Proportional Allocation of Indivisible Goods up to the Least Valued Good on Average

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Abstract

We study the problem of fairly allocating a set of indivisible goods to multiple agents and focus on the proportionality, which is one of the classical fairness notions. Since proportional allocations do not always exist when goods are indivisible, approximate notions of proportionality have been considered in the previous work. Among them, proportionality up to the maximin good (PROPm) has been the best approximate notion of proportionality that can be achieved for all instances. In this paper, we introduce the notion of *proportionality up to the least valued good on average* (PROPavg), which is a stronger notion than PROPm, and show that a PROPavg allocation always exists. Our results establish PROPavg as a notable non-trivial fairness notion that can be achieved for all instances. Our proof is constructive, and based on a new technique that generalizes the cut-and-choose protocol.

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1 Introduction

1.1 Proportional Allocation of Indivisible Goods

We study the problem of fairly allocating a set of indivisible goods to multiple agents under additive valuations. Fair division of indivisible goods is a fundamental and well-studied problem in Economics and Computer Science. We are given a set M of m indivisible goods and a set N of n agents with individual valuations. Under additive valuations, each agent $i \in N$ has value $v_i(\{g\}) \geq 0$ for each good g and her value for a bundle S of goods is equal to the sum of the value of each good $g \in S$, i.e., $v_i(S) = \sum_{g \in S} v_i(\{g\})$. An indivisible good can not be split among multiple agents and this causes finding a fair division to be a difficult task.

One of the standard notions of fairness is *proportionality*. Let $X = (X_1, X_2, \ldots, X_n)$ be an allocation, i.e., a partition of M into n bundles such that X_i is allocated to agent i. An allocation X is said to be *proportional (PROP)* if $v_i(X_i) \geq \frac{1}{n} v_i(M)$ holds for each agent i. In other words, in a proportional allocation, every agent receives a set of goods whose value is at least $1/n$ fraction of the value of the entire set. Unfortunately, proportional allocations do not always exist when goods are indivisible. For instance, when allocating a single indivisible good to more than one agents it is impossible to achieve any proportional allocation. Thus, several relaxations of proportionality such as PROP1, PROPx, and PROPm have been considered in the previous work.

Each of these notions requires that each agent $i \in N$ receives value at least $\frac{1}{n}v_i(M) - d_i(X)$, where $d_i(X)$ is appropriately defined for each notion. *Proportionality up to the largest valued good (PROP1)* is a relaxation of proportionality that was introduced by Conitzer et al. [17]. PROP1 requires $d_i(X)$ to be the largest value that agent i has for any good allocated to other agents, i.e., $d_i(X) = \max_{k \in N \setminus \{i\}} \max_{g \in X_k} v_i(\{g\})$. It is shown in [17] that there always exists a Pareto optimal[1] allocation that satisfies PROP1. Moreover, Aziz et al. [4] presented a polynomial-time algorithm that finds a PROP1 and Pareto optimal allocation even in the presence of chores, i.e., some items can have negative value.

Another relaxation is *proportionality up to the least valued good (PROPx)*, which is much stronger than PROP1. PROPx requires $d_i(X)$ to be the least value that agent i has for any good allocated to other agents, i.e., $d_i(X) = \min_{k \in N \setminus \{i\}} \min_{g \in X_k} v_i(\{g\})$. Moulin [26] gave an example for which no PROPx allocation exists, and Aziz et al. [4] gave a simpler example.

Recently, Baklanov et al. [5] introduced *proportionality up to the maximin good (PROPm)*. PROPm requires $d_i(X) = \max_{k \in N \setminus \{i\}} \min_{g \in X_k} v_i(\{g\})$, which shows that PROPm is the notion between PROP1 and PROPx. It is shown in [5] that a PROPm allocation always exists for instances with at most five agents, and later Baklanov et al. [6] showed that there always exists a PROPm allocation for any instance and it can be computed in polynomial time. To the best of our knowledge, PROPm has been the best approximate notion of proportionality that is shown to be achieved for all instances.

However, in some cases, PROPm is not a good enough relaxation of proportionality. Suppose that there exists a good $g \in M$ for which every agent has value at least $1/n$ fraction of the value of M. Then allocating g to some agent i and allocating all the goods in $M \setminus \{g\}$ to another agent achieves a PROPm allocation, whereas it will be better to allocate $M \setminus \{g\}$ to $N \setminus \{i\}$ in a fair manner (see Example 1). This motivates the study of better relaxations of proportionality than PROPm.

1.2 Our Contribution

In this paper, we introduce *proportionality up to the least valued good on average (PROPavg)*, a new relaxation of proportionality, and show that there always exists a PROPavg allocation for all instances. PROPavg requires $d_i(X)$ to be the average of minimum value that agent i has for any good allocated to other agents, i.e., $d_i(X) = \frac{1}{n-1}\sum_{k \in N \setminus \{i\}} \min_{g \in X_k} v_i(\{g\})$. It is easy to see that PROPavg implies PROPm. Note that a similar and slightly stronger notion was introduced by Baklanov et al. [5] with the name of Average-EFX (Avg-EFX), where $d_i(X) = \frac{1}{n}\sum_{k \in N \setminus \{i\}} \min_{g \in X_k} v_i(\{g\})$. Note that Avg-EFX is also an approximate notion of proportionality. It remains open whether an Avg-EFX allocation always exists. The following example demonstrates that PROPavg is a reasonable relaxation of proportionality compared to PROPm.

▶ **Example 1.** Suppose that $N = \{1, 2, 3\}$, $M = \{g_1, g_2, g_3, g_4\}$, and each agent has an identical additive valuation v such that $v(\{g_1\}) = 10, v(\{g_2\}) = v(\{g_3\}) = 7$, and $v(\{g_4\}) = 6$. As $v(\{g_1\}) \geq 10$, the allocation $(\{g_1, g_2, g_3, g_4\}, \emptyset, \emptyset)$ satisfies PROP1 even though agents 2 and 3 receive no good. Similarly, the allocation $(\{g_1\}, \{g_2, g_3, g_4\}, \emptyset)$ satisfies PROPm even though agent 3 receives no good. In contrast, every agent has to receive at least one good in any PROPavg allocation. Table 1 shows a comparison among some fairness notions (see Section 1.4 for the definition of EFX).

[1] An allocation $X = (X_1, \ldots, X_n)$ is Pareto optimal if there is no allocation $Y = (Y_1, \ldots, Y_n)$ such that $v_i(Y_i) \geq v_i(X_i)$ for any agent i, and there exists an agent j such that $v_j(Y_j) > v_j(X_j)$.

Table 1 Comparison among fairness notions in Example 1. The symbol "✓" (resp. "✗") indicates that the allocation satisfies (resp. does not satisfy) the corresponding fairness.

	EFX	PROPavg	PROPm	PROP1
$(\{g_1\}, \{g_2, g_4\}, \{g_3\})$	✓	✓	✓	✓
$(\{g_1\}, \{g_2, g_3\}, \{g_4\})$	✗	✓	✓	✓
$(\{g_1\}, \{g_2, g_3, g_4\}, \emptyset)$	✗	✗	✓	✓
$(\{g_1, g_2, g_3, g_4\}, \emptyset, \emptyset)$	✗	✗	✗	✓

Table 2 Relaxations of Proportionality.

	$d_i(X)$	Does it always exist?
PROPx	$\min_{k\in N\setminus\{i\}} \min_{g\in X_k} v_i(\{g\})$	No [4, 26]
Avg-EFX	$\frac{1}{n}\sum_{k\in N\setminus\{i\}} \min_{g\in X_k} v_i(\{g\})$	Open
PROPavg	$\frac{1}{n-1}\sum_{k\in N\setminus\{i\}} \min_{g\in X_k} v_i(\{g\})$	Yes **(our result)**
PROPm	$\max_{k\in N\setminus\{i\}} \min_{g\in X_k} v_i(\{g\})$	Yes [6]
PROP1	$\max_{k\in N\setminus\{i\}} \max_{g\in X_k} v_i(\{g\})$	Yes [17]

The main contribution of this paper is to show the existence of PROPavg allocations for all instances, which extends the existence of PROPm allocations shown by Baklanov et al. [6].

▶ **Theorem 2.** *There always exists a PROPavg allocation when each agent has a non-negative additive valuation.*

Known results on relaxations of proportionality are summarized in Table 2.

In order to prove Theorem 2, we provide an algorithm to find a PROPavg allocation. The running time of our algorithm is pseudo-polynomial, while Baklanov et al. [6] showed that a PROPm allocation can be computed in polynomial time. We discuss the time complexity in Section 5 in detail.

1.3 Our Techniques

Our algorithm can be seen as a generalization of *cut-and-choose* protocol, which is a well-known procedure to fairly allocate resources between two agents. In the cut-and-choose protocol, one agent partitions resources equally into two bundles for her valuation, and then the other agent chooses the best bundle of the two for her valuation. We generalize this protocol from two agents to n agents in the following way: some $n-1$ agents partition the goods into n bundles, and then the remaining agent chooses the best bundle among them for her valuation. To apply this protocol, it suffices to show that there exists a partition of the goods into n bundles such that no matter which bundle the remaining agent chooses, the remaining $n-1$ bundles can be allocated to the first $n-1$ agents fairly.

In our algorithm, we find such a partition by using an auxiliary graph called PROPavg-graph. A formal definition of the PROPavg-graph is given in Section 3, and our algorithm and its correctness proof are shown in Section 4. Let us emphasize that introducing the PROPavg-graph is a key technical ingredient in this paper. It is also worth noting that Hall's marriage theorem [21], a classical and famous theorem in discrete mathematics, plays an important role in our argument.

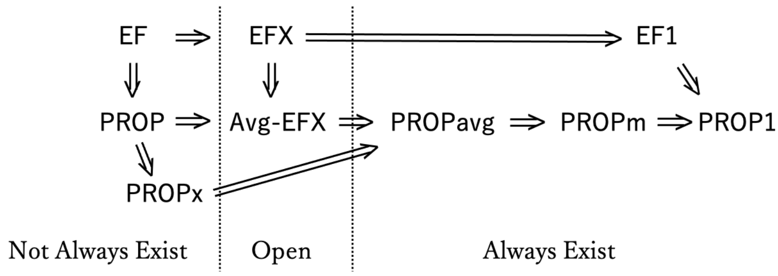

Figure 1 Relationship among some fairness notions. EF, PROP, or PROPx allocations do not always exist, while PROPavg, PROPm, EF1, and PROP1 can be achieved for all instances. It is not known whether EFX or Avg-EFX allocations always exist or not.

1.4 Related Work

Fair division of divisible resources is a classical topic starting from the 1940's [29] and has a long history in multiple fields such as Economics, Social Choice Theory, and Computer Science [9, 10, 25, 28]. In contrast, fair division of indivisible goods has actively studied in recent years (see, e.g., [2, 3]).

In the context of fair division, besides proportionality, *envy-freeness* is another well-studied notion of fairness. An allocation is called *envy-free (EF)* if for each agent, she receives a set of goods for which she has value at least value of the set of goods any other agent receives. As in the proportionality case, envy-free allocations do not always exist when goods are indivisible, and several relaxations of envy-freeness have been considered. Among them, a notable one is *envy-freeness up to one good (EF1)* [11]. It is known that there always exists an EF1 allocation, and it can be computed in polynomial time [22]. Another notable relaxation is *envy-freeness up to any good (EFX)* [13]. An allocation $X = (X_1, \ldots, X_n)$ is called EFX if for any pair of agents $i, j \in N$, $v_i(X_i) \geq v_i(X_j) - m_i(X_j)$, where $m_i(X_j)$ is the value of the least valuable good for agent i in X_j. It is one of the major open problems in fair division whether EFX allocations always exist or not. As mentioned in [5], it is easy to see that EFX implies Avg-EFX. As with EFX, it is not known whether Avg-EFX allocations always exist for instances with four or more agents. The relationship among notions mentioned above and the existence results are summarized in Figure 1.

There have been several studies on the existence of an EFX allocation for restricted cases. Plaut and Roughgarden [27] showed that an EFX allocation always exists for instances with two agents even when each agent can have more general valuations than additive valuations. Chaudhury et al. [14] showed that an EFX allocation always exists for instances with three agents. It is not known whether EFX allocations always exist even for instances with four agents having additive valuations. We can also consider the cases with restricted valuations. For example, there always exists an EFX allocation when valuations are identical [27], two types [23, 24], binary [7, 18], or bi-valued [1].

Another direction of research related to EFX is *EFX-with-charity*, in which unallocated goods are allowed. Obviously, without any constraints, the problem is trivial: leaving all goods unallocated results in an envy-free allocation. Thus, the goal here is to find allocations with better guarantees. For additive valuations, Caragiannis et al. [12] showed that there exists an EFX allocation with some unallocated goods where every agent receives at least half the value of her bundle in a maximum *Nash social welfare* allocation[2]. For normalized and monotone valuations, Chaudhury et al. [16] showed that there exist an EFX allocation and a set of unallocated goods U such that every agent has value for her own bundle at least value for U, and $|U| < n$. Berger et al. [8] showed that the number of the unallocated goods can be decreased to $n-2$, and to just one for the case of four agents having nice cancelable valuations, which are more general than additive valuations. Mahara [24] showed that the number of the unallocated goods can be decreased to $n-2$ for normalized and monotone valuations, which are more general than nice cancelable valuations. For additive valuations, Chaudhury et al. [15] presented a polynomial-time algorithm for finding an approximate EFX allocation with at most a sublinear number of unallocated goods and high Nash social welfare.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of n agents and M be a set of m goods. We assume that goods are indivisible: a good can not be split among multiple agents. Each agent $i \in N$ has a non-negative valuation $v_i : 2^M \to \mathbb{R}_{\geq 0}$, where 2^M is the power set of M. We assume that each valuation v_i is *additive*, i.e., $v_i(S) = \sum_{g \in S} v_i(\{g\})$ for any $S \subseteq M$. Note that since valuations are non-negative and additive, they have to be *normalized*: $v_i(\emptyset) = 0$ and *monotone*: $S \subseteq T$ implies $v_i(S) \leq v_i(T)$ for any $S, T \subseteq M$. For ease of explanation, we normalize the valuations so that $v_i(M) = 1$ for all $i \in N$.

To simplify notation, we denote $\{1, \ldots, k\}$ by $[k]$ for any positive integer k, write $v_i(g)$ instead of $v_i(\{g\})$ for $g \in M$, and use $S \setminus g$ and $S \cup g$ instead of $S \setminus \{g\}$ and $S \cup \{g\}$, respectively.

We say that $X = (X_1, X_2, \ldots, X_n)$ is an *allocation of M to N* if it is a partition of M into n disjoint subsets such that each set is indexed by $i \in N$. Each X_i is the set of goods given to agent i, which we call a *bundle*. It is simply called an *allocation to N* if M is clear from context. For $i \in N$ and $S \subseteq M$, let $m_i(S)$ denote the value of the least valuable good for agent i in S, that is, $m_i(S) = \min_{g \in S}\{v_i(g)\}$ if $S \neq \emptyset$ and $m_i(\emptyset) = 0$. For an allocation $X = (X_1, X_2, \ldots, X_n)$ to N, we say that an agent i is *PROPavg-satisfied* by X if

$$v_i(X_i) + \frac{1}{n-1} \sum_{k \in [n] \setminus i} m_i(X_k) \geq \frac{1}{n},$$

where we recall that $v_i(M) = 1$. In other words, agent i receives a set of goods for which she has value at least $1/n$ fraction of her total value minus the average of minimum value of the set of goods any other agent receives. An allocation X is called *PROPavg* if every agent $i \in N$ is PROPavg-satisfied by X.

Let $G = (V, E)$ be a graph. For $S \subseteq V$, let $\Gamma_G(S) = \{v \in V \setminus S \mid (s, v) \in E \text{ for some } s \in S\}$ denote the set of neighbors of S in G. For $v \in V$, let $G - v$ denote the graph obtained from G by deleting v. A *perfect matching* in G is a set of pairwise disjoint edges of G covering all the vertices of G.

3 Key Ingredient: PROPavg-Graph

In order to prove Theorem 2, we give an algorithm for finding a PROPavg allocation. As described in Section 1.3, our algorithm is a generalization of the cut-and-choose protocol that consists of the following three steps.

[2] This is an allocation that maximizes $\Pi_{i=1}^{n} v_i(X_i)$.

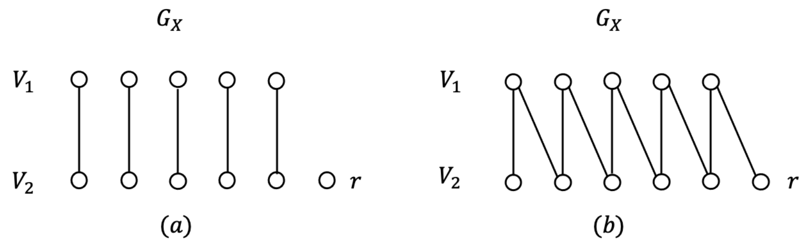

Figure 2 Examples of a PROPavg-graph G_X. If G_X is as in (a), then X satisfies (P1) but does not satisfy (P2). If G_X is as in (b), then X satisfies (P2).

1. We partition the goods into n bundles without assigning them to agents.
2. A specified agent, say n, chooses the best bundle for her valuation.
3. We determine an assignment of the remaining bundles to the agents in $N \setminus n$.

The partition given in the first step is represented by an allocation of M to a newly introduced set of size n, say V_2, and the assignment in the third step is represented by a matching in an auxiliary bipartite graph, which we call *PROPavg-graph*. In this section, we define the PROPavg-graph and its desired properties.

Let V_2 be a set of n elements and fix a specified element $r \in V_2$. We say that $X = (X_u)_{u \in V_2}$ is an *allocation to* V_2 if it is a partition of M into n disjoint subsets such that each set is indexed by an element in V_2, that is, $\bigcup_{u \in V_2} X_u = M$ and $X_u \cap X_{u'} = \emptyset$ for distinct $u, u' \in V_2$. For an allocation $X = (X_u)_{u \in V_2}$ to V_2, we define a bipartite graph $G_X = (V_1, V_2; E)$ called *PROPavg-graph* as follows. The vertex set consists of $V_1 = N \setminus n$ and V_2, and the edge set E is defined by

$$(i, u) \in E \iff v_i(X_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r, u\}} m_i(X_{u'}) \geq \frac{1}{n}$$

for $i \in V_1$ and $u \in V_2$. It should be emphasized that the summation is taken over $V_2 \setminus \{r, u\}$, i.e., $m_i(X_r)$ is not counted, in the above definition, which is crucial in our argument. The following lemma shows that the PROPavg-graph is closely related to the definition of PROPavg-satisfaction.

▶ **Lemma 3.** *Suppose that $G_X = (V_1, V_2; E)$ is the PROPavg-graph for an allocation $X = (X_u)_{u \in V_2}$ to V_2. Let σ be a bijection from N to V_2 and define an allocation $Y = (Y_1, \dots, Y_n)$ to N by $Y_i = X_{\sigma(i)}$ for $i \in N$. For $i^* \in V_1$, if $(i^*, \sigma(i^*)) \in E$, then i^* is PROPavg-satisfied by Y.*

Proof. Let $u^* = \sigma(i^*)$ and suppose that $(i^*, u^*) \in E$. We directly obtain

$$v_{i^*}(Y_{i^*}) + \frac{1}{n-1} \sum_{j \in [n] \setminus i^*} m_{i^*}(Y_j) \geq v_{i^*}(X_{u^*}) + \frac{1}{n-1} \sum_{u \in V_2 \setminus \{u^*, r\}} m_{i^*}(X_u) \geq \frac{1}{n},$$

where the first inequality follows from the definition of Y and $m_{i^*}(X_r) \geq 0$, and the second inequality follows from $(i^*, u^*) \in E$. ◀

As we will see in Section 4, throughout our algorithm, we always keep an allocation $X = (X_u)_{u \in V_2}$ to V_2 that satisfies the following property.

(P1) $G_X - r$ has a perfect matching.

By updating allocation X repeatedly while keeping (P1), we construct an allocation that satisfies the following stronger property.

(P2) For any $u \in V_2$, $G_X - u$ has a perfect matching.

Examples of a PROPavg-graph G_X are shown in Figure 2. We can rephrase these conditions by using the following classical theorem known as Hall's marriage theorem in discrete mathematics.

▶ **Theorem 4** (Hall's marriage theorem [21]). *Suppose that $G = (A, B; E)$ is a bipartite graph with $|A| = |B|$. Then, G has a perfect matching if and only if $|S| \leq |\Gamma_G(S)|$ for any $S \subseteq A$.*

The property (P1) is equivalent to $|S| \leq |\Gamma_{G_X - r}(S)|$ for any $S \subseteq V_1$ by this theorem. The property (P2) is equivalent to $|S| \leq |\Gamma_{G_X - u}(S)|$ for any $u \in V_2$ and $S \subseteq V_1$ by Hall's marriage theorem. By simple observation, we can obtain another characterization of property (P2).

▶ **Lemma 5.** *Let $X = (X_u)_{u \in V_2}$ be an allocation to V_2. Then, X satisfies (P2) if and only if $|S| + 1 \leq |\Gamma_{G_X}(S)|$ for any non-empty subset $S \subseteq V_1$.*

Proof. By Hall's marriage theorem, it is sufficient to show that the following two conditions are equivalent:

(i) $|S| \leq |\Gamma_{G_X - u}(S)|$ for any $u \in V_2$ and $S \subseteq V_1$, and

(ii) $|S| + 1 \leq |\Gamma_{G_X}(S)|$ for any non-empty subset $S \subseteq V_1$.

Suppose that (i) holds. Let S be a nonempty subset of V_1. Since (i) implies that $|\Gamma_{G_X}(S)| \geq |S| \geq 1$, we obtain $\Gamma_{G_X}(S) \neq \emptyset$. Let $u \in \Gamma_{G_X}(S)$. By (i) again, we obtain $|\Gamma_{G_X}(S)| = |\Gamma_{G_X - u}(S)| + 1 \geq |S| + 1$. This shows (ii).

Conversely, suppose that (ii) holds. Let $u \in V_2$ and let $S \subseteq V_1$. If $S = \emptyset$, then it clearly holds that $|S| \leq |\Gamma_{G_X - u}(S)|$. If $S \neq \emptyset$, then we have $|S| + 1 \leq |\Gamma_{G_X}(S)| \leq |\Gamma_{G_X - u}(S)| + 1$, which implies that $|S| \leq |\Gamma_{G_X - u}(S)|$. This shows (i). ◀

4 Existence of a PROPavg Allocation

We prove our main result, Theorem 2, in this section. Our algorithm begins with obtaining an initial allocation $X = (X_u)_{u \in V_2}$ to V_2 satisfying (P1). Unless X satisfies (P2), we appropriately choose a good in $\bigcup_{u \in V_2 \setminus r} X_u$ and move it to X_r while keeping (P1). Finally, we get an allocation $X^* = (X^*_u)_{u \in V_2}$ to V_2 satisfying (P2). As we will see later, we can obtain a PROPavg allocation to N by using this allocation.

4.1 Our Algorithm

In order to obtain an initial allocation $X = (X_u)_{u \in V_2}$ to V_2 satisfying (P1), we use the following previous result about EFX-with-charity.

▶ **Theorem 6** (Chaudhury et al. [16]). *For normalized and monotone valuations, there always exists an allocation $X = (X_1, \ldots, X_n)$ of $M \setminus U$ to N, where U is a set of unallocated goods, such that*

- *X is EFX, that is, $v_i(X_i) + m_i(X_j) \geq v_i(X_j)$ for any pair of agents $i, j \in N$,*
- *$v_i(X_i) \geq v_i(U)$ for any agent $i \in N$, and*
- *$|U| < n$.*

The following lemma shows that by applying Theorem 6 to agents $N \setminus n$, we can obtain an initial allocation $X = (X_u)_{u \in V_2}$ to V_2 satisfying (P1).

▶ **Lemma 7.** *There exists an allocation $X = (X_u)_{u \in V_2}$ to V_2 satisfying (P1).*

Proof. By applying Theorem 6 to agents $N\backslash n$, we can obtain an allocation $Y = (Y_1, \dots, Y_{n-1})$ of $M\backslash U$ to $N\backslash n$, where U is a set of unallocated goods, satisfying the conditions in Theorem 6. Let $V_2 = \{r, u_1, \dots, u_{n-1}\}$ and define an allocation $X = (X_u)_{u\in V_2}$ to V_2 as $X_{u_j} = Y_j$ for $j \in [n-1]$ and $X_r = U$. Let $G_X = (V_1, V_2; E)$ be the PROPavg-graph for X. We show that X satisfies (P1).

Fix any agent $i \in V_1$. We have $v_i(X_{u_i}) + m_i(X_{u_j}) \geq v_i(X_{u_j})$ for any $j \in [n-1] \setminus i$ since Y is EFX and $X_{u_j} = Y_j$. We also have $v_i(X_{u_i}) = v_i(Y_i) \geq v_i(U) = v_i(X_r)$ and a trivial inequality $v_i(X_{u_i}) \geq v_i(X_{u_i})$. By summing up these inequalities, we obtain $n \cdot v_i(X_{u_i}) + \sum_{j\in[n-1]\setminus i} m_i(X_{u_j}) \geq \sum_{u\in V_2} v_i(X_u) = 1$. This shows that

$$v_i(X_{u_i}) + \frac{1}{n-1} \sum_{u'\in V_2\setminus\{u_i,r\}} m_i(X_{u'}) \geq v_i(X_{u_i}) + \frac{1}{n} \sum_{j\in[n-1]\setminus i} m_i(X_{u_j}) \geq \frac{1}{n},$$

and hence $(i, u_i) \in E$. Therefore, $G_X - r$ has a perfect matching $\{(i, u_i) \mid i \in [n-1]\}$, which implies (P1). ◀

The following lemma shows that if we obtain an allocation $X = (X_u)_{u\in V_2}$ to V_2 satisfying (P2), then there exists a PROPavg allocation to N.

▶ **Lemma 8.** *Suppose that $X = (X_u)_{u\in V_2}$ is an allocation to V_2 satisfying (P2). Then, we can construct a PROPavg allocation to N.*

Proof. Let $X = (X_u)_{u\in V_2}$ be an allocation to V_2 satisfying (P2). First, agent n chooses the best bundle X_{u^*} for her valuation among $\{X_u \mid u \in V_2\}$ (if there is more than one such bundle, choose one arbitrarily). Since X satisfies (P2), there exists a perfect matching A in $G_X - u^*$. For each agent $i \in V_1 (= N \setminus n)$, the bundle corresponding to the vertex that matches i in A is allocated to i. By Lemma 3, i is PROPavg-satisfied for each agent $i \in V_1$. Furthermore, since we have $v_n(X_{u^*}) = \max_{u\in V_2} v_n(X_u) \geq \frac{1}{n}$, agent n is also PROPavg-satisfied. Therefore, the obtained allocation is a PROPavg allocation. ◀

The following proposition shows how we update an allocation in each iteration, whose proof is given in Section 4.2.

▶ **Proposition 9.** *Suppose that $X = (X_u)_{u\in V_2}$ is an allocation to V_2 that satisfies (P1) but does not satisfy (P2). Then, there exists another allocation $X' = (X'_u)_{u\in V_2}$ to V_2 satisfying (P1) such that $|X'_r| = |X_r| + 1$.*

We note that, as we will see in Section 4.2, the allocation X' in Proposition 9 is obtained by moving an appropriate good $g \in \bigcup_{u\in V_2\setminus r} X_u$ to X_r. If Proposition 9 holds, then we can show Theorem 2 as follows. See Algorithm 1 for the algorithm description.

Proof of Theorem 2. By Lemma 7, we first obtain an initial allocation $X = (X_u)_{u\in V_2}$ to V_2 satisfying (P1). By Proposition 9, unless X satisfies (P2), we can increase $|X_r|$ by one while keeping the property (P1). Since $|X_r| \leq |M|$, this procedure terminates in at most m steps, and we finally obtain an allocation X^* to V_2 satisfying (P2). Therefore, there exists a PROPavg allocation to N by Lemma 8. ◀

Algorithm 1 Algorithm for finding a PROPavg allocation.

Input: agents N, goods M, and a valuation v_i for each $i \in N$
Output: a PROPavg allocation to N

Apply Lemma 7 to obtain an allocation X to V_2 satisfying (P1).
while X does not satisfy (P2) **do**
  Apply Proposition 9 to X and obtain another allocation X' to V_2.
  $X \leftarrow X'$.
Apply Lemma 8 to obtain a PROPavg allocation to N.

4.2 Proof of Proposition 9

Let $X = (X_u)_{u \in V_2}$ be an allocation to V_2. For $u^* \in V_2 \setminus r$ and $g \in X_{u^*}$, we say that an allocation $X' = (X'_u)_{u \in V_2}$ to V_2 is obtained from X by *moving g to X_r* if

$$X'_u = \begin{cases} X_r \cup g & \text{if } u = r, \\ X_{u^*} \setminus g & \text{if } u = u^*, \\ X_u & \text{otherwise.} \end{cases}$$

The following lemma guarantees that if there exists an agent $i \in V_1$ such that $(i, r) \notin E$ in the PROPavg-graph $G_X = (V_1, V_2; E)$, then we can move some good in $\bigcup_{u \in V_2 \setminus r} X_u$ to X_r so that the edges incident to i do not disappear. This lemma is crucial in the proof of Proposition 9.

▶ **Lemma 10.** *Let $X = (X_u)_{u \in V_2}$ be an allocation to V_2 and let $i \in V_1$ be an agent such that $(i, r) \notin E$ in the PROPavg-graph $G_X = (V_1, V_2; E)$. Then, there exist $u^* \in V_2$ and $g \in X_{u^*}$ such that $(i, u^*) \in E$, $|X_{u^*}| \geq 2$, and the following property holds: if an allocation X' to V_2 is obtained from X by moving g to X_r, then the corresponding PROPavg-graph $G_{X'}$ has an edge (i, u^*).*

Proof. To derive a contradiction, assume that u^* and g satisfying the conditions in Lemma 10 do not exist. Then, we have the following claim.

▷ Claim 11. For any $u \in V_2$ with $(i, u) \in E$, we obtain

$$v_i(X_u) - m_i(X_u) + \frac{1}{n-1} \sum_{\substack{u' \in V_2 \setminus \{r, u\}: \\ (i, u') \in E}} m_i(X_{u'}) < \frac{1}{n}. \tag{1}$$

Proof of the Claim. Fix $u \in V_2$ with $(i, u) \in E$. If $X_u = \emptyset$, then we have

$$v_i(X_r) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus r} m_i(X_{u'}) \geq v_i(X_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r, u\}} m_i(X_{u'}) \geq \frac{1}{n},$$

where the first inequality follows from $v_i(X_u) = 0$ and the second inequality follows from $(i, u) \in E$. This contradicts $(i, r) \notin E$. Therefore, $X_u \neq \emptyset$.

Let g be a good in X_u that minimizes $v_i(g)$. Then, $v_i(g) = m_i(X_u)$. Define $X' = (X'_{u'})_{u' \in V_2}$ as the allocation to V_2 that is obtained from X by moving g to X_r. Let $G_{X'} = (V_1, V_2; E')$ be the PROPavg-graph corresponding to X'. Since u and g do not satisfy the conditions in Lemma 10 by our assumption, we have $(i, u) \notin E'$ or $|X_u| = 1$.

If $(i,u) \in E'$, then we obtain $|X_u| = 1$, and hence

$$v_i(X_r) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus r} m_i(X_{u'}) \geq \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X_{u'}) = v_i(X'_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X'_{u'}) \geq \frac{1}{n},$$

where the equality follows from $v_i(X'_u) = v_i(\emptyset) = 0$ and the last inequality follows from $(i,u) \in E'$. This contradicts $(i,r) \notin E$.

Thus, it holds that $(i,u) \notin E'$. Since $v_i(X_u) - m_i(X_u) = v_i(X'_u)$, we obtain

$$\begin{aligned} & v_i(X_u) - m_i(X_u) + \frac{1}{n-1} \sum_{\substack{u' \in V_2 \setminus \{r,u\}: \\ (i,u') \in E}} m_i(X_{u'}) \\ & \leq v_i(X_u) - m_i(X_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X_{u'}) \\ & = v_i(X'_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X'_{u'}) \\ & < \frac{1}{n}, \end{aligned}$$

where the last inequality follows from $(i,u) \notin E'$. ◁

By summing up inequality (1) for each $u \in V_2$ with $(i,u) \in E$, we obtain the following inequality:

$$\sum_{\substack{u \in V_2: \\ (i,u) \in E}} v_i(X_u) + \left(-1 + \frac{l-1}{n-1}\right) \sum_{\substack{u' \in V_2 \setminus r: \\ (i,u') \in E}} m_i(X_{u'}) < \frac{l}{n}, \tag{2}$$

where $l = |\{u \in V_2 \mid (i,u) \in E\}|$.

On the other hand, for any $u \in V_2$ with $(i,u) \notin E$, we have

$$v_i(X_u) + \frac{1}{n-1} \sum_{\substack{u' \in V_2 \setminus r: \\ (i,u') \in E}} m_i(X_{u'}) \leq v_i(X_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X_{u'}) < \frac{1}{n}, \tag{3}$$

where the both inequalities follow from $(i,u) \notin E$. Summing up inequality (3) for each $u \in V_2$ with $(i,u) \notin E$, we obtain

$$\sum_{\substack{u \in V_2: \\ (i,u) \notin E}} v_i(X_u) + \left(\frac{n-l}{n-1}\right) \sum_{\substack{u' \in V_2 \setminus r: \\ (i,u') \in E}} m_i(X_{u'}) < \frac{n-l}{n}, \tag{4}$$

where we note that $|\{u \in V_2 \mid (i,u) \notin E\}| = n - l$.

By taking the sum of inequalities (2) and (4), we obtain

$$\sum_{\substack{u \in V_2: \\ (i,u) \in E}} v_i(X_u) + \sum_{\substack{u \in V_2: \\ (i,u) \notin E}} v_i(X_u) < 1,$$

which contradicts $\sum_{u \in V_2} v_i(X_u) = 1$.

Therefore, there exist $u^* \in V_2$ and $g \in X_{u^*}$ satisfying the conditions in Lemma 10. ◀

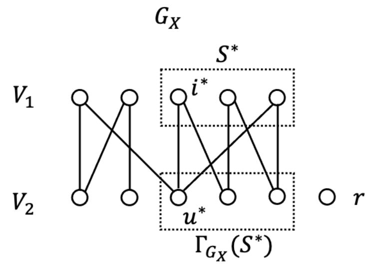

Figure 3 PROPavg-graph G_X corresponding to X in the proof of Proposition 9.

We are now ready to prove Proposition 9.

Proof of Proposition 9. Suppose that $X = (X_u)_{u \in V_2}$ is an allocation to V_2 that satisfies (P1) but does not satisfy (P2). Let $G_X = (V_1, V_2; E)$ be the PROPavg-graph corresponding to X. Since X does not satisfy (P2), there exists a non-empty set $S \subseteq V_1$ such that $|S| + 1 > |\Gamma_{G_X}(S)|$ by Lemma 5. Among such sets, let $S^* \subseteq V_1$ be an inclusion-wise minimal one. Then, $|S^*| \geq |\Gamma_{G_X}(S^*)|$ by the integrality of $|S^*|$ and $|\Gamma_{G_X}(S^*)|$, and $|S|+1 \leq |\Gamma_{G_X}(S)|$ for any non-empty proper subset $S \subsetneq S^*$. We now show some properties of S^*.

▷ Claim 12. For any $i \in S^*$, it holds that $(i, r) \notin E$.

Proof of the claim. Since X satisfies (P1), we have $|S^*| \leq |\Gamma_{G_X - r}(S^*)|$ by Hall's marriage theorem. Hence, we obtain $|S^*| \leq |\Gamma_{G_X - r}(S^*)| \leq |\Gamma_{G_X}(S^*)| \leq |S^*|$, where the last inequality follows from the definition of S^*. This shows that all the above inequalities are tight. Since $|\Gamma_{G_X - r}(S^*)| = |\Gamma_{G_X}(S^*)|$, we obtain $r \notin \Gamma_{G_X}(S^*)$, that is, $(i, r) \notin E$ for any $i \in S^*$. ◁

▷ Claim 13. For any $i \in S^*$ and $u \in \Gamma_{G_X}(S^*)$ with $(i, u) \in E$, $G_X - r$ has a perfect matching in which i matches u.

Proof of the claim. Fix any $i \in S^*$ and $u \in \Gamma_{G_X}(S^*)$ with $(i, u) \in E$. Note that $r \notin \Gamma_{G_X}(S^*)$ by Claim 12, and hence $u \neq r$.

Since X satisfies (P1), $G_X - r$ has a perfect matching A. In A, it is obvious that every vertex in S^* is matched to a vertex in $\Gamma_{G_X - r}(S^*)$. Conversely, every vertex in $\Gamma_{G_X - r}(S^*)$ is matched to a vertex in S^* as $|S^*| = |\Gamma_{G_X - r}(S^*)|$ (see the proof of Claim 12). Thus, by removing the edges between S^* and $\Gamma_{G_X}(S^*)$ from A, we obtain a matching $A_1 \subseteq A$ that exactly covers $V_1 \setminus S^*$ and $V_2 \setminus (\Gamma_{G_X}(S^*) \cup \{r\})$.

Let G'_X be the subgraph of G_X induced by $(S^* \setminus i) \cup (\Gamma_{G_X}(S^*) \setminus u)$. We now show that G'_X has a perfect matching. Consider any $S \subseteq S^* \setminus i$. If $S = \emptyset$, then it clearly holds that $|S| \leq |\Gamma_{G'_X}(S)|$. If $S \neq \emptyset$, then $|S| + 1 \leq |\Gamma_{G_X}(S)| \leq |\Gamma_{G'_X}(S) \cup u| = |\Gamma_{G'_X}(S)| + 1$, where the first inequality follows from the minimality of S^*. Therefore, $|S| \leq |\Gamma_{G'_X}(S)|$ holds for any $S \subseteq S^* \setminus i$, and hence G'_X has a perfect matching A_2 by Hall's marriage theorem.

Then, $A_1 \cup A_2 \cup \{(i, u)\}$ is a desired perfect matching in $G_X - r$. ◁

Fix any agent $i^* \in S^*$. Since $(i^*, r) \notin E$ by Claim 12, by applying Lemma 10 to agent i^*, we obtain $u^* \in V_2$ and $g \in X_{u^*}$ satisfying the conditions in Lemma 10 (see Figure 3). Let $X' = (X'_u)_{u \in V_2}$ be the allocation to V_2 obtained from X by moving g to X_r and let $G_{X'} = (V_1, V_2; E')$ be the PROPavg-graph corresponding to X'. Then, the conditions in Lemma 10 show that $(i^*, u^*) \in E \cap E'$ and $|X_{u^*}| \geq 2$. We also see that E' satisfies the following.

▷ Claim 14. For any $i \in V_1$ and $u \in V_2 \setminus u^*$, if $(i, u) \in E$ then $(i, u) \in E'$.

Proof of the claim. Since $|X_{u^*}| \geq 2$, we have $m_i(X'_{u^*}) = m_i(X_{u^*} \setminus g) \geq m_i(X_{u^*})$ for any agent $i \in V_1$. Hence, for any $i \in V_1$ and $u \in V_2 \setminus u^*$ with $(i, u) \in E$, we obtain

$$v_i(X'_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X'_{u'}) \geq v_i(X_u) + \frac{1}{n-1} \sum_{u' \in V_2 \setminus \{r,u\}} m_i(X_{u'}) \geq \frac{1}{n},$$

which shows that $(i, u) \in E'$. ◁

By Claim 13 and $(i^*, u^*) \in E$, there exists a perfect matching A in $G_X - r$ in which i^* matches u^*. Then, Claim 14 and $(i^*, u^*) \in E'$ show that $A \subseteq E'$, that is, A is a perfect matching also in $G_{X'} - r$. Therefore, X' satisfies (P1). Since $|X'_r| = |X_r| + 1$ clearly holds by definition, X' satisfies the conditions in Proposition 9. ◀

5 Discussion

In this paper, we have introduced PROPavg, which is a stronger notion than PROPm, and shown that a PROPavg allocation always exists.

As mentioned in Section 1.2, our algorithm runs in pseudo-polynomial time, and we do not know whether it can be improved to a polynomial-time algorithm. This is because we use Theorem 6 as a subroutine in order to obtain an initial allocation X to V_2 satisfying (P1). Actually, the proof of Theorem 6 given in [16] is constructive, but it only leads to a pseudo-polynomial time algorithm. We can see that the other parts of Algorithm 1 run in polynomial time as follows. In line 2, we can check (P2) in polynomial time by applying a maximum matching algorithm for each $G_X - u$. In line 3, it suffices to find a good $g \in \bigcup_{u \in V_2 \setminus r} X_u$ such that (P1) is kept after moving g. Since (P1) can be checked in polynomial time, this can be done in polynomial time by considering all g in a brute-force way. Finally, line 5 is executed in polynomial time by a maximum matching algorithm again. Note that we can speed up lines 2 and 3 by using the DM-decomposition of G_X [19, 20], but we do not go into details, because they are not the most time consuming part. We leave it open whether a PROPavg allocation can be found in polynomial time or not.

In order to devise our algorithm, we have developed a new technique that generalizes the cut-and-choose protocol. This technique is interesting by itself and seems to have a potential for further applications. In fact, we can define a bipartite graph like the PROPavg-graph for another fairness notion, and our argument works if we obtain an allocation satisfying a (P2)-like condition. We expect that this technique will be used in other contexts as well.

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