On Constrained Intersection Representations of Graphs and Digraphs

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Abstract

We study the problem of determining minimal directed intersection representations of DAGs in a model introduced by [Kostochka, Liu, Machado, and Milenkovic, ISIT2019]: vertices are assigned color sets, two vertices are connected by an arc if and only if they share at least one color and the tail vertex has a strictly smaller color set than the head, and the goal is to minimize the total number of colors. We show that the problem is polynomially solvable in the class of triangle-free and Hamiltonian DAGs and also disclose the relationship of this problem with several other models of intersection representations of graphs and digraphs.

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1 Introduction

Problem definition and state of the art. Given a digraph $D = (V, A)$, a *directed intersection representation* of D is a pair (U, φ) where U is a finite set of *colors* and φ is a *proper coloring* of D, that is, a mapping assigning to each vertex $v \in V$ a set $\varphi(v) \subseteq U$ such that for any two vertices $u, v \in V$, it holds that

$$(u, v) \in A \quad \text{if and only if} \quad \varphi(u) \cap \varphi(v) \neq \emptyset \text{ and } |\varphi(u)| < |\varphi(v)|\,. \tag{1}$$

The *cardinality* of a directed intersection representation (U, φ) of D is defined as the number of colors, that is, $|U|$. Note that if (U, φ) is a directed intersection representation of a digraph D and $W = (v_1, \ldots, v_k)$ is a walk in D, then $|\varphi(v_1)| < \ldots < |\varphi(v_k)|$, which implies that D is acyclic (that is, a DAG). On the other hand, Kostochka et al. [9] showed that every DAG admits a directed intersection representation. They initiated a study of the following invariant of DAGs. The *directed intersection number* of a DAG $D = (V, A)$ – denoted by $DIN(D)$ – is the smallest cardinality of a directed intersection representation of D. In [9, 11], the authors focused on characterizing the extremal values of DIN. They showed that:

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- for every DAG D with n vertices, it holds that $DIN(D) \leq \frac{5n^2}{8} + \mathcal{O}(n)$.[1]
- for every n there is a DAG D with n vertices such that $DIN(D) \geq \frac{9n^2}{16}(1 - o(1))$.

In [1, 2] Caucchiolo and Cicalese studied the computational complexity of determining $DIN(D)$. They showed that the problem of computing $DIN(D)$ is NP-hard even when D is an arborescence (a tree with all the edges oriented away from the root). Moreover, for general DAGs and any $\epsilon > 0$, the problem does not admit an $n^{1-\epsilon}$ approximation unless P = NP. Conversely, for the case of arborescences the problem is shown to be in APX, and in [2] the authors also provide an asymptotic fully polynomial time approximation scheme.

The main result. In this paper we continue the quest for islands of tractability in the complexity landscape for the problem of computing directed intersection representations of DAGs initiated in [1, 2]. We focus on a class of graphs that, in the analysis of [9], appears to include the instances that are the most demanding in terms of the *DIN* value. In fact, in [9], a key point in the construction of the lower bound on the extremal value of $DIN(D)$ is to consider DAGs that are both Hamiltonian and triangle-free.[2] Here we show that for every D in the class of Hamiltonian and triangle-free DAGs, computing the value of $DIN(D)$ is a tractable problem solvable in time $\mathcal{O}(n^3)$.

A key element for obtaining such a result is the fact that for any Hamiltonian and triangle-free DAG D it is possible to define a demand function b on the vertices such that the value of $DIN(D)$ can be exactly characterized in terms of the value of a maximum b-matching in the underlying graph of D, a parameter that can be computed in polynomial time [8, 14].

Additional results, related problems, and literature. The proof of the above result leads us to introduce several generalizations and variants of intersection representations of graphs and digraphs (defined in Section 2.1). We believe that these variants might be of independent interest and for which we are able to prove interesting results on their relations in terms of minimal intersection representations (summarized in Figure 1).

It is known that any finite undirected graph G admits an intersection representation given by a family of finite sets associated to its vertices, such that two vertices are adjacent if and only if their associated sets intersect. The minimum cardinality of the ground set of such a family is referred to as the *intersection number* of the graph G and denoted by $IN(G)$. Erdős, Goodman and Pósa [6] showed that $IN(G)$ equals the minimum number of cliques needed to cover the edges of G, i.e., the size of a minimum edge clique cover of G. Determining this value was proved to be NP-hard in [13] (see also [10]). By [7, 12], both problems are not approximable within a factor of $|V|^\epsilon$ for some $\epsilon > 0$ unless P = NP. Applications of intersection representations and clique covers are found in areas as diverse as computational geometry, matrix factorization, compiler optimization, applied statistics, resource allocations, etc; see, e.g., the survey papers [15, 16, 17], and the comprehensive introduction of [3].

In this paper several new variants are considered which contribute to this rich literature and in particular to the approach of [17] of studying constrained versions of intersection representation and their applications.

The following practical scenario can be modelled by a constrained variant of the intersection number of undirected graphs, which is considered in the series of reductions leading to the proof of our main result.

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[1] The precise bound given in [11] is $\frac{5n^2}{8} - \frac{3n}{4} + 1$.

[2] Liu et al. [11] leave as an open problem to show that for every n the maximum value of $DIN(D)$ among the DAGs with n vertices is attained by one that is also Hamiltonian.

There is a shared resource (for example, a wireless communication channel) and a set of participants who want to use the resource. However, no two participants are willing to share the resource at the same time unless they need to do it for accomplishing a common task. In the wireless communication example, you might imagine that using the channel at the same time means to be possibly eavesdropped while you do need to share temporally at least once with whomever you want to communicate with. We say that two participants are *compatible* (with each other) if they need to accomplish a common task. We assume that the resource can be used for an arbitrary number of time periods, where in each time period only a subset of pairwise compatible participants can share the resource. Furthermore, if a set S of pairwise compatible participants shares the resource in a certain time period, then the common task of any pair of participants in S can be carried out in this period. Since the use of the resource is expensive, our goal is to design a schedule of assigning the participants to time slots for using the resource that minimizes the total number of temporal slots, such that all pairs of participants can accomplish their tasks without ever using a resource together with an incompatible participant. This problem can be modeled as the problem of computing the intersection number of the compatibility graph. If we also assume that every participant requests to take part in at least a certain number of slots, we obtain the ℓ-constrained intersection number, where $\ell(v)$ is the desired lower bound on the number of slots for participant v (see Section 2.1 for definitions).

2 Notations and definitions

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers. All graphs in this paper will be finite and simple, but may be directed or undirected. We will use the term *graph* to refer to an undirected graph and the term *digraph* to refer to a directed graph.

Definitions for graphs. A *graph* is a pair $G = (V, E)$ where $V = V(G)$ is a finite set of *vertices* and $E = E(G)$ is a set of 2-element subsets of V called *edges*. Two vertices u and v in a graph $G = (V, E)$ are *adjacent* if $\{u, v\} \in E$. A vertex in a graph is *universal* if it is adjacent to all other vertices. A graph is said to be *nontrivial* if it contains more than one vertex. A *vertex cover* in G is a set C of vertices such that every edge has at least one endpoint in C. Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of G. A b-matching of G is a function $x : E \to \mathbb{Z}_+$ such that for each vertex v it holds that $\sum_{e\in E_v} x(e) \le b(v)$, where E_v denotes the set of edges incident with v. A maximum weight b-matching of G is a b-matching of G such that the total weight $\sum_{e\in E} x(e)$ is maximum among all b-matchings of G. We use $\nu(G, b)$ to denote the total weight of a maximum weight b-matching of G.

Definitions for digraphs. A *digraph* is a pair $D = (V, A)$ where $V = V(D)$ is a finite set of *vertices* and $A = A(D)$ is a set of ordered pairs of vertices called *arcs*. Given an arc $a = (u, v) \in A$, we call u the *tail* of a and v its *head*. Two vertices u and v in a digraph $D = (V, A)$ are *adjacent* if $(u, v) \in A$ or $(v, u) \in A$. A *walk* in a digraph D is a sequence $(v_1, \ldots, v_k)$ of vertices of D such that $(v_i, v_{i+1}) \in A$ for all $i \in \{1, \ldots, k-1\}$. A *path* in D is a walk in which all vertices are pairwise distinct. Given a positive integer k, a *cycle of length* k in D is a path $(v_1, \ldots, v_k)$ such that $(v_k, v_1) \in A$. Note that a cycle of length one consists of a vertex v at which D has a loop (that is, an arc of the form (v, v)), and a cycle of length two consists of a pair of vertices u, v such that both arcs (u, v) and (v, u) exist. A digraph is *acyclic* if it contains no cycles; a directed acyclic graph is referred to as a *DAG*. A digraph is *Hamiltonian* if it has a path containing every vertex. For a digraph $D = (V, A)$, the *underlying graph of* D is the undirected graph $U(D) = (V, E)$ in which two distinct vertices are adjacent if and only if they are adjacent in D.

Common definitions for graphs and digraphs. Let G be a graph or a digraph. An *independent set* in G is a set of pairwise nonadjacent vertices. Let f be a function from $V(G)$ to $\mathbb{Z}_+$. Given a set $S \subseteq V(G)$, we denote by $f(S)$ the value $\sum_{v \in S} f(v)$. Let $\alpha(G, f)$ denote the maximum value of $f(S)$ over all independent sets S in G. We say that G is *bipartite* if it admits a *bipartition*, that is, a partition of its vertex set into two (possibly empty) independent sets. More generally, G is *triangle-free* if it does not contain three pairwise adjacent vertices. Given a vertex $u \in V(G)$, the *degree* of u in G, denoted by $\deg_G(u)$ (or simply by $\deg(u)$ when the graph or digraph is clear from the context), is the number of vertices $v \in V(G)$ such that u and v are adjacent in G. Note that if D is a digraph without cycles of length one or two (in particular, if D is a DAG), then the vertex degrees are the same in D and in the underlying graph $U(D)$.

A *partially ordered set* (or: a *poset*) is a pair $(V, \preceq)$ where V is a finite set and $\preceq$ is a binary relation on V that is reflexive, antisymmetric, and transitive. We write $u \prec v$ if $u \preceq v$ and $u \neq v$. A poset element $v \in V$ is *minimal* if there is no element $u \in V \setminus \{v\}$ such that $u \prec v$.

2.1 Intersection representations of graphs and digraphs

In this section we introduce several definitions of intersection representation for graphs and digraphs. These variants will be used in the proof of our main result which will consist of a sequence of reductions from one variant to another.

A **weak directed intersection representation of a digraph** $D = (V, A)$ is a pair (U, φ) where U is a finite set of *colors* and φ is a *weak proper coloring* of D, that is, a mapping assigning to each vertex $v \in V$ a set $\varphi(v) \subseteq U$ such that for any two distinct vertices $u, v \in V$, it holds that $(u, v) \in A$ if and only if $\varphi(u) \cap \varphi(v) \neq \emptyset$ and $|\varphi(u)| \leq |\varphi(v)|$. Note that, with respect to the definition of a directed intersection representation given in Equation (1), the constraint on the cardinality is expressed by a weak inequality. Furthermore, if (U, φ) is a weak directed intersection representation of a digraph D and $W = (v_1, \ldots, v_k)$ is a walk in D, then it may happen that $|\varphi(v_1)| = \ldots = |\varphi(v_k)|$, in which case $(v_k, v_{k-1}, \ldots, v_1)$ is also a walk in D. In particular, D may contain cycles in which for each arc (u, v) the digraph D also contains the oppositely oriented arc (v, u). If G is a graph and D is the digraph obtained from G by replacing each edge with a pair of oppositely oriented arcs, then $DIN(D, \leq)$ equals the minimum cardinality of a set U such that G admits an intersection representation over the set U such that vertices in each component of G are assigned sets with the same cardinality (see, e.g., [4, 5, 19]).

The *cardinality* of a weak directed intersection representation (U, φ) of D is defined as the number of colors, that is, $|U|$. It is not hard to see that every DAG admits a weak directed intersection representation, e.g., (i) use a distinct color for each arc; (ii) fix a topological sorting (using the fact that it is a DAG); (iii) add to each vertex a distinct set of colors so that the cardinality of the color set of the vertices strictly increase along the chosen topological sorting[3]. The *weak directed intersection number* of a DAG $D = (V, A)$, denoted

[3] Note that the coloring defined by such procedure is also a (non-weak) directed intersection representation of the DAG. In fact, for a DAG, every weak directed intersection representation is also a directed intersection representation (see Lemma 7). Note, however, that the converse is not true as there are directed intersection representation of a DAG where non-adjacent vertices have intersecting color set of the same cardinality that are not weak directed intersection representation of the same graph.

by $DIN(D, \leq)$, is the smallest cardinality of a weak directed intersection representation of D. The weak directed intersection number has been previously considered in [20], where it was showed that the problem of computing $DIN(D, \leq)$ is NP-hard when D is an arbitrary DAG but polynomially solvable if D is an arborescence, which is in contrast with the NP-hardness of computing $DIN(D)$ for arborescences.

An **intersection representation of an undirected graph** $G = (V, E)$ is a pair (U, φ) where U is a finite set and φ is a mapping assigning to each vertex $v \in V$ a set $\varphi(v) \subseteq U$ such that for any two distinct vertices $u, v \in V$, it holds that $\{u, v\} \in E$ if and only if $\varphi(u) \cap \varphi(v) \neq \emptyset$. The *cardinality* of an intersection representation (U, φ) of G is defined as the cardinality of U. The *intersection number* of an undirected graph G, denoted by $IN(G)$, is the smallest cardinality of an intersection representation of G.

A *partially ordered graph* is a pair $(G, \preceq)$, where $G = (V, E)$ is an undirected graph and $\preceq$ is a partial order on the vertex set of G (that is, $(V, \preceq)$ is a poset). An **intersection representation of a partially ordered graph** $(G, \preceq)$ is a pair (U, φ) where U is a finite set of *colors* and φ is a *proper coloring* of $(G, \preceq)$, that is, a mapping assigning to each vertex $v \in V$ a set $\varphi(v) \subseteq U$ such that for any two distinct vertices $u, v \in V$, it holds that

- $\{u, v\} \in E$ if and only if $\varphi(u) \cap \varphi(v) \neq \emptyset$;
- if $u \prec v$ then $|\varphi(u)| < |\varphi(v)|$.

The *cardinality* of an intersection representation (U, φ) of a partially ordered graph $(G, \preceq)$ is defined as the cardinality of U. The *intersection number* of a partially ordered graph $(G, \preceq)$, denoted by $IN(G, \preceq)$, is the smallest cardinality of an intersection representation of $(G, \preceq)$.

Let $G = (V, E)$ be a graph and let $\ell : V \to \mathbb{Z}_+$ be a demand function on the vertices of G. An **ℓ-constrained intersection representation of an undirected graph** G is an intersection representation (U, φ) of G such that $|\varphi(v)| \geq \ell(v)$ for all $v \in V$. The *ℓ-constrained intersection number* of G, denoted by $IN(G, \ell)$, is the smallest cardinality of an ℓ-constrained intersection representation of G.

One can view the ℓ-constrained intersection number of a graph G and the intersection number of a partially ordered graph $(G, \preceq)$ as constrained variants of the intersection number of the graph G, placing it in a general framework proposed by Roberts in [16] along with numerous other variants of the intersection number studied in the literature.

When discussing algorithms on partially ordered graphs, we assume that a partially ordered graph $(G, \preceq)$ is represented with the adjacency lists of the graph G and an arbitrary DAG D on the same vertex set such that $u \preceq v$ if and only if there exists a directed u, v-path from u to v in D.

3 The DIN of triangle-free Hamiltonian DAGs

In this section, we present our main result about the polynomial computation of $DIN(D)$ (including a corresponding optimal directed intersection representation (U, φ) for D) for a triangle-free Hamiltonian DAG D. We prove the following theorem.

▶ **Theorem 1.** *Let $D = (V, A)$ be a triangle-free DAG with a Hamiltonian path $P = (v_1, \ldots, v_n)$. Let $w : V \to \mathbb{Z}_+$ be the vertex weight function on D defined recursively along P as follows: $w(v_1) = \deg(v_1)$, and for all $i \in \{2, \ldots, n\}$, we set $w(v_i) = \max\{w(v_{i-1}) + 1, \deg(v_i)\}$. Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of D defined by setting $b(v) = w(v) - \deg(v)$ for all $v \in V$ and let G be the underlying graph of D. Then $DIN(D) = |A| + b(V) - \nu(G, b)$ and a directed intersection representation (U, φ) of D with minimum cardinality can be computed in time $\mathcal{O}(|V|^3)$.*

3.1 Proof of Theorem 1

The proof of Theorem 1 involves several intermediate steps that allow us to state relationships among the variants of intersection representations introduced above. Here, we present these steps and the relationships they involve among variants of *IN* and *DIN*, and defer the proofs to the following subsections.

We first show that a weak directed intersection representation is equivalent to an intersection representation for any DAG that is Hamiltonian.

▶ **Lemma 2.** *Let $D = (V, A)$ be a Hamiltonian DAG. Then, $DIN(D) = DIN(D, \leq)$.*

We now show that computing a weak directed representation of a DAG D is equivalent to computing an intersection representation of a partially ordered graph $(G, \preceq)$, where G is the underlying graph of D and the partial order $\preceq$ is defined by the reachability relation in D.

▶ **Lemma 3.** *Let $D = (V, A)$ be a DAG. Let $G = (V, E)$ be the underlying graph of D and let $\preceq$ be the partial order on V defined by setting $u \preceq v$ if and only if there exists a u, v-path in D. Then, $DIN(D, \leq) = IN(G, \preceq)$.*

Then, we show that for a triangle-free graph G, and any given partial order $\preceq$ on the vertices of G, the computation of a weak intersection representation for the partially ordered graph $(G, \preceq)$ can be reduced to the computation of an ℓ-constrained intersection representation of G for a well defined and polynomially computable demand function ℓ.

▶ **Theorem 4.** *Let $(G, \preceq)$ be a partially ordered graph such that $G = (V, E)$ is triangle-free. Let M be the set of minimal elements in the poset $(V, \preceq)$. Let $\ell : V \to \mathbb{Z}_+$ be a demand function on the vertices of G defined by*

$$\ell(v) = \begin{cases} \deg(v) & \text{if } v \in M\,, \\ \max\left\{1 + \max_{u: u \prec v} \ell(u), \deg(v)\right\} & \text{if } v \notin M\,. \end{cases} \tag{2}$$

Then $IN(G, \preceq) = IN(G, \ell)$.

Figure 1 summarizes the relationships established by the above steps.

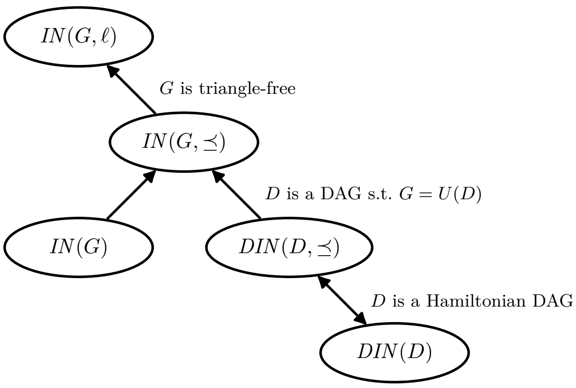

Figure 1 The relationships among the different types of intersection representations considered in this paper. The arc $Prob(a) \to Prob(b)$ is to be read "$Prob(a)$ is a special case of $Prob(b)$". A label on an arc specifies the restriction on the class of instances for which the relation is proved to hold.

Finally, we show that computing the ℓ-constrained intersection representation of a triangle-free graph G can be attained in polynomial time.

▶ **Theorem 5.** *Let $G = (V, E)$ be a triangle-free graph and $\ell : V \to \mathbb{Z}_+$ be a demand function on the vertices of G. Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of G defined by setting $b(v) = \max\{\ell(v) - \deg(v), 0\}$ for all $v \in V$. Then $IN(G, \ell) = |E| + b(V) - \nu(G, b)$ and an ℓ-constrained intersection representation (U, φ) of G with minimum cardinality can be computed in polynomial time, namely in time $\mathcal{O}(\min\{B|V|^2, |E|^2 \log |V| \log B\})$, where $B = \max_{v\in V} b(v)$.*

By following the above reductions in the opposite order, this final result extends to the computation of the directed intersection number of triangle-free Hamiltonian DAGs, i.e., it implies Theorem 1; we recall its statement and give a proof.

▶ **Theorem 1.** *Let $D = (V, A)$ be a triangle-free DAG with a Hamiltonian path $P = (v_1, \ldots, v_n)$. Let $w : V \to \mathbb{Z}_+$ be the vertex weight function on D defined recursively along P as follows: $w(v_1) = \deg(v_1)$, and for all $i \in \{2, \ldots, n\}$, we set $w(v_i) = \max\{w(v_{i-1}) + 1, \deg(v_i)\}$. Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of D defined by setting $b(v) = w(v) - \deg(v)$ for all $v \in V$ and let G be the underlying graph of D. Then $DIN(D) = |A| + b(V) - \nu(G, b)$ and a directed intersection representation (U, φ) of D with minimum cardinality can be computed in time $\mathcal{O}(|V|^3)$.*

Proof. Since D is a Hamiltonian DAG, Lemma 2 implies that $DIN(D) = DIN(D, \leq)$. Let $G = (V, E)$ denote the underlying graph of D and $\preceq$ the partial order on V such that $u \preceq v$ if and only if there exists a u, v-path in D. By Lemma 3, we obtain that $DIN(D, \leq) = IN(G, \preceq)$. We define the demand function $\ell : V \to \mathbb{Z}_+$ as in Theorem 4, which, by Theorem 4, implies that $IN(G, \preceq) = IN(G, \ell)$. Finally, Theorem 5 guarantees that $IN(G, \ell) = |E| + b(V) - \nu(G, b)$ and thus that $DIN(D) = |A| + b(V) - \nu(G, b)$, as claimed. ◀

As a corollary of Theorem 5, we also obtain a characterization of the intersection number of a partially ordered triangle-free graph.

▶ **Corollary 6.** *Let $(G, \preceq)$ be a partially ordered graph such that $G = (V, E)$ is triangle-free. Let M be the set of minimal elements in the poset $(V, \preceq)$. Let $w : V \to \mathbb{Z}_+$ be the vertex weight function on G defined by*

$$w(v) = \begin{cases} \deg(v) & \text{if } v \in M\,, \\ \max\left\{1 + \max\limits_{u:u\prec v} w(u), \deg(v)\right\} & \text{if } v \notin M\,. \end{cases} \tag{3}$$

Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of G defined by setting $b(v) = w(v) - \deg(v)$ for all $v \in V$. Then $IN(G, \preceq) = |E| + b(V) - \nu(G, b)$ and an intersection representation (U, φ) of $(G, \preceq)$ with minimum cardinality can be computed in time $\mathcal{O}(|V|^3)$.

3.2 On the relationships between DIN and weak DIN of DAGs

We first show that for arbitrary DAGs, the weak directed intersection number is always an upper bound on the directed intersection number.

▶ **Lemma 7.** *Let $D = (V, A)$ be a DAG. Then, $DIN(D) \leq DIN(D, \leq)$.*

Proof. The claim will follow from showing that any weak directed intersection representation of D is a directed intersection representation of D.

Let (U, φ) be a weak directed intersection representation of D. Then, $(u, v) \in A$ if and only if $\varphi(u) \cap \varphi(v) \neq \emptyset$ and $|\varphi(u)| \leq |\varphi(v)|$. Moreover, since D is a DAG, whenever $(u, v) \in A$, there is no arc (v, u), hence the inequality between the color sets' cardinalities must be strict, i.e., $|\varphi(u)| < |\varphi(v)|$. If $(u,v) \notin A$, then either $\varphi(u) \cap \varphi(v) = \emptyset$ or $|\varphi(u)| > |\varphi(v)|$, which implies that either $\varphi(u) \cap \varphi(v) = \emptyset$ or $|\varphi(u)| \geq |\varphi(v)|$. Hence, (U,φ) is also a directed intersection representation of D. ◀

▶ **Lemma 2.** *Let $D = (V,A)$ be a Hamiltonian DAG. Then, $DIN(D) = DIN(D, \leq)$.*

Proof. By Lemma 7, it suffices to show that if D is Hamiltonian, then any directed intersection representation of D is a weak directed intersection representation of D.

Let (U,φ) be a directed intersection representation of D. Clearly, we have that for each $(u,v) \in A$ the fact that $\varphi(u) \cap \varphi(v) \neq \emptyset$ and $|\varphi(u)| < |\varphi(v)|$ implies that

$$\varphi(u) \cap \varphi(v) \neq \emptyset \text{ and } |\varphi(u)| \leq |\varphi(v)|. \tag{4}$$

Moreover, the existence of a Hamiltonian path in D implies that for each pair of vertices u,v we have $|\varphi(u)| \neq |\varphi(v)|$. Therefore, if $(u,v) \notin A$, since (U,φ) is a directed intersection representation, at least one of the following must hold: (i) $|\varphi(u)| \geq |\varphi(v)|$, and hence $|\varphi(u)| > |\varphi(v)|$; (ii) $\varphi(u) \cap \varphi(v) = \emptyset$. The fact that (4) holds for each arc (u,v) and (i) or (ii) holds whenever $(u,v) \notin A$ implies that (U,φ) is a weak directed intersection representation of D. ◀

3.3 A relationship between weak directed intersection representations of DAGs and intersection representations of partially ordered graphs

▶ **Lemma 3.** *Let $D = (V,A)$ be a DAG. Let $G = (V,E)$ be the underlying graph of D and let $\preceq$ be the partial order on V defined by setting $u \preceq v$ if and only if there exists a u,v-path in D. Then, $DIN(D, \leq) = IN(G, \preceq)$.*

Proof. Let (U,φ) be a weak directed intersection representation of D.

▷ Claim 8. Fix distinct vertices u,v. If there is a u,v-path then $|\varphi(u)| < |\varphi(v)|$.

Proof. Let (u,v) be an arc of D. Then, we have $|\varphi(u)| \leq |\varphi(v)|$ and $\varphi(u) \cap \varphi(v) \neq \emptyset$. The latter, together with $(v,u) \notin A$ (since D is a DAG), implies $|\varphi(u)| < |\varphi(v)|$. The claim now follows by transitivity, repeatedly using the above argument on the arcs of a u,v-path. ◁

Let us show that (U,φ) satisfies the two properties defining an intersection representation of $(G,\preceq)$. Let u,v be a pair of distinct vertices in G.

1. Assume that $u \prec v$. By the definition of $\prec$ there is a u,v-path in D, hence by the claim above we have $|\varphi(u)| < |\varphi(v)|$.
2. Let $\{u,v\} \in E$. Then, either $(u,v) \in A$ or $(v,u) \in A$, and in either case, $\varphi(u) \cap \varphi(v) \neq \emptyset$. Similarly, if $\varphi(u) \cap \varphi(v) \neq \emptyset$, then either (u,v) or (v,u) is an arc of D, which implies that $\{u,v\} \in E$.

Let us now assume that (U,φ) is an intersection representation of $(G,\preceq)$. Then, for each pair of distinct vertices u,v of D we have the following.

- If $(u,v) \in A$ then: (i) $\{u,v\} \in E$; and (ii) $u \prec v$. Hence, from (i) and (ii) respectively, $\varphi(u) \cap \varphi(v) \neq \emptyset$, and $|\varphi(u)| < |\varphi(v)|$.
- Assume now that: (i) $\varphi(u) \cap \varphi(v) \neq \emptyset$, and (ii) $|\varphi(u)| < |\varphi(v)|$. From (i) we have $\{u,v\} \in E$. Since G is the underlying graph of D and D is a DAG, it follows that exactly one of the arcs $(u,v),(v,u)$ is in A. However, we cannot have $(v,u) \in A$, for otherwise $v \prec u$, which, together with (U,φ) being an intersection representation of $(G,\preceq)$, would contradict the hypothesis $|\varphi(u)| < |\varphi(v)|$. Therefore, we must have $(u,v) \in A$.

The two items imply that (U,φ) is a weak directed intersection representation of D and conclude the proof. ◀

3.4 From partially ordered triangle-free graphs to ℓ-constrained triangle-free graphs

▶ **Theorem 4.** *Let $(G, \preceq)$ be a partially ordered graph such that $G = (V, E)$ is triangle-free. Let M be the set of minimal elements in the poset $(V, \preceq)$. Let $\ell : V \to \mathbb{Z}_+$ be a demand function on the vertices of G defined by*

$$\ell(v) = \begin{cases} \deg(v) & \text{if } v \in M\,, \\ \max\left\{1 + \max_{u:u\prec v} \ell(u), \deg(v)\right\} & \text{if } v \notin M\,. \end{cases} \tag{2}$$

Then $IN(G, \preceq) = IN(G, \ell)$.

Proof. The claim will follow from showing that any intersection representation of $(G, \preceq)$ is an ℓ-constrained intersection representation of G, and that there exists an ℓ-constrained intersection representation of G with minimum cardinality that is also an intersection representation of $(G, \preceq)$.

Let (U, φ) be an intersection representation of $(G, \preceq)$. To show that (U, φ) is an ℓ-constrained intersection representation of G, we need to show that for each vertex $v \in V$, we have $|\varphi(v)| \geq \ell(v)$. By the definition of an intersection representation of a partially ordered graph, we have that

$$|\varphi(v)| > |\varphi(u)| \text{ for each } u \prec v. \tag{5}$$

Moreover, because of the triangle-free condition we have that the neighborhood of v is an independent set. Again, by the definition of intersection representation, this implies that the corresponding color sets are pairwise disjoint. On the other hand, each one of these color sets has a nonempty intersection with $\varphi(v)$. It follows that $|\varphi(v)| \geq \deg(v)$. This, together with Equation (5), implies that $|\varphi(v)| \geq \ell(v)$. Thus, (U, φ) is an ℓ-constrained intersection representation of G.

Assume now that (U, φ) is an ℓ-constrained intersection representation of G such that $|U| = IN(G, \ell)$. For each edge $e = \{u, v\} \in E$, we have $\varphi(u) \cap \varphi(v) \neq \emptyset$; in particular, there exists a color $c_e \in \varphi(u) \cap \varphi(v)$. We show next that for any two distinct edges $e, e' \in E$, we have $c_e \neq c_{e'}$. Suppose for a contradiction that there exist two different edges $e = \{u, v\} \in E$ and $e' = \{x, y\} \in E$ such that $c_e = c_{e'}$. Since $e \neq e'$ and G does not contain loops or duplicated edges, the set $\{u, v, x, y\}$ has cardinality at least 3. Therefore, there is a color shared by at least three vertices, and, since the coloring is proper, this implies that any such three vertices form a triangle, contradicting the hypothesis that G is triangle-free.

Let us write $U_E = \{c_e : e \in E\}$ and for all $v \in V$, denote $\varphi'(v) = \varphi(v) \setminus U_E$. Then $|\varphi'(v)| = |\varphi(v)| - \deg(v)$. Since the representation is ℓ-constrained, we have for any vertex $v \in V$ that $|\varphi'(v)| = |\varphi(v)| - \deg(v) \geq \ell(v) - \deg(v) \geq 0$. For each $v \in V$ such that $|\varphi'(v)| > \ell(v) - \deg(v)$, we select an arbitrary set $X_v \subseteq \varphi'(v)$ such that $|X_v| = \ell(v) - \deg(v)$. Then, we define, for each $v \in V$

$$\psi(v) = \begin{cases} (\varphi(v) \cap U_E) \cup X_v, & \text{if } |\varphi'(v)| > \ell(v) - \deg(v)\,, \\ \varphi(v), & \text{otherwise.} \end{cases}$$

Similarly as above, let us denote $\psi'(v) = \psi(v) \setminus U_E$ for all $v \in V$. By construction, we have $|\psi'(v)| = \ell(v) - \deg(v)$ for all $v \in V$ and consequently $|\psi(v)| = |\psi'(v)| + \deg(v) = \ell(v)$ for all $v \in V$. Note that the recursive definition of the function ℓ implies that $\ell(u) < \ell(v)$ whenever $u \prec v$, and hence $|\psi(u)| < |\psi(v)|$ whenever $u \prec v$. Next, we show that (U, ψ) is an intersection representation of $(G,\preceq)$. To this end, it remains to show that $\{u,v\}\in E$ if and only if $\psi(u)\cap\psi(v)\neq\emptyset$. If $\{u,v\}=e\in E$ then $c_e\in\varphi(u)\cap\varphi(v)\cap U_E\subseteq\psi(u)\cap\psi(v)$ since we obtained the mapping ψ from φ by only removing colors not in U_E. If $\{u,v\}\notin E$ then $\psi(u)\cap\psi(v)=\emptyset$ follows directly from the relations $\psi(u)\subseteq\varphi(u)$ and $\psi(v)\subseteq\varphi(v)$, using the fact that $\varphi(u)\cap\varphi(v)=\emptyset$. This shows that (U,ψ) is an intersection representation of $(G,\preceq)$. Furthermore, since $|\psi(v)|=\ell(v)$ for all $v\in V$ and $|U|=IN(G,\ell)$, the pair (U,ψ) is an ℓ-constrained intersection representation of G with minimum cardinality. This completes the proof. ◀

3.5 The ℓ-constrained intersection number of triangle-free graphs

▶ **Theorem 5.** *Let $G=(V,E)$ be a triangle-free graph and $\ell:V\to\mathbb{Z}_+$ be a demand function on the vertices of G. Let $b:V\to\mathbb{Z}_+$ be a capacity function on the vertices of G defined by setting $b(v)=\max\{\ell(v)-\deg(v),0\}$ for all $v\in V$. Then $IN(G,\ell)=|E|+b(V)-\nu(G,b)$ and an ℓ-constrained intersection representation (U,φ) of G with minimum cardinality can be computed in polynomial time, namely in time $\mathcal{O}(\min\{B|V|^2,|E|^2\log|V|\log B\})$, where $B=\max_{v\in V}b(v)$.*

Proof. First, we prove that $|E|+b(V)-\nu(G,b)$ is an upper bound on the ℓ-constrained intersection number of the graph G by constructing an ℓ-constrained intersection representation (U,φ) with cardinality at most $|E|+b(V)-\nu(G,b)$. We associate to each edge $e\in E$ a unique color c_e and define $U_E=\{c_e:e\in E\}$. Let $x:E\to\mathbb{Z}_+$ be a maximum weight b-matching in G. Then $\sum_{e\in E}x(e)=\nu(G,b)$. We denote by E_v the set of edges in G incident with a vertex v. To each edge e of G, we associate another set C_e of $x(e)$ new colors, and to each vertex v of G, we associate a set C_v of $b(v)-\sum_{e\in E_v}x(e)$ new colors, so that the sets U_E, C_e, $e\in E$, and C_v, $v\in V$, are pairwise disjoint. Note that this construction is indeed possible: for each vertex v, the value of $b(v)-\sum_{e\in E_v}x(e)$ is a non-negative integer since x is a b-matching in G. We define

$$U=U_E\cup\left(\bigcup_{e\in E}C_e\right)\cup\left(\bigcup_{v\in V}C_v\right)\quad\text{and}\quad\varphi(v)=\{c_e:e\in E_v\}\cup C_v\cup\left(\bigcup_{e\in E_v}C_e\right).\quad(6)$$

Next, we show that (U,φ) is an ℓ-constrained intersection representation of G. Clearly $\varphi(v)\subseteq U$ for all $v\in V$. Furthermore, for each vertex $v\in V$, by definition it holds that $b(v)+\deg(v)\geq\ell(v)$, and we have

$$\begin{aligned}|\varphi(v)|=|E_v|+|C_v|+\sum_{e\in E_v}|C_e|&=\deg(v)+b(v)-\sum_{e\in E_v}x(e)+\sum_{e\in E_v}x(e)\\&=\deg(v)+b(v)\geq\ell(v)\,.\end{aligned}\quad(7)$$

Therefore, the color sets assigned to the vertices by φ satisfy the lower bounds defined by the demand function ℓ.

We now prove that (U,φ) is an intersection representation of G, by showing that for any distinct vertices u and v of G, it holds that $(u,v)\in E$ if and only if $\varphi(v_i)\cap\varphi(v_j)\neq\emptyset$.

For this, assume first that $e=\{u,v\}\in E$. Then $e\in E_u\cap E_v$ and thus $c_e\in\varphi(u)\cap\varphi(v)$, implying that $\varphi(u)\cap\varphi(v)\neq\emptyset$. For the converse direction, assume that $\varphi(u)\cap\varphi(v)\neq\emptyset$. Let $c\in\varphi(u)\cap\varphi(v)$. Because for any two vertices $u,v\in V$, we have that $C_u\subseteq\varphi(v)$ if and only if $u=v$, the color c cannot belong to any set C_z for $z\in V$. Therefore, since the sets U_E, and C_e, $e\in E$, are pairwise disjoint, the color c must belong to either U_E or to some set C_e for $e\in E$. Assume first that $c\in U_E$. Then $c=c_e$ for some $e\in E$, which implies that

$e \in E_u \cap E_v$ and thus $e = \{u, v\}$, i.e., $\{u, v\}$ is an edge of E. Assume now that $c \in C_e$ for some $e \in E$. Since $c \in \varphi(u) \cap \varphi(v)$, we infer that $e \in E_u$ and $e \in E_v$, which again implies that $\{u, v\}$ is an edge of G. This concludes the proof that φ is a proper coloring of (G, φ).

It remains to show that $|U| = |E| + b(V) - \nu(G, b)$. Using the definition of U we infer that, as claimed, its cardinality is

$$\begin{aligned} |U| &= |U_E| + \sum_{e\in E} |C_e| + \sum_{v\in V} |C_v| = |E| + \sum_{e\in E} x_e + \sum_{v\in V} \left(b(v) - \sum_{e\in E_v} x(e) \right) \\ &= |E| + \nu(G, b) + \sum_{v\in V} b(v) - \sum_{v\in V} \sum_{e\in E_v} x(e) = |E| + \nu(G, b) + b(V) - 2\sum_{e\in E} x(e) \\ &= |E| + \nu(G, b) + b(V) - 2\nu(G, b) = |E| + b(V) - \nu(G, b). \end{aligned}$$

Second, we prove that $|E| + b(V) - \nu(G, b)$ is a lower bound for the ℓ-constrained intersection number of G. Let (U, φ) be an arbitrary ℓ-constrained intersection representation of G. We need to show that $|U| \geq |E| + b(V) - \nu(G, b)$. For each edge $e = \{u, v\} \in E$, we have $\varphi(u) \cap \varphi(v) \neq \emptyset$; in particular, there exists a color $c_e \in \varphi(u) \cap \varphi(v)$. We show next that for any two distinct edges $e, e' \in E$, we have $c_e \neq c_{e'}$. Suppose for a contradiction that there exist two different edges $e = \{u, v\} \in E$ and $e' = \{x, y\} \in E$ such that $c_e = c_{e'}$. Since $e \neq e'$ and G does not contain loops or duplicated edges, the set $\{u, v, x, y\}$ has cardinality at least 3. Therefore, there is a color shared by at least three vertices, and, since the coloring is proper, this implies that any such three vertices form a triangle, contradicting the hypothesis that G is triangle-free.

Because of the triangle-free condition we have that the neighborhood of v is an independent set. By the definition of ise , this implies that the corresponding color sets are pairwise disjoint. On the other hand, each one of these color sets has a nonempty intersection with $\varphi(v)$. It follows that $|\varphi(v)| \geq \deg(v)$. By the assumption that (U, φ) is an ℓ-constrained intersection representation of G we also have $|\varphi| \geq \ell(v)$, hence $|\varphi| \geq \max\{\ell(v), \deg(v)\}$.

For all $v \in V$, let us denote $\varphi'(v) = \varphi(v) \setminus U_E$. Then $|\varphi'(v)| = |\varphi(v)| - \deg(v) \geq 0$. It follows that for any vertex $v \in V$, we have that

$$|\varphi'(v)| = |\varphi(v)| - \deg(v) \geq \max\{0, \ell(v) - \deg(v)\} = b(v).$$

▷ Claim 9. We may assume without loss of generality that $|\varphi'(v)| = b(v)$ for all $v \in V$.

Proof. Suppose that this is not the case. Then, we choose for each $v \in V$ such that $|\varphi'(v)| > b(v)$, an arbitrary set $X_v \subseteq \varphi'(v)$ such that $|X_v| = b(v)$ and define, for all $v \in V$

$$\psi(v) = \begin{cases} (\varphi(v) \cap U_E) \cup X_v, & \text{if } |\varphi'(v)| > b(v)\,, \\ \varphi(v), & \text{otherwise.} \end{cases}$$

Similarly as above, let us denote $\psi'(v) = \psi(v) \setminus U_E$ for all $v \in V$. By construction, we have $|\psi'(v)| = b(v)$ for all $v \in V$ and consequently $|\psi(v)| = |\psi'(v)| + \deg(v) \geq \ell(v)$ for all $v \in V$. Hence, ψ satisfies the ℓ-constraints on the vertices. To see that (U, ψ) is indeed an ℓ-constrained intersection representation of G, it remains to show that $\{u, v\} \in E$ if and only if $\psi(u) \cap \psi(v) \neq \emptyset$. If $\{u, v\} = e \in E$ then $c_e \in \varphi(u) \cap \varphi(v) \cap U_E \subseteq \psi(u) \cap \psi(v)$ since we obtained the mapping ψ from φ by only removing colors not in U_E. If $\{u, v\} \notin E$ then $\psi(u) \cap \psi(v) = \emptyset$ follows directly from the relations $\psi(u) \subseteq \varphi(u)$ and $\psi(v) \subseteq \varphi(v)$, using the fact that $\varphi(u) \cap \varphi(v) = \emptyset$. This shows that (U, ψ) is an intersection representation of G and completes the proof of the claim. ◁

Let $U_E = \{c_e : e \in E\}$. Then $|U_E| = |E|$. To complete the proof of the lower bound, we need to show that $|U \setminus U_E| \geq b(V) - \nu(G, b)$, or, equivalently, that $b(V) \leq |U \setminus U_E| + \nu(G, b)$. Consider the function $x : E \to \mathbb{Z}_+$ defined by setting

$$x(e) = |\varphi'(u) \cap \varphi'(v)| \qquad \text{for each edge } e = \{u, v\} \in E\,.$$

We claim that x is a b-matching of G. Let v be a vertex of G. Let E_v be the set of edges in G that are incident with v and let $N(v)$ be the set of neighbors of v in G. Recall that by the assumption that the graph is triangle-free, each color $c \in U \setminus U_E$ appears in at most two of the color sets $\varphi'(u)$, $u \in V$. Therefore, for each vertex $v \in V$, the sets $\varphi'(u) \cap \varphi'(v)$, $u \in N(v)$, are pairwise disjoint. This implies that

$$\sum_{e \in E_v} x(e) = \sum_{u \in N(v)} |\varphi'(u) \cap \varphi'(v)| = \left|\varphi'(v) \cap \left(\bigcup_{u \in N(v)} \varphi'(u)\right)\right| \leq |\varphi'(v)| = b(v)\,,$$

and hence x is indeed a b-matching of D. For each $v \in V$, let us denote by C_v the set of *private* colors of v, that is, those colors $c \in \varphi'(v)$ such that $c \notin \varphi'(u)$ for all $u \in V \setminus \{v\}$. We have

$$\begin{aligned} b(V) &= \sum_{v \in V} b(v) = \sum_{v \in V} |\varphi'(v)| = \sum_{v \in V} (|C_v| + |\varphi'(v) \setminus C_v|) \\ &= \sum_{v \in V} |C_v| + \sum_{v \in V} \sum_{u \in N(v)} |\varphi'(u) \cap \varphi'(v)| = \sum_{v \in V} |C_v| + 2 \sum_{e \in E} x(e) \\ &= \left(\sum_{v \in V} |C_v| + \sum_{e \in E} x(e)\right) + \sum_{e \in E} x(e) \leq |U \setminus U_E| + \nu(G, b)\,, \end{aligned}$$

where the last inequality follows from the fact that each color in $U \setminus U_E$ is counted exactly once in the sum $\sum_{v \in V} |C_v| + \sum_{e \in E} x(e)$ and that $\sum_{e \in E} x(e) \leq \nu(G, b)$, since x is a b-matching in D.

Finally, we observe that an ℓ-constrained intersection representation (U, φ) of G with minimum cardinality can be computed in polynomial time. First, we compute the capacity function b according to the definition. Including also the time for the computation of the vertex degrees, this can be done in time $\mathcal{O}(|V| + |E|) = \mathcal{O}(|V|^2)$. Then we compute a maximum weight b-matching x in G for which we can either use the algorithm by Pulleyblank [14] (see also [18]), that requires time $\mathcal{O}(B|V|^2)$, or (when B is superpolynomial in $|V|$ and/or for the case of G being sparse) the algorithm of Gabow [8] which runs in $\mathcal{O}(|E|^2 \log |V| \log B)$. Finally, we use Equations (6) to compute an intersection representation (U, φ) of $(G, \preceq)$ with cardinality $|E| + b(V) - \nu(G, b)$. This can be done in time proportional to the total size of this representation, which is $\sum_{v \in V} |\varphi(v)| = \sum_{v \in V} w(v) = \mathcal{O}(|V|^2)$. Since we can choose the best of the two above options for the computation of maximum weight b-matching x, we conclude that the overall running time satisfies $\mathcal{O}(\min\{B|V|^2, |E|^2 \log |V| \log B\})$. ◀

3.6 Intersection number of triangle-free partially ordered graphs

▶ **Corollary 6.** *Let $(G, \preceq)$ be a partially ordered graph such that $G = (V, E)$ is triangle-free. Let M be the set of minimal elements in the poset $(V, \preceq)$. Let $w : V \to \mathbb{Z}_+$ be the vertex weight function on G defined by*

$$w(v) = \begin{cases} \deg(v) & \text{if } v \in M\,, \\ \max\left\{1 + \max\limits_{u : u \prec v} w(u), \deg(v)\right\} & \text{if } v \notin M\,. \end{cases} \tag{3}$$

Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of G defined by setting $b(v) = w(v) - \deg(v)$ for all $v \in V$. Then $IN(G, \preceq) = |E| + b(V) - \nu(G, b)$ and an intersection representation (U, φ) of $(G, \preceq)$ with minimum cardinality can be computed in time $\mathcal{O}(|V|^3)$.

Proof. The claim $IN(G, \preceq) = |E| + b(V) - \nu(G, b)$ follows directly from observing that by Theorem 4, upon defining a demand function ℓ on the vertices of G such that $\ell(v) = w(v)$ for each $v \in V$, we have $IN(G, \preceq) = IN(G, \ell)$. Moreover, by Theorem 5, we have $IN(G, \ell) = |E| + b(V) - \nu(G, b)$.

Finally, we show that a directed intersection representation (U, φ) of $(G, \preceq)$ with minimum cardinality can be computed in time $\mathcal{O}(|V|^3)$. For this, we first observe that the weight function w according to the definition given by Equation (3) and the capacity function b can be computed in time $\mathcal{O}(|V| + |A|) = \mathcal{O}(|V|^2)$ where $D = (V, A)$ is the DAG representing the poset $(V, \preceq)$. Then, we note that from the definition, it follows that for each vertex $v \in V$, we have $b(v) \leq w(v) \leq \Delta + L$ where Δ denotes the maximum vertex degree in G and L denotes the maximum length (that is, the number of arcs) in a directed path in D. Therefore $B \leq 2(|V| - 1)$, hence $B|V|^2 = O(|V|^3)$, which implies that, using the algorithm by Pulleyblank [14], we can compute a maximum weight b-matching in G in time $\mathcal{O}(|V|^3)$. And in particular, the bound on the construction of (U, φ) follows from Theorem 4, using $B|V|^2 = O(|V|^3)$. ◀

4 Some final observations and open questions

The bipartite case. In a bipartite graph G, the maximum size of a b-matching is equal to the minimum b-weight of a vertex cover (see, e.g., [18]). Using this, we can give an alternative expression of the result in Theorem 1 in the case of bipartite Hamiltonian DAGs.

▶ **Theorem 10.** *Let $D = (V, A)$ be a bipartite DAG with a Hamiltonian path $P = (v_1, \ldots, v_n)$. Let $w : V \to \mathbb{Z}_+$ be a vertex weight function on D defined recursively along P as follows: $w(v_1) = \deg(v_1)$, and for all $i \in \{2, \ldots, n\}$, we set $w(v_i) = \max\{w(v_{i-1}) + 1, \deg(v_i)\}$. Let $b : V \to \mathbb{Z}_+$ be a capacity function on the vertices of D defined by setting $b(v) = w(v) - \deg(v)$ for all $v \in V$. Then $DIN(D) = |A| + \alpha(D, b)$.*

$\mathcal{O}(1)$-approximations. Before this paper, the only class of graphs for which a constant approximation polynomial algorithm was known for computing the *DIN* was the class of arborescences. As a consequence of our results, we can significantly widen the class of DAGs where the problem admits a constant approximation as recorded in the following observation (the details are deferred to the full version of the paper).

▶ **Observation 11.** *The DIN can be approximated in polynomial time to a constant factor on DAGs obtained from graphs with bounded chromatic number and bounded path cover number by orienting the edges along a bounded path cover.*

Argument: such graphs need $\Omega(n^2)$ colors, which matches the algorithmic bound from Liu et al. [11] to within a constant. The lower bound can be argued by taking a path with linearly many vertices and finding a constant-fraction independent set along that path.

Other open questions are about the extent, in terms of graph classes, to which the relationships between the different intersection representations hold (see the diagram in Figure 1). For example, does the fact that the ℓ-constrained intersection number of a graph G generalizes the computation of intersection number over any partially ordered graph $(G, \preceq)$ hold beyond the class of triangle-free graphs? In particular, we can show that this is true in the larger class of diamond-free graphs, where the diamond is the graph obtained from the 4-vertex complete graph by removing an edge (details are deferred to the full version of the paper). However, for assessing the complexity of such a reduction one would need to determine the complexity of the following problem: Given a diamond-free graph G and a function $b : V(G) \to \mathbb{Z}_+$, find a function $x : \mathcal{C}(G) \to \mathbb{Z}_+$, where $\mathcal{C}(G)$ is the set of maximal cliques of G, such that for all $v \in V(G)$, the sum of the values $x(C)$ over all maximal cliques C containing v does not exceed $b(v)$, and the sum $\sum_{C \in \mathcal{C}(G)} x(C)$ is maximized.

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