A Fixed-Parameter Algorithm for the Schrijver Problem

Ishay Haviv
School of Computer Science, The Academic College of Tel Aviv-Yaffo, Israel

Abstract

The Schrijver graph $S(n,k)$ is defined for integers n and k with $n \geq 2k$ as the graph whose vertices are all the k-subsets of $\{1,2,\ldots,n\}$ that do not include two consecutive elements modulo n, where two such sets are adjacent if they are disjoint. A result of Schrijver asserts that the chromatic number of $S(n,k)$ is $n-2k+2$ (Nieuw Arch. Wiskd., 1978). In the computational SCHRIJVER problem, we are given an access to a coloring of the vertices of $S(n,k)$ with $n-2k+1$ colors, and the goal is to find a monochromatic edge. The SCHRIJVER problem is known to be complete in the complexity class PPA. We prove that it can be solved by a randomized algorithm with running time $n^{O(1)} \cdot k^{O(k)}$, hence it is fixed-parameter tractable with respect to the parameter k.

2012 ACM Subject Classification Theory of computation → Fixed parameter tractability; Mathematics of computing → Graph coloring; Mathematics of computing → Combinatorial algorithms; Mathematics of computing → Probabilistic algorithms

Keywords and phrases Schrijver graph, Kneser graph, Fixed-parameter tractability

Digital Object Identifier 10.4230/LIPIcs.IPEC.2022.16

Related Version *Full Version*: `http://arxiv.org/abs/2204.09009`

Funding Research supported in part by the Israel Science Foundation (grant No. 1218/20).

Acknowledgements We thank Gabriel Istrate for clarifications on [18] and the anonymous reviewers for their useful suggestions.

1 Introduction

The Kneser graph $K(n,k)$ is defined for integers n and k with $n \geq 2k$ as the graph whose vertices are all the k-subsets of $[n] = \{1,2,\ldots,n\}$ where two such sets are adjacent if they are disjoint. In 1955, Kneser [19] observed that the chromatic number of the graph $K(n,k)$ satisfies $\chi(K(n,k)) \leq n-2k+2$, that is, there exists a proper coloring of its vertices with $n-2k+2$ colors, and conjectured that this upper bound on the chromatic number is tight. The conjecture was proved in 1978 by Lovász [20] as an application of the Borsuk-Ulam theorem from algebraic topology [2]. Following this result, topological methods have become a powerful tool in combinatorics, discrete geometry, and theoretical computer science (see, e.g., [22]).

The Schrijver graph $S(n,k)$ is defined as the subgraph of $K(n,k)$ induced by the collection of all k-subsets of $[n]$ that do not include two consecutive elements modulo n (i.e., the k-subsets $A \subseteq [n]$ such that if $i \in A$ then $i+1 \notin A$, and if $n \in A$ then $1 \notin A$). Schrijver proved in [26], strengthening Lovász's result, that the chromatic number of $S(n,k)$ is equal to that of $K(n,k)$. His proof technique relies on a proof of Kneser's conjecture due to Bárány [1], which was obtained soon after the one of Lovász and combined the topological Borsuk-Ulam theorem with a lemma of Gale [12]. It was further proved in [26] that $S(n,k)$ is vertex-critical, that is, the chromatic number of any proper induced subgraph of $S(n,k)$ is strictly smaller than that of $S(n,k)$.

17th International Symposium on Parameterized and Exact Computation (IPEC 2022).
Editors: Holger Dell and Jesper Nederlof; Article No. 16; pp. 16:1–16:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In the computational Kneser and Schrijver problems, we are given an access to a coloring with $n - 2k + 1$ colors of the vertices of $K(n, k)$ and $S(n, k)$ respectively, and the goal is to find a monochromatic edge, i.e., two vertices with the same color that correspond to disjoint sets. Since the number of colors used by the input coloring is strictly smaller than the chromatic number of the graph [20, 26], it follows that every instance of these problems has a solution. However, the topological argument behind the lower bound on the chromatic number is not constructive, in the sense that it does not suggest an efficient algorithm for finding a monochromatic edge. By an efficient algorithm we mean that its running time is polynomial in n, whereas the number of vertices might be exponentially larger. Hence, it is natural to assume that the input coloring is given as an access to an oracle that given a vertex of the graph returns its color. The input can also be given by some succinct representation, e.g., a Boolean circuit that computes the color of any given vertex.

In recent years, it has been shown that the complexity class PPA perfectly captures the complexity of several total search problems for which the existence of the solution relies on the Borsuk-Ulam theorem. This complexity class belongs to a family of classes that were introduced in 1994 by Papadimitriou [25] in the attempt to characterize the mathematical arguments that lie behind the existence of solutions to search problems of TFNP. The complexity class TFNP, introduced in [24], is the class of total search problems in NP, namely, the search problems in which a solution is guaranteed to exist and can be verified in polynomial running time. Papadimitriou has introduced in [25] several subclasses of TFNP, each of which consists of the total search problems that can be efficiently reduced to a problem that represents some mathematical argument. One of those subclasses was PPA (Polynomial Parity Argument) that corresponds to the fact that every (undirected) graph with maximum degree 2 that has a vertex of degree 1 must have another degree 1 vertex. Hence, PPA is the class of all problems in TFNP that can be efficiently reduced to the Leaf problem, in which given a succinct representation of a graph with maximum degree 2 and given a vertex of degree 1 in the graph, the goal is to find another such vertex.

A prominent example of a PPA-complete problem whose totality is related to the Borsuk-Ulam theorem is the one associated with the Consensus Halving theorem [17, 27]. The PPA-completeness of the problem was proved for an inverse-polynomial precision parameter by Filos-Ratsikas and Goldberg [7, 8], and this was improved to a constant precision parameter in a recent work of Deligkas, Fearnley, Hollender, and Melissourgos [4]. The hardness of the Consensus Halving problem was used to derive the PPA-completeness of several other problems. This includes the Splitting Necklace problem with two thieves, the Discrete Sandwich problem [7, 8], the Fair Independent Set in Cycle problem, and the aforementioned Schrijver problem (with the input coloring given as a Boolean circuit) [14]. As for the Kneser problem, the question of whether it is PPA-complete was proposed by Deng, Feng, and Kulkarni [5]. It is interesting to mention that this question is motivated by connections of the Kneser problem to a resource allocation problem called Agreeable Set, that was introduced by Manurangsi and Suksompong [21] and further studied in [13, 15]. It is also motivated by the extension of the Kneser problem to Kneser hypergraphs, for which the complexity question was raised by Filos-Ratsikas, Hollender, Sotiraki, and Zampetakis [9].

In the area of parameterized complexity, a problem whose instances involve a parameter k is said to be fixed-parameter tractable with respect to k if it admits an algorithm whose running time is bounded by a polynomial in the input size multiplied by an arbitrary function of k (see, e.g., [3]). Adopting this notion to our setting, where the instance is not given explicitly but as an oracle access, we say that an algorithm for the Kneser and Schrijver problems is fixed-parameter with respect to k if its running time on an input coloring of, respectively, $K(n,k)$ and $S(n,k)$ is bounded by $n^{O(1)} \cdot f(k)$ for some function f. In the recent work [15], it was shown that the KNESER problem is fixed-parameter tractable with respect to the parameter k. More specifically, it was shown there that there exists a randomized algorithm that solves the KNESER problem on an input coloring of a Kneser graph $K(n,k)$ in running time $n^{O(1)} \cdot k^{O(k)}$.

1.1 Our Contribution

In the current work, we prove that the SCHRIJVER problem on graphs $S(n,k)$ is fixed-parameter tractable with respect to the parameter k.

▶ **Theorem 1.** *There exists a randomized algorithm that given integers n and k with $n \geq 2k$ and an oracle access to a coloring of the vertices of the Schrijver graph $S(n,k)$ with $n-2k+1$ colors, runs in time $n^{O(1)} \cdot k^{O(k)}$ and returns a monochromatic edge with high probability.*

A few remarks about Theorem 1 are in order here.

- The algorithmic task of finding a monochromatic edge in the Schrijver graph $S(n,k)$ given a coloring of its vertices with $n-2k+1$ colors is at least as hard as that of finding a monochromatic edge in the Kneser graph $K(n,k)$ given such a coloring. Indeed, $S(n,k)$ is an induced subgraph of $K(n,k)$ with the same chromatic number. Therefore, the KNESER problem can be solved by applying an algorithm for the SCHRIJVER problem to the restriction of a coloring of a Kneser graph to its Schrijver subgraph. This implies that Theorem 1 strengthens the fixed-parameter tractability result of [15], and yet achieves the same asymptotic dependence on k in the running time.
- In contrast to the current situation of the KNESER problem [5], the SCHRIJVER problem is known to be PPA-complete [14]. Hence, the study of its fixed-parameter tractability is motivated in a stronger sense.
- As mentioned earlier, the Schrijver graph $S(n,k)$ was shown in [26] to be vertex-critical. It follows that for every vertex A of the graph $S(n,k)$, there exists a coloring of its vertices with $n-2k+1$ colors, for which only edges that are incident with A are monochromatic. An algorithm for the SCHRIJVER problem, while running on such an input coloring, must be able to find an edge that is incident with this specified vertex A. Nevertheless, the algorithm given in Theorem 1 manages to do so in running time much smaller than the number of vertices, provided that n is sufficiently larger than k.
- Borrowing the terminology of the area of parameterized complexity, our algorithm for the SCHRIJVER problem can be viewed as a randomized polynomial Turing kernelization algorithm for the problem (see, e.g., [10, Chapter 22]). Namely, the problem of finding a monochromatic edge in a Schrijver graph $S(n,k)$ can essentially be reduced by a randomized efficient algorithm to finding a monochromatic edge in a Schrijver graph $S(n',k)$ for $n' = O(k^4)$. This aspect of the algorithm is common to the algorithm for the KNESER problem given in [15] (see [15, Section 3.4] for the details).

Our algorithm for the SCHRIJVER problem extends the approach developed in [15] for the KNESER problem. The adaptation to the SCHRIJVER problem relies on structural properties of induced subgraphs of Schrijver graphs (see Section 3). Their proofs involve some ideas that were applied in the context of Frege propositional proof systems by Istrate, Bonchis, and Craciun [18]. In the remainder of this section, we give an overview of the proof of Theorem 1.

1.2 Proof Overview

Our algorithm for the Schrijver problem is based on the strategy developed in [15] for the Kneser problem. We start by describing the algorithm of [15] for the Kneser problem and then present the modification in the algorithm and in its analysis needed for the Schrijver problem.

Suppose that we are given an oracle access to a coloring of the vertices of the Kneser graph $K(n, k)$ with $n - 2k + 1$ colors. In order to find a monochromatic edge in the graph, we use an efficient algorithm, called "element elimination", that reduces our problem to that of finding a monochromatic edge in a subgraph of $K(n, k)$ isomorphic to $K(n - 1, k)$ whose vertices are colored by $n - 2k$ colors. Since the chromatic number of the latter is $n - 2k + 1$ [20], such a coloring is guaranteed to have a monochromatic edge in the subgraph. By repeatedly applying this algorithm, we obtain a coloring with $n' - 2k + 1$ colors of a subgraph of $K(n, k)$ isomorphic to $K(n', k)$. When the size n' of the ground set is sufficiently small and depends only on k, a brute force algorithm that queries the oracle for the colors of all vertices allows us to find a monochromatic edge in running time that essentially depends only on k.

We turn to describe now the "element elimination" algorithm. This algorithm picks uniformly and independently polynomially many vertices of the graph $K(n, k)$ and queries the oracle for their colors. If the random samples include two vertices that form a monochromatic edge in the graph, then this edge is returned and we are done. Otherwise, the algorithm identifies a color $i \in [n - 2k + 1]$ that appears on a largest number of vertices among the random samples and an element $j \in [n]$ that is particularly popular on the sampled vertices colored i (say, that belongs to a constant fraction of them). The "element elimination" algorithm suggests to remove the element j from the ground set, and to keep looking for a monochromatic edge in the subgraph induced by the k-subsets of $[n] \setminus \{j\}$.

The correctness of the "element elimination" algorithm for Kneser graphs relies on structural properties of intersecting families of k-subsets of $[n]$. A stability result of Hilton and Milner [16] for the celebrated Erdös-Ko-Rado theorem [6] says that any sufficiently large intersecting family of k-subsets of $[n]$ satisfies that all of its members share a common element. This is used in [15], combined with an idea of Frankl and Kupavskii [11], to show that as long as $n \geq \Omega(k^4)$, if a large color class of the input coloring does not have an element that is quite popular on its members, then the sampled vertices include with high probability a monochromatic edge. Otherwise, for every color i of a large color class there exists some element j that is popular on its vertices, and the algorithm finds such a pair (i, j) with high probability. If this element j belongs to *all* the vertices colored i, then the restriction of the input coloring to the k-subsets of $[n] \setminus \{j\}$ is a coloring with $n - 2k$ colors of a graph isomorphic to $K(n - 1, k)$, as required.

However, although the element j belongs to a significant fraction of the vertices colored i, the coloring might use the color i for vertices that do not include the element j. This might lead to an elimination of the element j while the restriction of the coloring to the k-subsets of $[n] \setminus \{j\}$ still uses the color i, hence the corresponding subgraph is not guaranteed to have a monochromatic edge. This situation is handled in [15] by showing that every vertex colored i that does not include j is disjoint from a non-negligible fraction of the vertices colored i. Therefore, in case that the brute force algorithm that is applied to the subgraph obtained after all iterations of the "element elimination" algorithm finds a vertex A colored by a color i that is associated with an eliminated element j, we pick uniformly at random vertices from the subgraph of the corresponding iteration and with high probability find a neighbor of A colored i and thus a monochromatic edge. This completes the high-level description of the algorithm for the Kneser problem from [15].

Our algorithm for finding a monochromatic edge in a Schrijver graph $S(n,k)$ given a coloring of its vertices with $n-2k+1$ colors also uses an "element elimination" algorithm as a main ingredient. Observe, however, that whenever an element $j \in [n]$ is eliminated, the subgraph induced by the k-subsets of $[n] \setminus \{j\}$ is not isomorphic to a Schrijver graph. We therefore consider the subgraph of $S(n,k)$ that corresponds to the cyclic ordering of the elements of $[n] \setminus \{j\}$ which is induced by the cyclic ordering of the elements of $[n]$ (where $j-1$ precedes $j+1$). This allows the algorithm to proceed by looking for a monochromatic edge in a graph isomorphic to $S(n-1,k)$. As before, the eliminated element is chosen as an element $j \in [n]$ that is quite popular on the vertices colored by a color i that corresponds to a large color class of the input coloring. The pair of the color i and the element j is identified using polynomially many vertices chosen uniformly at random from the vertex set of $S(n,k)$.

The main contribution of the current work lies in the analysis of the "element elimination" algorithm for Schrijver graphs. Consider first the case where the input coloring has a large color class that does not have a popular element in its members. For this case we prove that the selected random vertices include a monochromatic edge with high probability. In contrast to the analysis used for Kneser graphs, here we cannot apply the Hilton-Milner theorem [16] that deals with intersecting families of general k-subsets of $[n]$. We overcome this issue using a Hilton-Milner-type result for stable sets, i.e., for vertices of the Schrijver graph, borrowing ideas that were applied by Istrate, Bonchis, and Craciun [18] in the context of Frege propositional proof systems (see Lemma 7). Note that this can be interpreted as an approximate stability result for the analogue of the Erdös-Ko-Rado theorem for stable sets that was proved in 2003 by Talbot [28]. The Hilton-Milner-type result is combined with the approach of [15] and with an idea of [11] to prove that if the vertices of a large color class do not have a popular element, then a pair of vertices chosen uniformly at random from $S(n,k)$ forms a monochromatic edge with a non-negligible probability (see Lemma 8). Hence, picking a polynomial number of them suffices to catch such an edge.

Consider next the case where every large color class of the input coloring has a popular element. Here, the "element elimination" algorithm identifies with high probability a color i of a large color class and an element j that is popular on its vertices. If all the vertices colored i include j then we can safely look for a monochromatic edge in the subgraph of $S(n,k)$ induced by the cyclic ordering of $[n]$ without the element j, as this means that the size of the ground set and the number of colors are both reduced by 1. However, for the scenario where the color class of i involves vertices A that do not include j, we prove, as in [15], that such an A is disjoint from a random set from the color class of i with a non-negligible probability. Note that the analysis in this case again employs the ideas applied by Istrate et al. [18] (see Lemma 10). Then, when such a set A is found by the algorithm, we can go back to the subgraph of the run of the "element elimination" algorithm that identified the color of A and find a neighbor of A from this color class using random samples.

1.3 Outline

The rest of the paper is organized as follows. In Section 2, we gather several definitions and results that will be used throughout the paper. In Section 3, we present and prove several structural results on induced subgraphs of Schrijver graphs needed for the analysis of our algorithm. Finally, in Section 4, we present and analyze our randomized fixed-parameter algorithm for the SCHRIJVER problem and prove Theorem 1.

2 Preliminaries

2.1 Kneser and Schrijver Graphs

Consider the following definition.

▶ **Definition 2.** *For a family $\mathcal{F}$ of non-empty sets, let $K(\mathcal{F})$ denote the graph on the vertex set $\mathcal{F}$ in which two vertices are adjacent if they represent disjoint sets.*

For a set X and an integer k, let $\binom{X}{k}$ denote the family of all k-subsets of X. Note that the *Kneser graph* $K(n,k)$ can be defined for integers n and k with $n \geq 2k$ as the graph $K(\binom{[n]}{k})$.

A set $A \subseteq [n]$ is said to be *stable* if it does not include two consecutive elements modulo n, that is, it forms an independent set in the n-vertex cycle with the numbering from 1 to n along the cycle. For integers n and k with $n \geq 2k$, let $\binom{[n]}{k}_{\text{stab}}$ denote the collection of all stable k-subsets of $[n]$. The *Schrijver graph* $S(n,k)$ is defined as the graph $K(\binom{[n]}{k}_{\text{stab}})$. Equivalently, it is the subgraph of $K(n,k)$ induced by the vertex set $\binom{[n]}{k}_{\text{stab}}$.

For a set $X \subseteq [n]$ consider the natural cyclic ordering of the elements of X induced by that of $[n]$, and let $\binom{X}{k}_{\text{stab}}$ denote the collection of all k-subsets of X that do not include two consecutive elements according to this ordering. More formally, letting $j_1 < j_2 < \cdots < j_{|X|}$ denote the elements of X, $\binom{X}{k}_{\text{stab}}$ stands for the collection of all independent sets of size k in the cycle on the vertex set X with the numbering $j_1, \ldots, j_{|X|}$ along the cycle. Note that $\binom{X}{k}_{\text{stab}} \subseteq \binom{X'}{k}_{\text{stab}}$ whenever $X \subseteq X' \subseteq [n]$. Note further that the graph $K(\binom{X}{k}_{\text{stab}})$ is isomorphic to the Schrijver graph $S(|X|,k)$.

The chromatic number of the graph $S(n,k)$ was determined by Schrijver [26], strengthening a result of Lovász [20].

▶ **Theorem 3** ([26]). *For all integers n and k with $n \geq 2k$, $\chi(S(n,k)) = n - 2k + 2$.*

A family $\mathcal{F} \subseteq \binom{[n]}{k}$ of k-subsets of $[n]$ is called *intersecting* if for every two sets $F_1, F_2 \in \mathcal{F}$ it holds that $F_1 \cap F_2 \neq \emptyset$. Note that such a family forms an independent set in the graph $K(n,k)$. If the members of $\mathcal{F}$ share a common element, then we say that the intersecting family $\mathcal{F}$ is *trivial*.

The computational search problem associated with the Schrijver graph is defined as follows.

▶ **Definition 4.** *In the* Schrijver *problem, the input is a coloring $c : \binom{[n]}{k}_{\text{stab}} \to [n-2k+1]$ of the vertices of the Schrijver graph $S(n,k)$ with $n-2k+1$ colors for integers n and k with $n \geq 2k$, and the goal is to find a monochromatic edge.*

The existence of a solution to every instance of the Schrijver problem follows from Theorem 3. In our algorithm for the Schrijver problem, we consider the black-box input model, where the input coloring is given as an oracle access that for a vertex A returns its color $c(A)$. This reflects the fact that the algorithm does not rely on the representation of the input coloring.

2.2 Chernoff-Hoeffding Bound

We need the following concentration result (see, e.g., [23, Theorem 2.1]).

▶ **Theorem 5** (Chernoff-Hoeffding Bound). *Let $0 < p < 1$, let $X_1, \ldots, X_m$ be m independent binary random variables satisfying $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$ for all i, and put $\overline{X} = \frac{1}{m} \cdot \sum_{i=1}^{m} X_i$. Then, for any $\mu \geq 0$,*

$$\Pr\left[|\overline{X} - p| \geq \mu\right] \leq 2 \cdot e^{-2m\mu^2}.$$

3 Induced Subgraphs of Schrijver Graphs

In this section, we provide a couple of lemmas on induced subgraphs of Schrijver graphs that will play a central role in the analysis of our algorithm for the Schrijver problem. We start with some preliminary claims related to counting stable sets.

3.1 Counting Stable Sets

The following claim employs an argument of Istrate, Bonchis, and Craciun [18]. Its proof is omitted and can be found in the full version of the paper.

▷ Claim 6. For integers $k \geq 2$ and $n \geq 2k$ and for every two distinct integers $a, b \in [n]$, the number of stable k-subsets F of $[n]$ satisfying $\{a, b\} \subseteq F$ is at most $\binom{n-k-2}{k-2}$.

The following result stems from Claim 6 and can be viewed as an approximate variant of the Hilton-Milner theorem of [16] for stable sets (see [18]).

▶ **Lemma 7.** *For integers $k \geq 2$ and $n \geq 2k$, every non-trivial intersecting family $\mathcal{F}$ of stable k-subsets of $[n]$ satisfies $|\mathcal{F}| \leq k^2 \cdot \binom{n-k-2}{k-2}$.*

Proof. Let $\mathcal{F}$ be a non-trivial intersecting family of stable k-subsets of $[n]$. Consider an arbitrary set $A = \{a_1, \ldots, a_k\}$ in $\mathcal{F}$. Since $\mathcal{F}$ is non-trivial, for every $t \in [k]$, there exists a set $B_t \in \mathcal{F}$ satisfying $a_t \notin B_t$. Since $\mathcal{F}$ is intersecting, every set in $\mathcal{F}$ intersects A, and therefore includes the element a_t for some $t \in [k]$. Such a set further intersects B_t, hence it also includes some element $b \in B_t$ (which is different from a_t). By Claim 6, the number of stable k-subsets of $[n]$ that include both a_t and b does not exceed $\binom{n-k-2}{k-2}$. Since there are at most k^2 ways to choose the elements a_t and b, this implies that $|\mathcal{F}| \leq k^2 \cdot \binom{n-k-2}{k-2}$, as required. ◀

3.2 Induced Subgraphs of Schrijver Graphs

We are ready to prove the lemmas that lie at the heart of the analysis of our algorithm for the Schrijver problem. The first lemma, given below, shows that in a large induced subgraph of the Schrijver graph $S(n, k)$ whose vertices do not have a popular element, a random pair of vertices forms an edge with a non-negligible probability.

▶ **Lemma 8.** *For integers $k \geq 2$ and $n \geq 2k$, let $\mathcal{F}$ be a family of stable k-subsets of $[n]$ whose size satisfies $|\mathcal{F}| \geq k^2 \cdot \binom{n-k-2}{k-2}$ and let $\gamma \in (0, 1]$. Suppose that every element of $[n]$ belongs to at most γ fraction of the sets of $\mathcal{F}$. Then, the probability that two random sets chosen uniformly and independently from $\mathcal{F}$ are adjacent in $K(\mathcal{F})$ is at least*

$$\frac{1}{2} \cdot \left(1 - \gamma - \frac{k^2}{|\mathcal{F}|} \cdot \binom{n-k-2}{k-2}\right) \cdot \left(1 - \frac{k^2}{|\mathcal{F}|} \cdot \binom{n-k-2}{k-2}\right).$$

Proof. Let $\mathcal{F} \subseteq \binom{[n]}{k}_{\text{stab}}$ be a family of sets as in the statement of the lemma. We first claim that every subfamily $\mathcal{F}' \subseteq \mathcal{F}$ whose size satisfies

$$|\mathcal{F}'| \geq \gamma \cdot |\mathcal{F}| + k^2 \cdot \binom{n-k-2}{k-2} \tag{1}$$

spans an edge in $K(\mathcal{F})$. To see this, consider such an $\mathcal{F}'$, and notice that the assumption that every element of $[n]$ belongs to at most γ fraction of the sets of $\mathcal{F}$, combined with the fact that $|\mathcal{F}'| > \gamma \cdot |\mathcal{F}|$, implies that $\mathcal{F}'$ is not a trivial family, that is, its sets do not share a common element. In addition, using $|\mathcal{F}'| > k^2 \cdot \binom{n-k-2}{k-2}$, it follows from Lemma 7 that $\mathcal{F}'$ is not a non-trivial intersecting family. We thus conclude that $\mathcal{F}'$ is not an intersecting family, hence it spans an edge in $K(\mathcal{F})$.

We next show a lower bound on the size of a maximum matching in $K(\mathcal{F})$. Consider the process that maintains a subfamily $\mathcal{F}'$ of $\mathcal{F}$, initiated as $\mathcal{F}$, and that removes from $\mathcal{F}'$ the two endpoints of some edge spanned by $\mathcal{F}'$ as long as its size satisfies the condition given in (1). The pairs of vertices that are removed during the process form a matching $\mathcal{M}$ in $K(\mathcal{F})$, whose size satisfies

$$\begin{aligned} |\mathcal{M}| &\geq \frac{1}{2}\cdot\left(|\mathcal{F}|-\left(\gamma\cdot|\mathcal{F}|+k^2\cdot\binom{n-k-2}{k-2}\right)\right) \\ &= \frac{1}{2}\cdot\left((1-\gamma)\cdot|\mathcal{F}|-k^2\cdot\binom{n-k-2}{k-2}\right). \end{aligned} \tag{2}$$

We now consider the sum of the degrees of adjacent vertices in the graph $K(\mathcal{F})$. Let $A, B \in \mathcal{F}$ be any adjacent vertices in $K(\mathcal{F})$. Since A and B are adjacent, they satisfy $A \cap B = \emptyset$, hence every vertex of $\mathcal{F}$ that is not adjacent to A nor to B must include two distinct elements $a \in A$ and $b \in B$. For every two such elements, it follows from Claim 6 that the number of stable k-subsets of $[n]$ that include them both is at most $\binom{n-k-2}{k-2}$. Therefore, the number of vertices of $\mathcal{F}$ that are not adjacent to A nor to B does not exceed $k^2 \cdot \binom{n-k-2}{k-2}$. This implies that the degrees of A and B in $K(\mathcal{F})$ satisfy

$$d(A) + d(B) \geq |\mathcal{F}| - k^2 \cdot \binom{n-k-2}{k-2}.$$

Let $\mathcal{E}$ denote the edge set of $K(\mathcal{F})$. We combine the above bound with the lower bound given in (2) on the size of the matching $\mathcal{M}$, to obtain that

$$\begin{aligned} 2\cdot|\mathcal{E}| = \sum_{F\in\mathcal{F}} d(F) &\geq \sum_{\{A,B\}\in\mathcal{M}} (d(A)+d(B)) \geq |\mathcal{M}|\cdot\left(|\mathcal{F}|-k^2\cdot\binom{n-k-2}{k-2}\right) \\ &\geq \frac{1}{2}\cdot\left((1-\gamma)\cdot|\mathcal{F}|-k^2\cdot\binom{n-k-2}{k-2}\right)\cdot\left(|\mathcal{F}|-k^2\cdot\binom{n-k-2}{k-2}\right). \end{aligned}$$

Finally, consider a pair of random vertices chosen uniformly and independently from $\mathcal{F}$. The probability that they form an edge in $K(\mathcal{F})$ is twice the number of edges in $K(\mathcal{F})$ divided by $|\mathcal{F}|^2$. Hence, the above bound on $2 \cdot |\mathcal{E}|$ completes the proof. ◀

As a corollary of Lemma 8, we obtain the following. The proof is omitted.

▶ **Corollary 9.** *For integers $k \geq 2$ and $n \geq 10k^4$, let $\mathcal{F}$ be a family of stable k-subsets of $[n]$ of size $|\mathcal{F}| \geq \frac{1}{2n} \cdot \left|\binom{[n]}{k}_{\text{stab}}\right|$ and let $\gamma \in (0, 1]$. Suppose that every element of $[n]$ belongs to at most γ fraction of the sets of $\mathcal{F}$. Then, the probability that two random sets chosen uniformly and independently from $\mathcal{F}$ are adjacent in $K(\mathcal{F})$ is at least $\frac{3}{8} \cdot (\frac{3}{4} - \gamma)$.*

The following lemma shows that if a large collection of vertices of $S(n, k)$ has a quite popular element, then every k-subset of $[n]$ that does not include this element is disjoint from many of the vertices in the collection.

▶ **Lemma 10.** *For integers $k \geq 2$ and $n \geq 2k$, let $X \subseteq [n]$ be a set, let $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ be a family, and let $\gamma \in (0,1]$. Let $j \in X$ be an element that belongs to at least γ fraction of the sets of $\mathcal{F}$, and suppose that $A \in \binom{[n]}{k}$ is a set satisfying $j \notin A$. Then, the probability that a random set chosen uniformly from $\mathcal{F}$ is disjoint from A is at least*

$$\gamma - \frac{k}{|\mathcal{F}|} \cdot \binom{|X|-k-2}{k-2}.$$

Proof. Let $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ be a family as in the lemma, and put $\mathcal{F}' = \{F \in \mathcal{F} \mid j \in F\}$. By assumption, it holds that $|\mathcal{F}'| \geq \gamma \cdot |\mathcal{F}|$. Suppose that $A \in \binom{[n]}{k}$ is a set satisfying $j \notin A$. We claim that for every $i \in A$, the number of sets $B \in \binom{X}{k}_{\text{stab}}$ satisfying $\{i, j\} \subseteq B$ does not exceed $\binom{|X|-k-2}{k-2}$. Indeed, if $i \notin X$ then there are no such sets, so the bound trivially holds, and if $i \in X$, the bound follows from Claim 6, using the one-to-one correspondence between $\binom{X}{k}_{\text{stab}}$ and the vertex set of $S(|X|, k)$. This implies that the number of sets $B \in \binom{X}{k}_{\text{stab}}$ with $j \in B$ that intersect A does not exceed $k \cdot \binom{|X|-k-2}{k-2}$. It thus follows that the number of sets of $\mathcal{F}$ that are disjoint from A is at least

$$|\mathcal{F}'| - k \cdot \binom{|X|-k-2}{k-2} \geq \gamma \cdot |\mathcal{F}| - k \cdot \binom{|X|-k-2}{k-2}.$$

Hence, a random set chosen uniformly from $\mathcal{F}$ is disjoint from A with the desired probability. ◀

As a corollary of Lemma 10, we obtain the following. The proof is omitted.

▶ **Corollary 11.** *For integers $k \geq 2$ and n, let $X \subseteq [n]$ be a set of size $|X| \geq 10k^3$, let $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ be a family of size $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$, and let $\gamma \in (0,1]$. Let $j \in X$ be an element that belongs to at least γ fraction of the sets of $\mathcal{F}$, and suppose that $A \in \binom{[n]}{k}$ is a set satisfying $j \notin A$. Then, the probability that a random set chosen uniformly from $\mathcal{F}$ is disjoint from A is at least $\gamma - \frac{1}{4}$.*

4 A Fixed-Parameter Algorithm for the Schrijver Problem

In this section we provide our randomized fixed-parameter algorithm for the Schrijver problem. We first describe the "element elimination" algorithm and then use it to present the final algorithm and to prove Theorem 1.

4.1 The Element Elimination Algorithm

The "element elimination" algorithm, given by the following theorem, will be used to repeatedly reduce the size of the ground set of a Schrijver graph while looking for a monochromatic edge.

▶ **Theorem 12.** *There exists a randomized algorithm that given integers n and k, a set $X \subseteq [n]$ of size $|X| \geq 10k^4$, a parameter $\varepsilon > 0$, a set of colors $C \subseteq [n-2k+1]$ of size $|C| = |X| - 2k + 1$, and an oracle access to a coloring $c : \binom{X}{k}_{\text{stab}} \to [n-2k+1]$ of the vertices of $K(\binom{X}{k}_{\text{stab}})$, runs in time $\text{poly}(n, \ln(1/\varepsilon))$ and returns, with probability at least $1 - \varepsilon$,*

(a). *a monochromatic edge of $K(\binom{X}{k}_{\text{stab}})$, or*

(b). *a vertex $A \in \binom{X}{k}_{\text{stab}}$ satisfying $c(A) \notin C$, or*

(c). *a color $i \in C$ and an element $j \in X$ such that for every $A \in \binom{[n]}{k}_{\text{stab}}$ with $j \notin A$, a random vertex B chosen uniformly from $\binom{X}{k}_{\text{stab}}$ satisfies $c(B) = i$ and $A \cap B = \emptyset$ with probability at least $\frac{1}{9n}$.*

Proof. For integers n and k, let $X \subseteq [n]$, $C \subseteq [n-2k+1]$, and $c : \binom{X}{k}_{\text{stab}} \to [n-2k+1]$ be an input satisfying $|X| \geq 10k^4$ and $|C| = |X| - 2k + 1$ as in the statement of the theorem. It can be assumed that $k \geq 2$. Indeed, Theorem 3 guarantees that the graph $K(\binom{X}{k}_{\text{stab}})$, which is isomorphic to $S(|X|, k)$, has either a monochromatic edge or a vertex whose color does not belong to C. Hence, for $k = 1$, an output of type (a) or (b) can be found by querying the oracle for the colors of all the vertices in time polynomial in n. For $k \geq 2$, consider the algorithm that given an input as above acts as follows (see Algorithm 1).

The algorithm first selects uniformly and independently m random sets $A_1, \ldots, A_m \in \binom{X}{k}_{\text{stab}}$ for $m = b \cdot n^2 \cdot \ln(n/\varepsilon)$, where b is some fixed constant to be determined later (see lines 1–2), and queries the oracle for their colors. If the sampled sets include two vertices that form a monochromatic edge in $K(\binom{X}{k}_{\text{stab}})$, then the algorithm returns such an edge (output of type (a); see line 5). If they include a vertex whose color does not belong to C, then the algorithm returns it (output of type (b); see line 10). Otherwise, the algorithm defines $i^* \in C$ as a color that appears on a largest number of sampled sets A_t (see lines 13–16). It further defines $j^* \in X$ as an element that belongs to a largest number of sampled sets A_t with $c(A_t) = i^*$ (see lines 17–20). Then, the algorithm returns the pair (i^*, j^*) (output of type (c); see line 21).

The running time of the algorithm is clearly polynomial in n and in $\ln(1/\varepsilon)$. We turn to prove that for every input, the algorithm returns a valid output, of type (a), (b), or (c), with probability at least $1 - \varepsilon$. We start with the following lemma that shows that if the input coloring has a large color class with no popular element, then the algorithm returns a valid output of type (a) with the desired probability.

▶ **Lemma 13.** *Suppose that the input coloring c has a color class $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ of size $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$ such that every element of X belongs to at most half of the sets of $\mathcal{F}$. Then, Algorithm 1 returns a monochromatic edge with probability at least $1 - \varepsilon$.*

Proof. Let $\mathcal{F}$ be as in the lemma. Using the assumptions $k \geq 2$ and $|X| \geq 10k^4$ and using the one-to-one correspondence between $\binom{X}{k}_{\text{stab}}$ and the vertex set of $S(|X|, k)$, we can apply Corollary 9 with $\gamma = \frac{1}{2}$ to obtain that two random sets chosen uniformly and independently from $\mathcal{F}$ are adjacent in $K(\mathcal{F})$ with probability at least $\frac{3}{8} \cdot (\frac{3}{4} - \gamma) = \frac{3}{32}$. Further, since the family $\mathcal{F}$ satisfies $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$, a random vertex chosen uniformly from $\binom{X}{k}_{\text{stab}}$ belongs to $\mathcal{F}$ with probability at least $\frac{1}{2|X|}$. Hence, for two random vertices chosen uniformly and independently from $\binom{X}{k}_{\text{stab}}$, the probability that they both belong to $\mathcal{F}$ is at least $(\frac{1}{2|X|})^2$, and conditioned on this event, their probability to form an edge in $K(\mathcal{F})$ is at least $\frac{3}{32}$. This implies that the probability that two random vertices chosen uniformly and independently from $\binom{X}{k}_{\text{stab}}$ form a monochromatic edge in $K(\binom{X}{k}_{\text{stab}})$ is at least $(\frac{1}{2|X|})^2 \cdot \frac{3}{32} = \frac{3}{128|X|^2}$.

Now, by considering $\lfloor m/2 \rfloor$ pairs of the random sets chosen by Algorithm 1 (line 2), it follows that the probability that no pair forms a monochromatic edge does not exceed

$$\left(1 - \frac{3}{128|X|^2}\right)^{\lfloor m/2 \rfloor} \leq e^{-3 \cdot \lfloor m/2 \rfloor / (128|X|^2)} \leq \varepsilon,$$

where the last inequality follows by $|X| \leq n$ and by the choice of m, assuming that the constant b is sufficiently large. It thus follows that with probability at least $1 - \varepsilon$, the algorithm returns a monochromatic edge, as required. ◀

Algorithm 1 Element Elimination Algorithm (Theorem 12).

Input: Integers n and $k \geq 2$, a set $X \subseteq [n]$ of size $|X| \geq 10k^4$, a set of colors $C \subseteq [n-2k+1]$ of size $|C| = |X| - 2k + 1$, and an oracle access to a coloring $c : \binom{X}{k}_{\text{stab}} \to [n-2k+1]$ of $K(\binom{X}{k}_{\text{stab}})$.

Output: (a) A monochromatic edge of $K(\binom{X}{k}_{\text{stab}})$, or (b) a vertex $A \in \binom{X}{k}_{\text{stab}}$ satisfying $c(A) \notin C$, or (c) a color $i \in C$ and an element $j \in X$ such that for every $A \in \binom{[n]}{k}_{\text{stab}}$ with $j \notin A$, a random vertex $B \in \binom{X}{k}_{\text{stab}}$ chosen uniformly satisfies $c(B) = i$ and $A \cap B = \emptyset$ with probability at least $\frac{1}{9n}$.

1: $m \leftarrow b \cdot n^2 \cdot \ln(n/\varepsilon)$ for a sufficiently large constant b
2: pick uniformly and independently at random sets $A_1, \dots, A_m \in \binom{X}{k}_{\text{stab}}$
3: **for all** $t, t' \in [m]$ **do**
4:     **if** $c(A_t) = c(A_{t'})$ and $A_t \cap A_{t'} = \emptyset$ **then**
5:         **return** $\{A_t, A_{t'}\}$ ▷ output of type (a)
6:     **end if**
7: **end for**
8: **for all** $t \in [m]$ **do**
9:     **if** $c(A_t) \notin C$ **then**
10:         **return** A_t ▷ output of type (b)
11:     **end if**
12: **end for**
13: **for all** $i \in C$ **do**
14:     $\widetilde{\alpha}_i \leftarrow \frac{1}{m} \cdot |\{t \in [m] \mid c(A_t) = i\}|$
15: **end for**
16: $i^* \leftarrow$ an $i \in C$ with largest value of $\widetilde{\alpha}_i$
17: **for all** $j \in X$ **do**
18:     $\widetilde{\gamma}_{i^*,j} \leftarrow \frac{1}{m} \cdot |\{t \in [m] \mid c(A_t) = i^* \text{ and } j \in A_t\}|$
19: **end for**
20: $j^* \leftarrow$ a $j \in X$ with largest value of $\widetilde{\gamma}_{i^*,j}$
21: **return** (i^*, j^*) ▷ output of type (c)

We next handle the case in which every large color class of the input coloring has a popular element. To do so, we first show that the samples of the algorithm provide a good estimation for the fraction of vertices in each color class as well as for the fraction of vertices that share any given element in each color class. For every color $i \in C$, let α_i denote the fraction of vertices of $K(\binom{X}{k}_{\text{stab}})$ colored i, that is,

$$\alpha_i = \frac{|\{A \in \binom{X}{k}_{\text{stab}} \mid c(A) = i\}|}{|\binom{X}{k}_{\text{stab}}|},$$

and let $\widetilde{\alpha}_i$ denote the fraction of the vertices sampled by the algorithm that are colored i (see line 14). Similarly, for every $i \in C$ and $j \in X$, let $\gamma_{i,j}$ denote the fraction of vertices of $K(\binom{X}{k}_{\text{stab}})$ colored i that include j, that is,

$$\gamma_{i,j} = \frac{|\{A \in \binom{X}{k}_{\text{stab}} \mid c(A) = i \text{ and } j \in A\}|}{|\binom{X}{k}_{\text{stab}}|},$$

and let $\widetilde{\gamma}_{i,j}$ denote the fraction of the vertices sampled by the algorithm that are colored i and include j. Let E denote the event that

$$|\alpha_i - \widetilde{\alpha}_i| \leq \frac{1}{2|X|} \quad \text{and} \quad |\gamma_{i,j} - \widetilde{\gamma}_{i,j}| \leq \frac{1}{2|X|} \quad \text{for all } i \in C,\ j \in X. \tag{3}$$

By a standard concentration argument, we obtain the following lemma.

▶ **Lemma 14.** *The probability of the event E is at least $1-\varepsilon$.*

Proof. By the Chernoff-Hoeffding bound (Theorem 5) applied with $\mu = \frac{1}{2|X|}$, the probability that an inequality from (3) does not hold is at most

$$2 \cdot e^{-2m/(4|X|^2)} \leq \frac{\varepsilon}{n^2},$$

where the inequality follows by $|X| \leq n$ and by the choice of m, assuming that the constant b is sufficiently large. By the union bound over all the colors $i \in C$ and all the pairs $(i,j) \in C \times X$, that is, over $|C| + |C| \cdot |X| = |C| \cdot (1+|X|) \leq n^2$ events, we get that all the inequalities in (3) hold with probability at least $1 - n^2 \cdot \frac{\varepsilon}{n^2} = 1-\varepsilon$, as required. ◀

We now show that if every large color class of the input coloring has a popular element and the event E occurs, then the algorithm returns a valid output.

▶ **Lemma 15.** *Suppose that the coloring c satisfies that for every color class $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ whose size satisfies $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$ there exists an element of X that belongs to more than half of the sets of $\mathcal{F}$. Then, if the event E occurs, Algorithm 1 returns a valid output.*

Proof. Assume that the event E occurs. If Algorithm 1 returns an output of type (a) or (b), i.e., a monochromatic edge or a vertex whose color does not belong to C, then the output is verified before it is returned and is thus valid. So suppose that the algorithm returns a pair $(i^*, j^*) \in C \times X$. Recall that the color i^* is defined by Algorithm 1 as an $i \in C$ with largest value of $\widetilde{\alpha}_i$ (see line 16). Since the colors of all the sampled sets belong to C, it follows that $\sum_{i \in C} \widetilde{\alpha}_i = 1$, and thus

$$\widetilde{\alpha}_{i^*} \geq \frac{1}{|C|} \geq \frac{1}{|X|}, \tag{4}$$

where the last inequality follows by $|C| = |X| - 2k + 1 \leq |X|$.

Let $\mathcal{F}$ be the family of vertices of $K(\binom{X}{k}_{\text{stab}})$ colored i^*, i.e.,

$$\mathcal{F} = \{A \in \binom{X}{k}_{\text{stab}} \mid c(A) = i^*\}.$$

Since the event E occurs (see (3)), it follows from (4) that

$$|\mathcal{F}| = \alpha_{i^*} \cdot \left|\binom{X}{k}_{\text{stab}}\right| \geq \left(\widetilde{\alpha}_{i^*} - \frac{1}{2|X|}\right) \cdot \left|\binom{X}{k}_{\text{stab}}\right| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|.$$

Hence, by the assumption of the lemma, there exists an element $j \in X$ that belongs to more than half of the sets of $\mathcal{F}$, that is, $\gamma_{i^*,j} > \frac{1}{2}$. Since the event E occurs, it follows that this j satisfies $\widetilde{\gamma}_{i^*,j} > \frac{1}{2} - \frac{1}{2|X|}$. Recalling that the element j^* is defined by Algorithm 1 as a $j \in X$ with largest value of $\widetilde{\gamma}_{i^*,j}$ (see line 20), it must satisfy $\widetilde{\gamma}_{i^*,j^*} > \frac{1}{2} - \frac{1}{2|X|}$, and using again the fact that the event E occurs, we derive that $\gamma_{i^*,j^*} \geq \widetilde{\gamma}_{i^*,j^*} - \frac{1}{2|X|} > \frac{1}{2} - \frac{1}{|X|}$.

By $k \geq 2$ and $|X| \geq 10k^4$, we can apply Corollary 11 with $\mathcal{F}$, j^*, and $\gamma = \frac{1}{2} - \frac{1}{|X|}$ to obtain that for every set $A \in \binom{[n]}{k}_{\text{stab}}$ with $j^* \notin A$, the probability that a random set chosen uniformly from $\mathcal{F}$ is disjoint from A is at least $\gamma - \frac{1}{4}$. Since the probability that a random set chosen uniformly from $\binom{X}{k}_{\text{stab}}$ belongs to $\mathcal{F}$ is at least $\frac{1}{2|X|}$, it follows that the probability that a random set B chosen uniformly from $\binom{X}{k}_{\text{stab}}$ satisfies $c(B) = i^*$ and $A \cap B = \emptyset$ is at least

$$\frac{1}{2|X|} \cdot \left(\gamma - \frac{1}{4}\right) = \frac{1}{2|X|} \cdot \left(\frac{1}{4} - \frac{1}{|X|}\right) \geq \frac{1}{9|X|} \geq \frac{1}{9n},$$

where the first inequality holds because $k \geq 2$ and $|X| \geq 10k^4$. This implies that (i^*, j^*) is a valid output of type (c). ◀

Equipped with Lemmas 13, 14, and 15, we are ready to derive the correctness of Algorithm 1 and to complete the proof of Theorem 12. If the input coloring c has a color class $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ of size $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$ such that every element of X belongs to at most half of the sets of $\mathcal{F}$, then, by Lemma 13, the algorithm returns with probability at least $1 - \varepsilon$ a monochromatic edge, i.e., a valid output of type (a). Otherwise, the input coloring c satisfies that for every color class $\mathcal{F} \subseteq \binom{X}{k}_{\text{stab}}$ of size $|\mathcal{F}| \geq \frac{1}{2|X|} \cdot \left|\binom{X}{k}_{\text{stab}}\right|$ there exists an element of X that belongs to more than half of the sets of $\mathcal{F}$. By Lemma 14, the event E occurs with probability at least $1 - \varepsilon$, implying by Lemma 15 that with such probability, the algorithm returns a valid output. It thus follows that for every input coloring the algorithm returns a valid output with probability at least $1 - \varepsilon$, and we are done. ◀

4.2 The Fixed-Parameter Algorithm for the Schrijver Problem

We turn to present our fixed-parameter algorithm for the Schrijver problem and to complete the proof of Theorem 1.

Proof of Theorem 1. Suppose that we are given, for integers n and k with $n \geq 2k$, an oracle access to a coloring $c : \binom{[n]}{k}_{\text{stab}} \to [n - 2k + 1]$ of the vertices of the Schrijver graph $S(n, k)$. It suffices to present an algorithm with success probability at least $1/2$, because the latter can be easily amplified by repetitions. Our algorithm has two phases, as described below (see Algorithm 2).

In the first phase, the algorithm repeatedly applies the "element elimination" algorithm given in Theorem 12 (Algorithm 1). Initially, we define

$$s = \max(n - 10k^4, 0), \quad X_0 = [n], \quad \text{and} \quad C_0 = [n - 2k + 1].$$

In the lth iteration, $0 \leq l < s$, we call Algorithm 1 with n, k, X_l, C_l, $\varepsilon = \frac{1}{4n}$ and with the restriction of the given coloring c to the vertices of $\binom{X_l}{k}_{\text{stab}}$ to obtain with probability at least $1 - \varepsilon$,

(a). a monochromatic edge $\{A, B\}$ of $K(\binom{X_l}{k}_{\text{stab}})$, or
(b). a vertex $A \in \binom{X_l}{k}_{\text{stab}}$ satisfying $c(A) \notin C_l$, or
(c). a color $i_l \in C_l$ and an element $j_l \in X_l$ such that for every $A \in \binom{[n]}{k}_{\text{stab}}$ with $j_l \notin A$, a random vertex B chosen uniformly from $\binom{X_l}{k}_{\text{stab}}$ satisfies $c(B) = i_l$ and $A \cap B = \emptyset$ with probability at least $\frac{1}{9n}$.

As will be explained shortly, if the output of Algorithm 1 is of type (a) or (b) then we either return a monochromatic edge or declare "failure", and if the output is a pair (i_l, j_l) of type (c) then we define $X_{l+1} = X_l \setminus \{j_l\}$ and $C_{l+1} = C_l \setminus \{i_l\}$ and, as long as $l < s$, proceed to the next call of Algorithm 1. Note that the sizes of the sets X_l and C_l are reduced by 1 in every iteration, hence we maintain the equality $|C_l| = |X_l| - 2k + 1$ for all l. We now describe how the algorithm acts in the lth iteration for each type of output returned by Algorithm 1.

Algorithm 2 The Algorithm for the Schrijver Problem (Theorem 1).

Input: Integers n, k with $n \geq 2k$ and an oracle access to a coloring $c : \binom{[n]}{k}_{\text{stab}} \to [n-2k+1]$.
Output: A monochromatic edge of $S(n, k)$.

1: $s \leftarrow \max(n - 10k^4, 0)$, $X_0 \leftarrow [n]$, $C_0 \leftarrow [n - 2k + 1]$ ▷ $|C_0| = |X_0| - 2k + 1$
2: **for all** $l = 0, 1, \ldots, s - 1$ **do** ▷ first phase
3: **call** Algorithm 1 with n, k, X_l, C_l, $\varepsilon = \frac{1}{4n}$ and with the restriction of c to $\binom{X_l}{k}_{\text{stab}}$
4: **if** Algorithm 1 returns an edge $\{A, B\}$ with $c(A) = c(B)$ **then** ▷ output of type (a)
5: **return** $\{A, B\}$
6: **end if**
7: **if** Algorithm 1 returns a vertex $A \in \binom{X_l}{k}_{\text{stab}}$ with $c(A) = i_r \notin C_l$ **then** ▷ output of type (b)
8: **for all** $t \in [18n]$ **do**
9: pick uniformly at random a set $B_t \in \binom{X_r}{k}_{\text{stab}}$
10: **if** $c(B_t) = i_r$ and $A \cap B_t = \emptyset$ **then**
11: **return** $\{A, B_t\}$
12: **end if**
13: **end for**
14: **return** "failure"
15: **end if**
16: **if** Algorithm 1 returns a pair $(i_l, j_l) \in C_l \times X_l$ **then** ▷ output of type (c)
17: $X_{l+1} \leftarrow X_l \setminus \{j_l\}$, $C_{l+1} \leftarrow C_l \setminus \{i_l\}$ ▷ $|C_{l+1}| = |X_{l+1}| - 2k + 1$
18: **end if**
19: **end for**
20: query the oracle for the colors of all the vertices of $K(\binom{X_s}{k}_{\text{stab}})$ ▷ second phase
21: **if** there exists a vertex $A \in \binom{X_s}{k}_{\text{stab}}$ of color $c(A) = i_r \notin C_s$ **then**
22: **for all** $t \in [18n]$ **do**
23: pick uniformly at random a set $B_t \in \binom{X_r}{k}_{\text{stab}}$
24: **if** $c(B_t) = i_r$ and $A \cap B_t = \emptyset$ **then**
25: **return** $\{A, B_t\}$
26: **end if**
27: **end for**
28: **return** "failure"
29: **else**
30: find $A, B \in \binom{X_s}{k}_{\text{stab}}$ satisfying $c(A) = c(B)$ and $A \cap B = \emptyset$ ▷ exist by Theorem 3 [20]
31: **return** $\{A, B\}$
32: **end if**

If the output is of type (a), then the returned monochromatic edge of $K(\binom{X_l}{k}_{\text{stab}})$ is also a monochromatic edge of $S(n, k)$, so we return it (see lines 4–6).

If the output is of type (b), then we are given a vertex $A \in \binom{X_l}{k}_{\text{stab}}$ satisfying $c(A) = i_r \notin C_l$ for some $r < l$. Since $i_r \notin C_l$, it follows that $j_r \notin X_l$, and thus $j_r \notin A$. In this case, we pick uniformly and independently $18n$ random sets from $\binom{X_r}{k}_{\text{stab}}$ and query the oracle for their colors. If we find a vertex B that forms together with A a monochromatic edge in $S(n, k)$, we return the monochromatic edge $\{A, B\}$, and otherwise we declare "failure" (see lines 7–15).

If the output of Algorithm 1 is a pair (i_l, j_l) of type (c), then we define, as mentioned above, the sets $X_{l+1} = X_l \setminus \{j_l\}$ and $C_{l+1} = C_l \setminus \{i_l\}$ (see lines 16–18). Observe that for $0 \leq l < s$, it holds that $|X_l| = n - l > n - s = 10k^4$, allowing us, by Theorem 12, to call Algorithm 1 in the lth iteration.

In case that all the s calls to Algorithm 1 return an output of type (c), we arrive to the second phase of the algorithm. Here, we are given the sets X_s and C_s that satisfy $|X_s| = n - s \leq 10k^4$ and $|C_s| = |X_s| - 2k + 1$, and we query the oracle for the colors of each and every vertex of the graph $K(\binom{X_s}{k}_{\text{stab}})$. If we find a vertex $A \in \binom{X_s}{k}_{\text{stab}}$ satisfying $c(A) = i_r \notin C_s$ for some $r < s$, then, as before, we pick uniformly and independently $18n$ random sets from $\binom{X_r}{k}_{\text{stab}}$ and query the oracle for their colors. If we find a vertex B that forms together with A a monochromatic edge in $S(n,k)$, we return the monochromatic edge $\{A, B\}$, and otherwise we declare "failure" (see lines 21–28). Otherwise, all the vertices of $K(\binom{X_s}{k}_{\text{stab}})$ are colored by colors from C_s. By Theorem 3, the chromatic number of the graph $K(\binom{X_s}{k}_{\text{stab}})$, which is isomorphic to $S(|X_s|, k)$, is $|X_s| - 2k + 2 > |C_s|$. Hence, there must exist a monochromatic edge in $K(\binom{X_s}{k}_{\text{stab}})$, and by checking all the pairs of its vertices we find such an edge and return it (see lines 30–31).

We turn to analyze the probability that Algorithm 2 returns a monochromatic edge. Note that whenever the algorithm returns an edge, it checks that it is monochromatic and thus ensures that it forms a valid solution. Hence, it suffices to show that the algorithm declares "failure" with probability at most 1/2. To see this, recall that the algorithm calls Algorithm 1 at most $s < n$ times, and that by Theorem 12 the probability that its output is not valid is at most $\varepsilon = \frac{1}{4n}$. By the union bound, the probability that any of the calls to Algorithm 1 returns an invalid output does not exceed 1/4. The only situation in which Algorithm 2 declares "failure" is when it finds, for some $r < s$, a vertex $A \in \binom{[n]}{k}_{\text{stab}}$ with $c(A) = i_r$ and $j_r \notin A$, and none of the $18n$ sampled sets $B \in \binom{X_r}{k}_{\text{stab}}$ satisfies $c(B) = i_r$ and $A \cap B = \emptyset$ (see lines 7–15, 21–28). However, assuming that all the calls to Algorithm 1 return valid outputs, the rth run guarantees, by Theorem 12, that a random vertex B uniformly chosen from $\binom{X_r}{k}_{\text{stab}}$ satisfies $c(B) = i_r$ and $A \cap B = \emptyset$ for the given A with probability at least $\frac{1}{9n}$. Hence, the probability that the algorithm declares "failure" does not exceed $(1 - \frac{1}{9n})^{18n} \leq e^{-2} < \frac{1}{4}$. Using again the union bound, it follows that the probability that Algorithm 2 either gets an invalid output from Algorithm 1 or fails to find a vertex that forms a monochromatic edge with a set A as above is at most 1/2. Therefore, the probability that Algorithm 2 successfully finds a monochromatic edge is at least 1/2, as desired.

We finally analyze the running time of Algorithm 2. In its first phase, the algorithm calls Algorithm 1 at most $s < n$ times, where the running time needed for each call is, by Theorem 12 and by our choice of ε, polynomial in n. It is clear that the other operations made throughout this phase can also be implemented in time polynomial in n. In its second phase, the algorithm enumerates all the vertices of $K(\binom{X_s}{k}_{\text{stab}})$. This phase can be implemented in running time polynomial in n and in the number of vertices of this graph. The latter is $|\binom{X_s}{k}_{\text{stab}}| \leq |X_s|^k \leq (10k^4)^k = k^{O(k)}$. It thus follows that the total running time of Algorithm 2 is $n^{O(1)} \cdot k^{O(k)}$, completing the proof. ◀

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