia C. Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michal Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Neighborhood complexity and kernelization for nowhere dense classes of graphs. In *ICALP*, volume 80 of *LIPICs*, pages 63:1–63:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
- 13 Grzegorz Fabianski, Michal Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. Progressive Algorithms for Domination and Independence. In Rolf Niedermeier and Christophe Paul, editors, *36th International Symposium on Theoretical Aspects of Computer Science (STACS 2019)*, volume 126 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–27:16, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi: 10.4230/LIPIcs.STACS.2019.27.
- 14 Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Daniel Lokshtanov, and MS Ramanujan. A new perspective on model checking of dense graph classes. *ACM Transactions on Computational Logic (TOCL)*, 21(4):1–23, 2020.
- 15 Jakub Gajarský, Stephan Kreutzer, Jaroslav Nešetřil, Patrice Ossona De Mendez, Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. First-order interpretations of bounded expansion classes. *ACM Transactions on Computational Logic (TOCL)*, 21(4):1–41, 2020.
- 16 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *Journal of the ACM (JACM)*, 64(3):1–32, 2017.
- 17 Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, 1997.
- 18 Yiting Jiang, Jaroslav Nešetřil, Patrice Ossona de Mendez, and Sebastian Siebertz. Regular partitions of gentle graphs. *Acta Mathematica Hungarica*, 161(2):719–755, 2020.
- 19 Stephan Kreutzer, Roman Rabinovich, and Sebastian Siebertz. Polynomial kernels and wideness properties of nowhere dense graph classes. *ACM Transactions on Algorithms (TALG)*, 15(2):1–19, 2018.
- 20 M. Malliaris and S. Shelah. Regularity lemmas for stable graphs. *Transactions of the American Mathematical Society*, 366(3):1551–1585, 2014. URL: <http://www.jstor.org/stable/23813167>.
- 21 Jaroslav Nešetřil and P Ossona de Mendez. Structural sparsity. *Russian Mathematical Surveys*, 71(1):79, 2016.
- 22 Jaroslav Nešetřil and Patrice Ossona De Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 32(4):600–617, 2011.
- 23 Jaroslav Nešetřil and Patrice Ossona De Mendez. *Sparsity: graphs, structures, and algorithms*, volume 28. Springer Science & Business Media, 2012.
- 24 Jaroslav Nešetřil, Patrice Ossona de Mendez, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Rankwidth meets stability. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2014–2033. SIAM, 2021.
- 25 Jaroslav Nešetřil, Patrice Ossona de Mendez, and Sebastian Siebertz. Structural properties of the first-order transduction quasiorder. In *30th EACSL Annual Conference on Computer Science Logic (CSL 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.

- 26 Jaroslav Nešetřil, Roman Rabinovich, Patrice Ossona de Mendez, and Sebastian Siebertz. Linear rankwidth meets stability. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1180–1199. SIAM, 2020.
- 27 Benjamin Oye. Stable graphs.
- 28 Norbert Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.
- 29 Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.
- 30 Endre Szemerédi. Regular partitions of graphs. *Colloq. Int. CNRS*, 260:399–401, 1978.
- 31 Vladimir Naumovitch Vapnik and Alexei Iakovlevitch Červonenkis. On the uniform convergence of relative sequences of events to their probabilities. *Theory Probab. Appl.*, 16:264–280, 1971.